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Job contact networks

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Abstract

Many workers hear about or obtain their jobs through friends and relatives. The aim of this paper is twofold. First, we relate both individual and aggregate labor market outcomes to the network structure of personal contacts. Second, we study strategic network formation. To this purpose, we develop a model specifying at the individual level both the decision to form contacts with other agents, and the process by which information about jobs is obtained and transmitted. We show that equilibrium networks always exist and that only moderate levels of network asymmetry can be sustained at equilibrium. Also, we establish a general non-monotonicity result on information flow and unemployment with respect to network size in symmetric networks.

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1. Introduction

It is widely known and documented that many workers obtain their jobs through friends and relatives. A large number of studies, going back to [6] and [10] conclude that about half of all jobs are filled through contacts.¹ The aim of this paper is twofold. First, we relate both individual and aggregate labor market outcomes to the network structure of contacts. Second, we study strategic network formation.

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¹Of course, the precise figures vary with the type of occupation considered and with the socio-economic characteristics of the worker. See [9] for more details.

Our model specifies at the individual level both the decision to create links and the process by which information about jobs is obtained and transmitted. Ex ante identical players choose to form links with others. Links are costly and created by mutual consent. When players become unemployed, links may provide access to job offers the following way. Each player receives a job offer with some exogenous probability. While unemployed players immediately take any such offer, employed players pass it along to the unemployed players they are linked with, if any. Players thus partly rely on their contacts to gather information about jobs.

We first show that the information inflow to any player is shaped by the network structure of his direct and two-links-away contacts. Direct contacts are beneficial whereas two-links-away contacts are detrimental. The net balance of these two effects is very much dependent on the details of the network structure. In particular, two networks with the same total number of links but different geometry may induce different aggregate unemployment levels. Also, there is a general non-monotonicity result on information flow and unemployment with respect to network size in symmetric networks (where each player has the same number of links).

We then analyze the Nash equilibria of a non-cooperative game of network formation, with the added requirement that any mutually beneficial link be formed at equilibrium. When information can only flow up to one link away from the sender, a simple geometric characterization of equilibrium networks is provided. Equilibrium networks always exist. They are not unique and need not be symmetric, but only moderate levels of asymmetry can arise. For instance, stars encompassing ‘too many’ players are excluded. In general, because of the payoff spillovers that players (and their surrounding local networks) exert on each other, strategic link formation leads to inefficient network structures.² When information can flow further away in the network, the analysis for the case $n = 4$ shows that even small stars can now arise at equilibrium.

Although the central role of personal contacts in labor markets has long been recognized by sociologists and labor economists, there are few formal models that analyze them. Following the intuitions in Granovetter’s [6] field study, Boorman [3] first provides a model of the communication process among social contacts for job-seeking purposes. This paper relates network geometry to communication processes and takes up the issue of strategic network formation.³ A recent and flourishing new strand of research on network formation, pioneered by Bala and Goyal [1] and Jackson and Wolinsky [8], provides the adequate tools and methods for our analysis.⁴

²We measure network efficiency by considering the employment prospects arising in some network relative to the network’s aggregate cost of creation.

³Our focus here is on the emerging structure of bilateral contacts in labor markets, the effectiveness of the associated communication pattern and the corresponding aggregate unemployment. There are of course other reasons for why networks of contacts are important in the labor market. For instance, Montgomery [9] analyzes a hiring model with adverse selection where network contacts partially solve the information asymmetry between the firms and the applicants.

⁴See [4] for further references and an overview of this literature.

2. Job transmission on contact networks

Contact networks: $N = \{1, \dots, n\}$ is a set of players. Players are connected by a network, modeled by a non-directed graph. Let g^N be the complete graph. $\mathcal{G} = \{g \mid g \subseteq g^N\}$ is the set of graphs on N .

Let $g \in \mathcal{G}$ and $ij \in g$ a link. The graph g is *connected* if there is a *path* in g connecting all $i \neq j$, that is, if there exists $\{i_0 = i, \dots, i_m = j\} \subset N$ such that $i_{k-1}i_k \in g$, $m \geq k \geq 1$. We denote by $g + ij$ (resp. $g - ij$) the network obtained by adding (resp. deleting) ij to (resp. from) g . The set of i 's *direct contacts* is $N_i(g) = \{j \neq i \mid ij \in g\}$, of size $n_i(g)$. The *size* of g is $n(g) = \sum_{i \in N} n_i(g)/2$. If $n_i(g) = v$, for all $i \in N$, g is a *symmetric network* of degree v , denoted $g(v)$.

Job transmission: Initially, all players are employed. The labor market is subject to the following turnover. First, players lose their job with breakdown probability $b \in (0, 1)$. Second, they receive a job offer with arrival probability $a \in (0, 1)$. Unemployed players immediately take any offer; employed players pass it along to their unemployed direct contacts, if any, with uniform randomization.^{5,6} If an unemployed player receives multiple job offers, all but one randomly selected offer remain unfilled.

Let $\alpha = a(1 - b)$ and $\beta = b(1 - a)$. An employed player has an extra job slot for his contacts with probability α , and an unemployed player needs his contacts to find a job with probability β .

Proposition 1. *The probability that i gets a job through contacts is $P_i(g) = 1 - \prod_{j \in N_i(g)} q(n_j(g))$, where $q(n_j(g)) = 1 - \alpha \frac{1 - (1 - b)^{n_j(g)}}{bn_j(g)}$ is the probability that i does not get a job from $j \in N_i(g)$.*

Proof. Assume that $j \in N_i(g)$ hears of a vacant job and does not need it. Player j transmits this information to player i with conditional probability

$$\sum_{k=0}^{n_j(g)-1} \binom{n_j(g) - 1}{k} \frac{(1 - b)^{n_j(g)-1-k} b^k}{k + 1} = \frac{1 - (1 - b)^{n_j(g)}}{bn_j(g)}.$$

Therefore, the probability that i does not get a job from j is $1 - \alpha [1 - (1 - b)^{n_j(g)}] / bn_j(g)$. \square

⁵Implicitly, we are thus assuming that job information is passed at the initiative of the informed and employed contact, which is consistent with empirical findings: “For 57.9 percent of the individuals finding their new job through contacts, the interaction during which job information was passed was, in fact, initiated by the contact. In another 20.9 percent of the instances, the respondent contacted his friend, asked him if he knew of anything, and was told about the job he subsequently took.” ([7, p. 33].

⁶Contrarily to [3] that makes a sharp distinction between strong and weak contacts—thus introducing a priority ranking over contacts—, we assume here that pairwise links are all of the same kind, which motivates this equal-treatment-of-neighbors assumption.

The information inflow to player i is shaped by the structure of his direct and two-links-away contacts. The other indirect contacts in the network do not affect this flow.

Remark 1. $P_i(g)$ increases with⁷ $N_i(g)$ and decreases with $n_j(g)$, $j \in N_i(g)$.

In other words, direct contacts are beneficial whereas two-links-away contacts are detrimental. Direct contacts constitute information channels that increase individual employment prospects. Two-links-away contacts are competitors for information and harm these prospects.⁸

Consider a symmetric network $g(v)$, and let $P(v) = P_i(g(v))$, for all $i \in N$.

Proposition 2. *There exists $\bar{v} \geq 2$ such that $P(v)$ increases on $\bar{v} \geq v \geq 1$ and decreases on $v \geq \bar{v}$.*

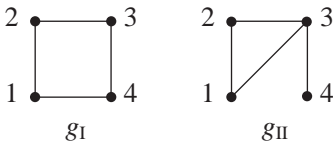
In other words, $P(v)$ has a unique global maximum. The proof uses the fact that $\Delta P(v) = P(v + 1) - P(v)$ decreases on $\kappa \geq v \geq 1$, for some $\kappa \geq 2$. $\Delta P(v)$ accounts for the effect of simultaneously adding one direct contact and $2v + 1$ two-links-away contacts. When $v \geq \bar{v}$, the joint negative effect of $2v + 1$ two-links-away contacts overwhelms the isolated positive effect of one direct contact.

Aggregate unemployment: Let $g \in \mathcal{G}$. $(P_1(g), \dots, P_n(g))$ determines how job information flows through personal contacts and, ultimately, the resulting unemployment rate, denoted by $u(g)$.

Proposition 3. *The unemployment rate is $u(g) = \beta[1 - \sum_{i \in N} P_i(g)/n]$.*

Networks that mediate job information fluently are such that the aggregate probability $\sum_{i \in N} P_i(g)$ of finding a job through contacts is high. These networks result in low unemployment $u(g)$. Proposition 2 implies that $u(g(v))$ decreases with v on $\bar{v} \geq v \geq 1$ and increases on $v \geq \bar{v}$. Therefore, the size of a network affects the unemployment rate.⁹ Remark 1 suggests that the geometry of the network also matters. The following example illustrates this point.

Example 1. Let $N = \{1, 2, 3, 4\}$, g_I and g_{II} , with $n(g_I) = n(g_{II}) = 4$ but different geometries:



⁷For the inclusion ordering on sets.

⁸Indeed, when some direct contact j of player i becomes better connected, j 's information is shared among a bigger group of players. The information i could have acceded privately to is more likely to reach someone else instead.

⁹It is readily checked that $n(g(v)) = nv/2$. Therefore, the size of a symmetric network is proportional to its degree v .

The patterns of information transmission differ across networks. By Remark 1, $P_1(g_I) > P_1(g_{II})$, $P_3(g_I) < P_3(g_{II})$ and $P_4(g_I) > P_4(g_{II})$. Unemployment rates also differ. Here, $u(g_{II}) > u(g_I)$.¹⁰

3. Equilibrium networks

3.1. The non-cooperative game of network formation

Players and strategies: Players individually announce all the links they wish to form. For all $i, j \in N$, $s_{i,j} = 1$ if i wants to link with j , and $s_{i,j} = 0$ otherwise. By convention, $s_{i,i} = 0$. A strategy of player i is $s_i = (s_{i,1}, \dots, s_{i,n})$, and $S^i = \{0, 1\}^{n-1}$ is the set of strategies available to i . The link ij is created if and only if $s_{i,j}s_{j,i} = 1$. Links are thus created by *mutual consent*.

A strategy profile $s = (s_1, \dots, s_n)$ induces a non-directed graph $g(s)$. We omit s in the sequel.

Individual payoffs: By assumption, all jobs are identical. Wages are exogenous and we set $w = 1$. The assumption of identical jobs is consistent with our model of information transmission. Indeed, search for information can have either an extensive or an intensive margin.¹¹ Search at the intensive margin consists on getting additional information about an offer already received, while search at the extensive margin consists on broadening the access to highly standardized information. By creating links connections, players here simply increase their access to job offers (extensive margin).

Each link ij results in a cost $c > 0$ to both i and j , equal across players.¹²

Let $g \in \mathcal{G}$. The expected net payoff $Y_i(g)$ of player i is

$$Y_i(g) = \underbrace{(1 - b)}_{i \text{ keeps job}} + b \underbrace{\left[\underbrace{a}_{\text{market}} + \underbrace{(1 - a)P_i(g)}_{\text{contact}} \right]}_{i \text{ fired and reemployed}} - cn_i(g).$$

With some algebra, $Y_i(g) = 1 - \beta \prod_{j \in N_i(g)} q(n_j(g)) - cn_i(g)$.

The equilibrium concept: A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash equilibrium of the network formation game if and only if $Y_i(g(s_i^*, s_{-i}^*)) \geq Y_i(g(s_i, s_{-i}^*))$, for all $s_i \in S^i$, $i \in N$. Nash equilibrium is too weak an equilibrium concept. Indeed, the empty network is always a Nash equilibrium.¹³ Following [5], we refine Nash equilibrium

¹⁰ Indeed, $\sum_{i \in N} P_i(g_I) > \sum_{i \in N} P_i(g_{II})$ (see Claim 1) and Proposition 3 then implies that $u(g_I) < u(g_{II})$.

¹¹ This idea is developed in [10].

¹² Given our normalization for the wage value, $w = 1$, the per capita cost of a link c is termed here in wage units, and has to be interpreted as the ratio of the monetary cost of a link to the wage level.

¹³ When nobody announces any link.

building upon the pairwise stability concept in [8]. We require that any mutually beneficial link be formed at equilibrium.¹⁴

Definition 1. $g \in \mathcal{G}$ is a pairwise-equilibrium network if and only if there is a Nash equilibrium strategy profile which supports g and, for all $ij \notin g$, $Y_i(g + ij) > Y_i(g)$ implies $Y_j(g) > Y_j(g + ij)$.

Pairwise-equilibrium networks are such that no player gains by altering the current configuration of links, neither by adding a new link nor by eliminating any subset of the existing links.

3.2. Equilibrium networks

Lemma 1. Let $j^*(i) = \arg \max \{n_j(g) \mid j \in N_i(g)\}$. If $Y_i(g) \geq Y_i(g - ij^*(i))$, then $Y_i(g) \geq Y_i(g - ij_1 \dots - ij_\ell)$, for all $j_1, \dots, j_\ell \in N_i(g)$.

In other words, if a player does not benefit from severing the link with his *most connected* direct contact, then he does not gain from cutting any link or group of links simultaneously. Therefore, a network is a pairwise-equilibrium if and only one-link deviations are not profitable.¹⁵

Proposition 4. If $c > \alpha\beta$, the empty graph is the only pairwise-equilibrium network. Suppose that $\alpha\beta \geq c$. A network $g \in \mathcal{G}$, $g \neq \emptyset$, is a pairwise-equilibrium network if and only if both:

- (a) $\min_{ij \in g} \{P_i(g) - P_i(g - ij)\} \geq c/\beta$;
- (b) $P_i(g + ij) - P_i(g) > c/\beta$ implies $c/\beta > P_j(g + ij) - P_j(g)$, for all $ij \notin g$.

Equilibrium networks are non-empty if the net value of a first link, $\alpha\beta - c$, is non-negative.

Note that $Y_i(g) - Y_i(g - ij) = \beta[P_i(g) - P_i(g - ij)]$. Condition (a) is an individual rationality constraint. It states that any existing link is necessarily worth its cost for the two involved players. Condition (b) reflects lack of mutual consent between any two players not directly linked at equilibrium. Both conditions characterize pairwise-equilibrium networks geometrically, only in terms of the number of links adjacent to the nodes of a graph.

We provide a direct characterization for symmetric equilibrium networks. Recall that $q(v)$ is the probability of *not* hearing of a job from some contact with v direct connections. Denote by $c_{\max}(v) = \beta q(v)^{v-1} [1 - q(v)]$ the current individual per-link

¹⁴By definition, a network is pairwise stable if no player gains by unilaterally severing one existing link, and no pair of players gain by forming a new link.

¹⁵With our payoffs, the set of pairwise-equilibrium networks thus coincides with the set of pairwise stable networks.

reward and by $c_{\min}(v) = \beta q(v)^v [1 - q(v + 1)]$ the individual marginal benefit from an extra link.

Corollary 1. For all $n - 2 \geq v \geq 1$, $g(v)$ is a pairwise-equilibrium network if and only if $c_{\max}(v) \geq c > c_{\min}(v)$, and $g(n - 1) = g^N$ is a pairwise-equilibrium network if and only if $c_{\max}(n - 1) \geq c$.

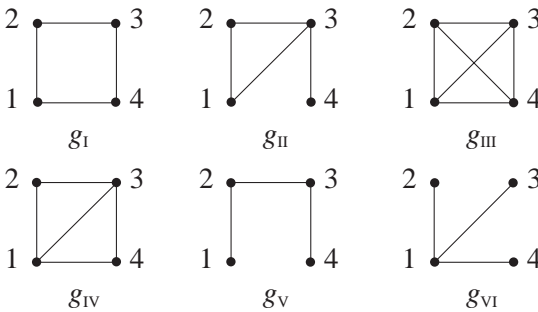
The geometry of pairwise-equilibrium networks: For all $n - 2 \geq v \geq 1$, $c_{\max}(v + 1) > c_{\min}(v)$. Therefore, the cost ranges for which $g(v)$ and $g(v + 1)$ arise at equilibrium overlap. Moreover, for identical parameter values, both symmetric and asymmetric networks may arise.

The following result identifies a particular feature of asymmetric equilibrium networks. We say that $ij \in g$ is a *loose-end* if either $n_i(g) = 1$ or $n_j(g) = 1$. When a loose-end is cut, (at least) one player gets completely disconnected. Let $m^*(a, b) = \arg \max \{m \in \mathbb{N} | 1 - q(m) > q(m)[1 - q(2)]\}$.¹⁶

Remark 2. Let $j \in \arg \max \{n_k(g) | k \in N\}$. If $ij \in g$ is a loose-end and $n_j(g) > n_k(g)$, for some $k \in N_j(g)$, $k \neq i$, then $m^*(a, b) \geq n_j(g)$.

The proof amounts to check that, when j is willing to invest in at least $m^* + 1$ connections then, necessarily, any loose-end appended to j and any other of j 's direct contacts mutually consent to link each other. Remark 2 implies that stars encompassing $m \geq m^* + 1$ players are ruled out at equilibrium.¹⁷ More generally, this result rules out too asymmetric equilibrium networks and simplifies their characterization, as illustrated below.

Example 2. Let $N = \{1, 2, 3, 4\}$. There are six different connected network architectures on N :



¹⁶ Given that $q(m)$ is an increasing function of m and that $\lim_{m \rightarrow \infty} q(m) = 1$, the existence of $m^*(a, b)$ is always guaranteed. Given that $q(m)$ depends on a and b , so does $m^*(a, b)$. Also, it is readily checked that $m^*(a, b) \geq 1$, for all $a, b \in (0, 1)$. Suppose not. Then, $q(1)[1 - q(2)] \geq 1 - q(1)$, which implies that $q(1)[1 - q(1)] \geq 1 - q(1)$, impossible as $1 > q(1) > 0$.

¹⁷ A network $g \in \mathcal{G}$ is a star if $g \neq \emptyset$ and there exists $i \in N$ such that, if $jk \in g$, then either $j = i$ or $k = i$.

Suppose that $a = b = \frac{1}{2}$. Then, $m^* = 2$. By Remark 2, g_{II} and g_{VI} are never equilibrium networks. Suppose that g_{IV} is an equilibrium network. Then, 1 does not gain by cutting 13, that is, $c_{\min}(2) > c$, nor does 2 benefit from creating 24, that is, $c \geq c_{\max}(3)$, impossible as $c_{\max}(3) > c_{\min}(2)$. Similarly, g_V being an equilibrium network implies both $c_{\min}(1) > c$ and $c \geq c_{\max}(2)$, which is impossible. The only equilibrium networks are g_I and g_{III} . Suppose that $a = \frac{1}{2}$ and $b = \frac{1}{4}$. Now $m^* = 4$, and the only equilibrium networks are g_I , g_{II} and g_{III} .¹⁸

Proposition 5. *Pairwise-equilibrium networks always exist.*

Existence is established in two steps. First, there always exists one v^* satisfying the conditions of Corollary 1. Second, we check that a geometric pattern of links constituting an equilibrium network can always be constructed with such a v^* .¹⁹

4. Discussion

Pairwise-equilibrium networks and efficiency: We first define a welfare measure.

Definition 2. The welfare measure of $g \in \mathcal{G}$ is $W(g) = \sum_{i \in N} Y_i(g) = [1 - u(g)]n - 2cn(g)$.

Pairwise-equilibrium networks are generally inefficient, for three different reasons.

First, *externalities*. When i and j link each other, they exert a negative externality to their direct contacts. Individual incentives to form contacts may thus sometimes be excessive in relation to what is socially desirable. For low values of the per-link cost, at equilibrium, the degree v in Corollary 1 lies above the threshold \bar{v} in Proposition 2, where unemployment increases with network size. Pairwise-equilibrium networks are then over-connected from an efficient viewpoint.

Second, *the distribution of the joint link gains to the linked players*. Players i and j pay the same fixed cost c for ij , but the individual marginal returns from ij depend on i and j 's shape of contacts. If i and j hold very asymmetric positions in the network, ij may well have a total positive net value, that is, $Y_i(g + ij) + Y_j(g + ij) > Y_i(g) + Y_j(g)$, while some player gets a negative return from it, for instance, $Y_i(g) > Y_i(g + ij)$. Absent mutual consent, ij is not formed although socially desirable. Pairwise-equilibrium networks are then under-connected from an efficient viewpoint. The following remark provides an example.

¹⁸ When $a = b = \frac{1}{2}$, g_I and g_{III} are equilibrium networks for the cost ranges $(0.0241, 0.0381]$ and $[0, 0.0266]$, respectively. When $a = \frac{1}{2}$ and $b = \frac{1}{4}$, g_I , g_{II} and g_{III} are equilibrium networks for the cost ranges $(0.0163, 0.0276]$, $(0.0196, 0.0197]$, and $[0, 0.0183]$, respectively.

¹⁹ If n or v^* are even, we can always connect the n players such that they all have v^* direct contacts. Otherwise, we can connect the n players such that all but one have v^* direct contacts, the remaining player having only $v^* - 1$ direct contacts. In each case, the graph obtained is a pairwise-equilibrium network.

Remark 3. There exists $\bar{c} > c$ such that separate pairs, g_{pairs} , are the only pairwise-equilibrium networks when $\bar{c} \geq c > c$ while efficient networks have at least $n(g_{\text{pairs}}) + 1$ links in the same cost range when n is odd.

Third, *the network geometry*. Individual payoffs are shaped by the network geometry, and two networks of identical size but different geometry may lead to different values for W . In Example 1, g_I and g_{II} have the same aggregate cost of link formation, as $n(g_I) = n(g_{II})$. Yet, $u(g_{II}) > u(g_I)$. Therefore, $W(g_I) > W(g_{II})$. Note that Example 2 shows that both networks may arise at equilibrium for a certain cost range and values for a and b . Note also that g_I can easily be obtained from g_{II} by a simple rewiring procedure (cut 13 and create 14). Therefore, rewiring may act as a welfare-enhancing redistribution scheme.

Pairwise-equilibrium networks and unemployment: We now focus on information flows. Adding a link has a positive effect on the newly connected players but a negative effect on two-links-away contacts. The global effect on unemployment depends on the balance between these two opposite effects.

It is readily checked that $W(g + ij) > W(g)$ implies that $u(g) > u(g + ij)$. Therefore, when link addition is welfare enhancing, it reduces unemployment. Yet, link addition may sometimes increase unemployment (and decrease welfare).

Remark 4. Suppose that $n > \bar{v} + 1$.²⁰ There exists $g \in \mathcal{G}$ and $ij \notin g$ such that $u(g + ij) > u(g)$.

The proof is constructive. We find two nested networks $g^e \subset g^*$ such that g^* (resp. g^e) is an equilibrium (resp. efficient) network for the same value of c , and $u(g^*) > u(g^e)$. The results then follow as g^* is obtained from g^e by a finite iterative addition of one link.

Extensions: So far, information can only flow from employed to unemployed workers. We now allow for employed-to-employed flows, with two restrictions. First, information goes to an employed contact *only* when no direct contacts are unemployed. Second, we just allow for one such relay.²¹ We assume that information is lost during relays with probability $1 \geq 1 - \delta \geq 0$.

The *geodesic distance* $d_{ij}(g)$ is the length of the shortest path between i and j on g . For all $i \in N$, $j \in N_i(g)$ and $n_j(g) - 1 \geq \ell \geq 0$, let

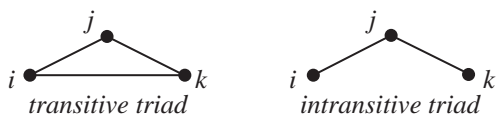
$$\phi_{ij}(g, \ell) = \binom{n_j(g) - 1}{\ell} + \delta(1 - a) \sum_{\substack{S \subset N_j(g) \setminus \{i\} \\ |S| = n_j(g) - 1 - \ell}} \sum_{k \in S, d_{ik}(g) = 2} \frac{(1 - b)^{n_k(g) - 1}}{n_k(g)}.$$

²⁰ Recall that \bar{v} is equal to the highest integer smaller or equal than the unique solution to $P'(x) = 0$, when P is extended to $[1, +\infty)$. Therefore, \bar{v} only depends on a and b , and is independent of n .

²¹ Information transmission with at most one relay already accounts for 84.4% of all the cases encountered in Granovetter's [7] field study: "Chains of length [one] accounted for 39.1 percent of the cases; 45.3 percent had length [two]" (p. 57).

enough so that direct links with each other are not worth their cost, and stars can be sustained at equilibrium.

Information relays also have implications for network efficiency, as illustrated below:



When information relays are permitted, any player accedes the information held by any other player in both triads.²³ Transitive triads, though, are more costly to create than intransitive ones. When the per-link cost c is high enough, a third link is redundant and transitive triads are inefficient.²⁴

Assume, more generally, that information can be relayed with no restrictions. Now, chains of contacts of arbitrary length provide access to job information in distant parts of the social setting. As before, when c is high, cycles have one redundant link. Efficient networks are thus minimally connected.²⁵ Also, when $\delta \neq 1$, intermediate links in a chain of contacts add frictions. Efficient networks should thus also minimize geodesic distances. Stars are the only minimally connected networks which minimize geodesic distances.

5. Conclusion

In this paper, ex ante identical players choose to form links with others so as to access valuable information the latter may possess. When information cannot travel further away than one link from the initial recipient, only moderate levels of network asymmetry can emerge at equilibrium. When information can flow further away in the network, the case $n = 4$ suggests that equilibrium network geometries need not follow simple patterns and may be highly asymmetric.²⁶

²³ Sociologists roughly distinguish between two types of network connections, *strong* and *weak* ties, depending on their transitivity properties. Strong ties (with close friends) are *transitive* and constitute cohesive subgroups whose members are densely tied to each other. Weak ties (with acquaintances) are *intransitive* and constitute far-reaching and open-ended networks that penetrate the social boundaries impermeable to stronger ties. Weak and strong ties play different roles in communication processes: “whatever is to be diffused can reach a larger number of people, and traverse greater social distance (i.e. path length), when passed through weak ties rather than strong ties” [6, p. 1366].

²⁴ Let $\underline{c} = \alpha(1 - b/2) - \alpha^2(1 - b/2)^2$. If information relays are permitted, transitive triads are inefficient whenever $c \geq \underline{c} - \alpha(1 - a)(1 - b)$. When, $\underline{c} \geq c \geq \underline{c} - \alpha(1 - a)(1 - b)$, transitive triads are efficient if information relays are not permitted.

²⁵ A *cycle* is a path $\{i_0, \dots, i_m\}$, $m \geq 1$ such that $i_0 = i_m$. A graph g is *minimally connected* if g is connected but $g - ij$ is not, for all $ij \in g$.

²⁶ Goyal and Joshi [5] provide a neat characterization of equilibrium network architectures for a broad class of payoff functions with network spillovers. Yet, they already acknowledge that ascertaining the nature of equilibrium geometries for certain classes of network spillovers, like those encountered in our model, need not always be a possible task. On the contrary, the informal discussion below suggests that the

The general model of information transmission is reminiscent of the two-way flow model with decay in [1] and the connection model in [8], with a crucial difference. Here, frictions do not only depend on the length of the shortest path between the initial sender and the final receiver. Rather, frictions are tailored to the details of the network geometry (links' ramifications) encountered along this path. This is so because, contrarily to previous models, our payoff function (build on the network) reflects *rivalry* in the access through network links to the resources held by distant players in the network. As a result, both the sign and the intensity of the payoff spillovers that players exert on each other are very much dependent on the details of the network geometry surrounding those players. Given that the information flows associated with two different chains of contacts of identical length but different topologies are generally different, the entire network structure has to be considered when specifying best responses. This constitutes a major obstacle to a more general analysis.

6. Proofs

Proof of Remark 1. The function $g(x) = (1 - b)^x$ is strictly convex for $x > 1$. Therefore,

$$\frac{g(x) - g(0)}{x - 0} = \frac{(1 - b)^x - 1}{x}$$

increases on $[1, +\infty)$, implying that q increases. Also, $q(1) = 1 - \alpha$ and $\lim_{n \rightarrow +\infty} q(n) = 1$. \square

Claim 1. $\sum_{i \in N} P_i(g_I) > \sum_{i \in N} P_i(g_{II})$

Proof. $\sum_{i \in N} P_i(g_I) = 4(1 - q(2)^2)$ and $\sum_{i \in N} P_i(g_{II}) = 4 - q(3) - 2q(2)q(3) - q(1)q(3)^2$. We are left to prove that $q(3)[1 + 2q(2) + q(1)q(3)] > 4q(2)^2$. We have $q(3) = 1 - \alpha(1 - b + b^2/3) > 1 - \alpha(1 - 2b/3)$, and $q(3) > q(2)$. It suffices to show that $1 + 2q(2) + q(1)[1 - \alpha(1 - 2b/3)] > 4q(2)$. With some algebra, this is equivalent to $\alpha + (1 - \alpha)2b/3 > 0$, which is true. \square

Proof of Proposition 2. For all $x \geq 1$, let $Q(x) = [q(x)]^x$. We show that Q decreases on $[1, \bar{x}]$ and increases on $[\bar{x}, +\infty)$, for some uniquely defined \bar{x} , and is strictly convex on $[1, K]$, for some $K > \bar{x}$. Then, $\bar{v} = \max\{[1, \bar{x}] \cap \mathbb{N}^*\}$ and $\kappa = \max\{[1, K] \cap \mathbb{N}^*\}$.

We have $Q'(x) = \Phi(x)Q(x)$, where $\Phi(x) = \ln q(x) + xq'(x)/q(x)$.

(footnote continued)

geometry of efficient networks could be narrowed down to stars, at least for some parameter values. Note that stars are already identified as the unique efficient networks (for some parameters' values) in [1, Propositions 4.3 and 5.5] and in [8, Proposition 1].

Step 1. We show that $0 > Q'(1)$. With some algebra, $\Phi(1) = \alpha(1 - \alpha)\rho_b(a)$, where

$$\rho_b(a) = a(1 - b) \left[1 + \frac{(1 - b)\ln(1 - b)}{b} \right] + [1 - a(1 - b)]\ln[1 - a(1 - b)].$$

We show that $0 > \rho_b(a)$, for all $a, b \in (0, 1)$. Fix $b \in (0, 1)$. $\rho_b''(a) = (1 - b)^2/[1 - a(1 - b)] > 0$. Therefore, ρ_b' increases on $(0, 1)$ with supremum $\rho_b'(1) = (1 - b)[(1 - b)\ln(1 - b) - b \ln b]/b$. It is straightforward to see that $g(x) = (1 - x)\ln(1 - x) - x \ln x$ takes positive values on $(0, \frac{1}{2})$ and negative values on $(\frac{1}{2}, 1)$. If $b \geq \frac{1}{2}$, $0 \geq \rho_b'(1)$ and ρ_b decreases on $(0, 1)$. As $\rho_b(0) = 0$, $0 > \rho_b(a)$, for all $a \in (0, 1)$. If $\frac{1}{2} > b$, $\rho_b'(1) > 0$. As $0 > \rho_b'(0)$, there exists some $a_b^* \in (0, 1)$ such that ρ_b decreases on $(0, a_b^*)$ and increases on $(a_b^*, 1)$. We show that $0 > \rho_b(1) = b \ln b + (1 - b)[1 + (1 - b)\ln(1 - b)/b]$. As $(1 - b)\ln(1 - b) > b \ln b$ when $\frac{1}{2} > b$, $[b + \ln(1 - b)](1 - b)/b > \rho_b(1)$. It is easy to check that $h(x) = x + \ln(1 - x)$ is negative on $(0, 1)$.

Step 2. It is easy to check that $\Phi(x) \sim_{x \rightarrow +\infty} (\alpha/bx)^2$, and $Q'(x) > 0$ for high values of x .

Step 3. We show that $0 \geq Q'$ implies that $Q'' > 0$. We have $\lim_{x \rightarrow +\infty} \Phi(x) = 0$. Hence, $0 \geq \Phi'$ implies $\Phi > 0$. Reciprocally, $0 \geq \Phi$ implies $\Phi' > 0$, which in turn implies $Q'' > 0$. But $0 \geq \Phi$ is equivalent to $0 \geq Q'$.

Step 4. Steps 1 and 2 imply that $Q'(x) = 0$ for some $\bar{x} > 1$. Step 3 implies that $Q''(\bar{x}) > 0$, and \bar{x} is uniquely defined. By continuity, Q is strictly convex on $[1, K]$, for some $K > \bar{v}$. \square

Claim 2. Let $1 \geq x_i \geq 0, i \in \{1, \dots, \ell\}$. Then, $1 - \prod_{i \in \{1, \dots, \ell\}} x_i \geq \sum_{i=1}^{\ell} (1 - x_i) \prod_{j \in \{1, \dots, \ell\}, j \neq i} x_j$.

Proof. The claim is trivially satisfied for $\ell = 2$. Suppose that it holds at $\ell \geq 2$. We establish it for $\ell + 1$. Let $x'_1 = \prod_{i \in \{1, \dots, \ell\}} x_i$ and $x'_2 = x_{\ell+1}$. The inequality applied to these two values gives

$$1 - \left(\prod_{i \in \{1, \dots, \ell\}} x_i \right) x_{\ell+1} \geq \left(1 - \prod_{i \in \{1, \dots, \ell\}} x_i \right) x_{\ell+1} + (1 - x_{\ell+1}) \prod_{i \in \{1, \dots, \ell\}} x_i.$$

Using the inequality at ℓ , we can conclude. \square

Proof of Lemma 1. Let $j^*(i) = \arg \max\{n_j(g) \mid j \in N_i(g)\}$. Let $j \in N_i(g)$. Remark 1 implies that $Y_i(g - ij^*(i)) \geq Y_i(g - ij)$. Therefore, $Y_i(g) \geq Y_i(g - ij^*(i))$ implies $Y_i(g) \geq Y_i(g - ij)$. Let $j_1, \dots, j_\ell \in N_i(g)$. We prove that

$$Y_i(g) - Y_i(g - ij_1 - \dots - ij_\ell) \geq \sum_{p \in \{1, \dots, \ell\}} [Y_i(g) - Y_i(g - ij_p)], \tag{1}$$

and the result follows. We have

$$Y_i(g) - Y_i(g - ij_1 - \dots - ij_\ell) = \beta \left[1 - \prod_{p \in \{1, \dots, \ell\}} q(n_{j_p}(g)) \right] \prod_{\substack{k \in N_i(g) \\ k \notin \{j_1, \dots, j_\ell\}}} q(n_k(g)) - c\ell.$$

Therefore, (1) is equivalent to

$$1 - \prod_{p \in \{1, \dots, \ell\}} q(n_{j_p}(g)) \geq \sum_{p=1}^{\ell} [1 - q(n_{j_p}(g))] \prod_{q \in \{1, \dots, \ell\}, q \neq p} q(n_{j_q}(g)),$$

which is true by Claim 2. \square

Proof of Remark 2. Let $m^*(a, b) = \arg \max \{m \in \mathbb{N} \mid 1 - q(m) > q(m)[1 - q(2)]\}$. Note that $m^*(a, b) \geq 1$. Let g be a pairwise-equilibrium network, $ij \in g$ such that $n_i(g) = 1$ and $j \in \arg \max \{n_k(g) \mid k \in N\}$. Suppose that $n_j(g) > m^*(a, b)$. Then, $q(n_j(g))[1 - q(2)] \geq 1 - q(n_j(g))$. Let $l \in N_j(g)$, $l \neq i$. At equilibrium, player l is compensated by the link lj , that is,

$$\beta \prod_{k \in N_l(g), k \neq j} q(n_k(g))[1 - q(n_j(g))] \geq c.$$

Suppose first that $n_l(g) = 1$. Then, i and l hold symmetric positions in the network, and il should be formed at equilibrium, which is a contradiction. Suppose now that $n_l(g) \geq 2$. Combining the previous inequality and the fact that $n_j(g) > m^*(a, b)$ implies that

$$\beta \prod_{k \in N_l(g), k \neq j} q(n_k(g))q(n_j(g))[1 - q(2)] \geq c,$$

that is, il is worth its cost to l . Given that $il \notin g$, necessarily il is not worth its cost to i , that is,

$$c > \beta q(n_j(g))[1 - q(n_l(g) + 1)].$$

Suppose first that $n_j(g) > n_l(g)$. Then

$$c > \beta q(n_j(g))[1 - q(n_l(g) + 1)] \geq \beta \prod_{k \in N_l(g), k \neq j} q(n_k(g))[1 - q(n_j(g))] \geq c,$$

which is a contradiction. \square

Proof of Proposition 5. We first show that for all $\alpha\beta \geq c$, there always exist some $v \in \mathbb{N}^*$ such that $c_{\max}(v) \geq c > c_{\min}(v)$. First, both c_{\min} and c_{\max} extended to $[1, +\infty)$ are continuous. Second, $c_{\max}(1) = \alpha\beta$ and $\lim_{x \rightarrow +\infty} c_{\max}(x) = 0$. Finally, $c_{\max}(x) > c_{\min}(x)$, for all $x \geq 1$. Therefore, if $\alpha\beta \geq c$, there exists some v' such that $c_{\max}(v') \geq c > c_{\min}(v')$. Denote by v^* the smallest of such v' . If $n - 2 \geq v^*$, the conditions of Corollary 1 hold. If $v^* \geq n - 1$, necessarily $\beta q(n - 1)^{n-2}[1 - q(n - 1)] \geq c$. Otherwise, we can find $v^* > v'$ satisfying the inequalities, which is a contradiction. Therefore, the conditions of Corollary 1 also hold. The existence of

a geometric pattern of links constituting a pairwise-equilibrium network follows from the following theorem.

Theorem 1 (Erdős and Gallai).²⁷ Let $d_1 \geq d_2 \geq \dots \geq d_n$ a sequence of n integers such that $\sum_{1 \leq i \leq n} d_i$ is even. The two following conditions are equivalent: (a) $\exists g \in \mathcal{G}$ such that $n_i(g) = d_i, \forall i \in N$; (b) $\sum_{1 \leq i \leq k} d_i \leq k(k-1) + \sum_{k+1 \leq j \leq n} \min\{k, d_j\}, \forall 1 \leq k \leq n$.

Claim 3.1. $1 - q(2) > q(1)[1 - q(1)]$.

Proof. The inequality is equivalent to $1 - \alpha(1 - b/2) > \alpha(1 - \alpha)$, implied by the fact that $1 - \alpha(1 - b/2) > 1 - \alpha$ and $1 > \alpha > 0$. \square

Proof of Remark 3. Separate pairs, g_{pairs} ,²⁸ is a pairwise-equilibrium network whenever (a) players prefer to be in pairs than isolated, and (b) there are no mutual gains from additional links. Condition (a) is equivalent to $\beta[1 - q(1)] \geq c$. We establish (b). We have, $Y_j(\{ij, jk\}) > Y_j(\{jk\})$ equivalent to $\beta q(1)[1 - q(1)] > c$, and $Y_i(\{ij, jk\}) > Y_i(\{jk\})$ equivalent to $\beta[1 - q(2)] > c$. With Claim 3, (b) is equivalent to

$$\beta[1 - q(2)] \geq c > \beta q(1)[1 - q(1)]. \tag{2}$$

Given that $q(2) > q(1)$, (a) is implied by (b), and g_{pairs} is the only pairwise-equilibrium network whenever (2) holds. Suppose that n is odd. Then, g_{pairs} contains an isolated player, k . Let $ij \in g$. If $W(g_{\text{pairs}} + ik) > W(g_{\text{pairs}})$, then g_{pairs} is inefficient. This is equivalent to $Y_i(\{ij, ik\}) + 2Y_j(\{ij, ik\}) > 2Y_i(\{ij\}) \Leftrightarrow c > c_{\text{pairs}}^e$, where $c_{\text{pairs}}^e = 1/2 + \beta[2q(1) - 2q(2) - q(1)^2]/2$. The reader can check that c_{pairs}^e satisfies (2) with strict inequalities. \square

Claim 4. For all $\alpha\beta \geq c$, there exists a unique v^e such that symmetric networks of size v^e are efficient. Moreover, v^e is a non-increasing function of c and $v^e(0) = \bar{v}$.²⁹

Proof. The welfare measure of $g(v)$ is $W(v)/n = (1 - \beta) + \beta P(v) - cv$. Proposition 2 implies that W extended to $[1, +\infty)$ is strictly concave on $[1, K)$ and decreases on $[K, +\infty)$. The efficient network size v^e is thus equal to (the highest integer smaller or equal than) the unique solution to $W'(x) = 0 \Leftrightarrow \beta P'(x) = c$. Recall that \bar{v} is given by (the highest integer smaller or equal than) the solution to $P'(x) = 0$. Therefore, $\bar{v} \geq v^e$. Note also that P' decreases on $[1, K)$. Therefore, x^e is a decreasing function of c , implying that v^e is a non-increasing function of c . Finally, when $c \rightarrow 0$, $P'(x^e) \rightarrow 0$, implying that $v^e(0) = \bar{v}$. \square

²⁷ See [2].

²⁸ We identify the class of all networks constituted by separate pairs by any of such networks, denoted by g_{pairs} .

²⁹ Here again (as in Proposition 5), there does not always exist a pattern of links connecting players such that they all have exactly v^e direct contacts. In some cases, we can only construct an *almost*-symmetric efficient network where all players but one have v^e direct contacts, whereas one player holds only $v^e - 1$ direct links.

Proof of Proposition 4. Let $c = 0$. By Corollary 1, g^N is a pairwise-equilibrium network. Let g^e be an efficient symmetric network. By Claim 4, g^e has exactly \bar{v} links. Suppose that $n - 1 > \bar{v}$. Then, by Propositions 2 and 3, $u(g^N) > u(g^e)$. Let $\{g_0, \dots, g_p\}$, $p \geq 1$ be a sequence of adjacent networks such that, for all $p - 1 \geq k \geq 0$, $g_k \subset g_{k+1}$, $g_0 = g^e$, and $g_p = g^N$. Suppose that for all $g \in \mathcal{G}$, $ij \notin g$, $u(g) \geq u(g + ij)$. Then, for all $p - 1 \geq k \geq 0$, $u(g_k) \geq u(g_{k+1})$, implying that $u(g^e) \geq u(g^N)$, which is a contradiction. \square

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