Efficient Networks in Games with Local Complementarities*

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Abstract

We address the problem of a planner looking for the efficient network (i.e. maximizing social welfare) when agents play a network game with local complementarities, as in Ballester, Calvó-Armengol and Zenou (2006), and links are costly. Starting from an arbitrary network, we identify specific link reallocations and agent permutations that guarantee an increase of social welfare. By exhausting these reallocations and permutations, we prove that for very general network cost functions, efficient networks belong to the class of *Nested-Split Graphs*. This result echoes the results of König, Tessone and Zenou (2014) who show that Nested-Split Graphs are stable structures of the same game with a decentralized network formation.

Next we specify some network cost functions and refine our results. Prominent members of the class of Nested-Split Graphs appear to play an important role: the Complete network, the Core-Periphery networks, the Dominant Group Architecture, the Quasi-Star and the Quasi-Complete networks.

Keywords: Social and Economic Networks, Strategic Complementarity, Nested Split Graphs.

JEL: C72, D85

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1 Introduction

In many economic contexts, individual outcomes are affected by social contacts. As a consequence, the social network plays an important role in shaping agents' decisions, and its structure needs to be taken fully into account. A natural question that arises is therefore that of finding the efficient network. Our paper addresses this problem in the context of games with linear interactions[1] introduced in Ballester, Calvó-Armengol and Zenou (2006), restricted to the case of pure complementarities. In order to address efficiency, we introduce a cost of forming links between agents and determine the network that maximizes social welfare, given by the sum of utilities net of the cost of the network.

In this setting, equilibrium utilities are proportional to the square of agents' Bonacich centralities. The cost of forming links is defined in a general way: agents have an individual linking type, reflecting how costly it is to link agents to each other, and the network cost is increasing in the sum of the value of all the links in the network. Our general formulation includes the standard case of a constant cost per link, as well as many other cases that are described below.

One way to tackle the problem of finding the efficient network is to compare how social welfare changes when a network is modified. However, changing a network's structure may have an ambiguous impact on agents' utilities because direct effects can be weaker than feedback effects. As a consequence, comparing the sum of the utilities in two different networks implies comparing the infinite number of paths that constitute the Bonacich centralities. This gives rise to non-trivial trade-offs, where a network can dominate another network on one path length while itself being dominated on another path length.

The simplest modification imaginable consists of reallocating a single link in a given network, from a neighbor of an agent with low centrality to a link between this same neighbor and an agent with higher centrality. However, this single link reallocation does not guarantee that all utilities will increase. To show this we use an example in which such a single link reallocation leads to a strict decrease of the sum of utilities.

Instead, we resort to multiple-link reallocations. Starting from an arbitrary network, we identify a specific link reallocation, called a *Neighborhood-switch* (N-switch in the remaining), that strictly increases the sum of utilities without changing the number of links in the network (Lemma 1). This link reallocation consists of deleting a specific set of links between an agent with low centrality and his neighbors, and re-creating them between these neighbors and another agent with higher centrality. The N-switch alone guarantees an increase of the sum of utilities, but because the network cost may also increase after this modification, we need to consider a second type of link reallocation that we call an agent permutation. This reallocation consists of changing the links in the network in such a way that the positions of two agents in the resulting network are exchanged. We show that when these two reallocations (N-switches and agent permutations) are combined, social welfare strictly increases although the number of links in the network remains constant.

We then start from an arbitrary network and exhaust all possible N-switches and agent

[1] The linear interaction setting has been widely used in the growing network literature because it offers an appropriate approximation of a wide range of social and economic phenomena, such as education decisions, crime, technology adoption, R&D races etc.

permutations to prove our main result (Theorem 1): efficient networks are *Nested-Split Graphs* (NSGs hereafter). An NSG is a hierarchical structure such that the neighborhood of an agent with low centrality is a subset of the neighborhood of another agent with higher centrality (neighborhoods are nested). These graphs have long been studied in mathematics (see for instance Madahev and Peled (1995) for an overview), but have only recently been identified in economics (to the best of our knowledge, König, Tessone and Zenou (2014) is the first contribution on NSGs in economics. The reader will find that their paper amply illustrates the importance of NSGs). The properties of NSGs induce a high degree of structure on the set of potentially efficient networks. This structure is driven by the accumulation of links around a subset of agents, which is desirable in the context of complementarities.

Our finding complements that of König et al. (2014), who obtain the NSGs as the result of a decentralized dynamic link formation process in the same game. In other words, while *stable*[2] structures of this game are NSGs, efficient networks are also NSGs. In the decentralized case described in König et al. (2014), an agent is picked at random at each step. His best decision is to delete a link to an agent with low centrality, replacing it by a link to an agent with high centrality. The authors show that this repeated single-link reallocation mechanism leads to the NSG class. However, as mentioned earlier, single-link reallocation may decrease the sum of utilities, due to negative externalities. In the efficiency problem addressed here, the multiple-link reallocation we identify can be thought of as a series of simultaneous single-link reallocations targeted on a specific group of agents, and it is both the simultaneity and the targeting of a specific group that guarantee that the sum of utilities will increase. This parallel between the two types of reallocation may explain why both stable and efficient networks are NSGs.

This class of networks contains prominent members, such as the Complete network or the Star, but also more complex structures. In the second part of the paper we specify some network cost functions in order to discriminate between these different NSGs. Before proceeding, we identify a specific process of link addition that we use throughout the second part. This process guarantees that the marginal contribution to the sum of utilities of the new links is increasing (Lemma 2).

We first analyze a case where all individual linking types are equal, so the network cost is simply given by an increasing transformation of the number of links in the network. First, we show that when the intensity of interactions goes to zero, the network maximizing social welfare is either empty, a *Quasi-Star* or a *Quasi-Complete* network (Proposition 1). A Quasi-Star network is built by forming a star network, in which a central agent is involved in as many links as possible. If some links are left unattributed, a second central agent is created, and so on. On the contrary, a Quasi-Complete network is built by accumulating links around a subset of agents, so as to create the largest possible complete component. To prove this result, we show that the efficient network is given by the graph maximizing the sum of the square of the degrees. We then rely on Abrego, Fernandez-Merchant, Neubauer, and Watkins (2009), who recently solved this problem.

Second, we consider general levels of interaction intensity and distinguish two subcases: either the network cost is a concave or linear transformation of the number of links it contains,

[2]The stability notion we refer to is the one used in König et al. (2014), where agents dynamically revise their linking strategies. We refer the reader to their paper for more details.

in which case we show that the efficient network is either the empty or the complete network (Proposition 2); or the network cost is another type of transformation of the number of links, and in this case it appears from numerical simulations that complex NSG structures can be efficient.

When individual linking types are heterogeneous, we first consider two polar cases. In the first polar case, the cost of a link between two individuals depends only on the lowest individual linking type of the two. We prove that the efficient network is necessarily a *Core-Periphery* network (Proposition 3). Core-Periphery networks were introduced by Galeotti and Goyal (2010) as a generalization of the star network with several central agents, and constitute a subclass of the NSGs.

Conversely, in the second polar case we assume that the cost of a link between two agents depends only on the highest individual linking type of the two. We then show that the efficient network is necessarily a *Dominant Group Architecture* (Proposition 4), a family of networks introduced by Goyal and Joshi (2003) that consists of a complete component of any size and isolated agents.

Finally, we consider general cases with heterogeneous individual linking types and we illustrate how efficient NSGs can have complex structures.

Related literature. The issue of finding efficient networks has been analyzed in the context of strategic network formation games. A pioneering literature addressed this question in a setting where agents derive utility from their connections (for seminal contributions, see Jackson and Wolinsky, 1996 and Bala and Goyal, 2000). However in this paper, agents derive utility from a chosen action.

Some recent papers have addressed efficiency in models of network formation with endogenous choice of action. In the context of R&D networks, Goyal and Moraga (2001) restrict their attention to *regular* networks (i.e. where agents all have the same number k of links) and derive the number k that maximizes social welfare. In a dynamical setting, König, Battiston, Napoletano and Schweitzer (2012), show that the efficient network structure depends on the marginal cost of collaborations. When cost is high, it is asymmetric and has a nested structure. In Westbrock (2010) and Billand, Bravard, Durieu and Sarangi (2014), the authors study efficiency in games with global spillovers à la Goyal and Joshi (2003), whereas we consider a network with bilateral influences.

In the context of local public goods, Galeotti and Goyal (2010) focus on games where actions are strategic *substitutes* and show that either the star network or the empty network is efficient.

More closely related to our work, the paper by Corbo, Calvó-Armengol and Parkes (2006) analyzes efficiency in the same game as we do. They restrict their attention to *connected graphs* and show that, when the level of interaction tends to its upper bound, and when the number of links is equal to $n-1$, the star is the unique connected network maximizing social welfare. Our paper takes the analysis further: the number of links is not restricted to $n-1$, being determined endogenously; we do not restrict to connected graphs; and we also examine every possible level of interaction.

In a recent paper, Hiller (2014) explores the case of non-linear interactions in a game with local complementarities, and although he mainly focuses on stability issues, he also finds that the efficient network is either empty or complete when the cost of links is constant. We also find this result as a particular case in our analysis.

Finally, there is an extensive literature examining the design of optimal communication structures in teams and organizations where the objective is the efficiency of information transmission. Some important references are Marshak and Radner (1972), Sah and Stiglitz (1987), Radner (1993), Bolton and Dewatripont (1994), Garicano (2000) or more recently Renou and Tomala (2012). However, in our setting, networks do not serve as communication channels, rather as a channel for peer effects.

The rest of the paper is organized as follows. Section 2 introduces the model. In section 3 we characterize efficient networks as NSGs while in section 4 we specify some cost functions and refine our results. We conclude in section 5. All proofs can be found in the Appendix.

2 The model

We consider a fixed and finite set of agents $N = \{1, 2, \cdots, n\}$ who interact on a network and choose some effort level. An agent's payoff is determined both by his effort and by the effort of the agents he is linked to.

2.1 The Network

The networks we consider are collections of binary and symmetric relationships, represented by an adjacency matrix $G = (g_{ij})_{i,j\in(1,n)\times(1,n)}$ where $g_{ij} = 1$ when there is a link between agents i and j, and $g_{ij} = 0$ otherwise. By convention $g_{ii} = 0$. By abuse of notation, G will alternatively stand for the network or its adjacency matrix. When $g_{ij} = 1$ (resp. $g_{ij} = 0$) we will say $ij \in G$ (resp. $ij \notin G$). We let $N_i(G) = \{j \in N; g_{ij} = 1\}$ denote the set of neighbors of agent i in network G and we let $deg(i, G) = \#N_i(G)$ denote the number of these neighbors (the degree).

A component is defined as a set of individuals such that there is a path between every pair of individuals belonging to the component and there is no path between individuals inside the component and individuals outside the component. A component is said to be non-trivial if it contains strictly more than one agent. Let $\mathcal{G}^n$ denote the set of all networks with n agents and let $\mathcal{G}^n(l)$ denote the set of all networks in $\mathcal{G}^n$ with l links ($0 \le l \le \frac{n(n-1)}{2}$). Finally denote by $\mu(G)$ the largest eigenvalue (or the *index*) of the adjacency matrix G.

2.2 Bonacich Centralities and Social Welfare

Agent i chooses effort level $x_i \in \mathbb{R}_+$. Let $X \in \mathbb{R}^n_+$ be the profile of individual efforts and let x_{-i} denote the profile of the efforts of all agents other than i. We consider a standard linear quadratic utility function with synergies as in Ballester et al. (2006). It is formed of an idiosyncratic component resulting from own effort and a term reflecting strategic complementarities between neighbors.

$$u_i(x_i, x_{-i}, G, \delta) = x_i - \frac{1}{2}x_i^2 + \delta \sum_{j=1}^{n} g_{ij} x_i x_j$$

where $\delta > 0$ measures the intensity of interactions between agents.

A (pure) Nash equilibrium X^* of this game satisfies the following first order conditions:

$$(I - \delta G)X^* = \mathbf{1} \tag{1}$$

As shown in Ballester et al. (2006), an equilibrium exists if and only if $\delta\mu(G) < 1$. In that case the inverse matrix $M = (I - \delta G)^{-1}$ is non-negative and the linear system admits a unique solution. This solution coincides with the Bonacich centralities of agents (Bonacich, 1987), i.e.

$$X^* = B(G, \delta) = \sum_{k=0}^{+\infty} \delta^k G^k \mathbf{1}$$

Agent i's Bonacich centrality can be interpreted as the weighted sum of paths of any length starting from agent i in network G. By denoting $B(G, \delta) = (b_i(G, \delta))_{i \in N}$ we have:

$$b_i(G, \delta) = \sum_{k=0}^{+\infty} \delta^k P_k(i, G)$$

where $P_k(i, G)$ is the number of paths of length k, including loops, starting from agent i (formally, $P_k(i, G) = G^k \mathbf{1}$). Given the linear quadratic specification under consideration, agents' utility at equilibrium is given by:

$$u_i(X^*, G, \delta) = \frac{1}{2} b_i^2(G, \delta)$$

We address the problem of a social planner looking for the network maximizing social welfare. When doing so, we implicitly assume that on any given network G and for any given level of interactions δ, agents exert their equilibrium effort. Because of the uniqueness of the equilibrium efforts, we drop X^* from the arguments of the utility function ($u_i(G, \delta) \equiv u_i(X^*, G, \delta)$). In order to guarantee that equilibrium exists on every network, we impose $\delta \in [0, 1/(n-1)[$.[3]

We assume that the total cost of a network (called the *network cost*) is given by:

$$\Phi\left(\sum_{i,j \in N} g_{ij}.c_{ij}\right) \tag{2}$$

where $\Phi(.)$ is an increasing function, with $\Phi(0) = 0$, and where c_{ij} is a positive number characterizing the link between agents i and j, defined as a function of i and j's *individual*

[3]Indeed, $1/(n-1)$ is, among all possible networks with n agents, the largest possible index. It is associated with the complete network. If $\delta \geq 1/(n-1)$, the equilibrium efforts in the complete network are infinite, so that the problem at hand is trivial.

linking types: let $C = (c_i)_{i\in N}$ be a vector in $\mathcal{R}^n_+$ of *individual linking types*, then $c_{ij} = f(c_i, c_j)$, where $f(.,.)$ is positive, increasing in both arguments and symmetric (i.e. $f(c_i, c_j) = f(c_j, c_i)$).

Individual linking types may reflect individuals' capacity for social life, ability to communicate with peers, etc. In turn, c_{ij} can be interpreted as the cost of forming a link between agents i and j when $\Phi(x) = x$. More general formulations of $\Phi(.)$ indicate that the contribution of a link to the network cost depends on the structure of the current network.

This formulation covers numerous situations. It includes the standard case where the network cost is proportional to the number of links it contains when $\Phi(x) = x$ and $c_{ij} = c$ for all i and j. Then the cost of a network with l links is simply $c.l$.

It also allows for increasing transformations of the number of links (in some contexts any additional link may be more costly - or conversely, less costly - than the previous link), and it allows for more general situations where c_{ij} may be heterogeneous, such as $c_{ij} = c_i + c_j$, $c_{ij} = c_i.c_j$, $c_{ij} = Min\{c_i; c_j\}$, $c_{ij} = Max\{c_i; c_j\}$ etc.

We define social welfare as a function of network, of interaction level and of network cost:

$$W(G,\delta,C) = \sum_{i\in N} u_i(G,\delta) - \Phi\left(\sum_{i,j\in N} g_{ij}.c_{ij}\right) \tag{3}$$

A network $G \in \mathcal{G}^n$ is efficient whenever

$$W(G,\delta,C) \geq W(G',\delta,C) \text{ for all } G' \in \mathcal{G}^n \tag{4}$$

Remark 1. *We assume throughout that the social planner lets agents choose their equilibrium effort level. In this sense the problem at hand is a second-best investigation. However, we could assume that the social planner imposes both the network and agent efforts. It can be shown that the efficient level of effort in this game, when the intensity of interactions is δ, is given by the Bonacich centralities of agents playing the same game with interaction intensity $\delta^* = 2\delta$. The solution to this first-best problem is thus given by the network maximizing $W(G, 2\delta, C)$.*

2.3 Some specific network structures

Before turning to the analysis, we present some network structures that will play a prominent role: the class of Nested Split Graphs (NSGs). NSGs were first introduced in graph theory by Chvatal and Hammer (1977) as *threshold graphs*. They propose several equivalent definitions, among which:

Definition 1 (Nested Split Graph)**.** *A graph G is called a Nested Split Graph if*

$$[ij \in G \text{ and } deg(k,G) \geq deg(j,G)] \Longrightarrow ik \in G$$

This definition, while not standard in the graph theory literature, will be useful in our context. A non-empty NSG is a network with one non-trivial component, in which agents' neighborhoods are nested. A non-empty NSG can also have additional isolated agents. Agents in the non-trivial component can be partitioned into p classes, such that: (i) two agents in the same class have the same degree and (ii) agents in class i are linked to every agent in classes 1 to $p-i+1$.[4]

Out of their many interesting properties (see for instance König et al. (2014) for an extensive analysis of the properties of NSGs), the four following are particularly worthy of note. First, the non-trivial component of the NSG is of diameter two. Second, agents are ordered by degree. Class 1 agents have degree $n_C - 1$ (where n_C is the size of the non-trivial component) and as the class index increases, degree strictly decreases. Thus in an NSG, $deg(i, G) > deg(j, G)$ implies that i is in a higher class than j. Third, the set of agents belonging to the first $E[\frac{p+1}{2}]$ classes forms a complete subgraph (also called a clique). Last, for every interaction level δ, the Bonacich centrality ranking of agents is aligned with their degree ranking. Figure 1 presents an NSG with 11 agents and 5 classes (dotted circles represent classes).

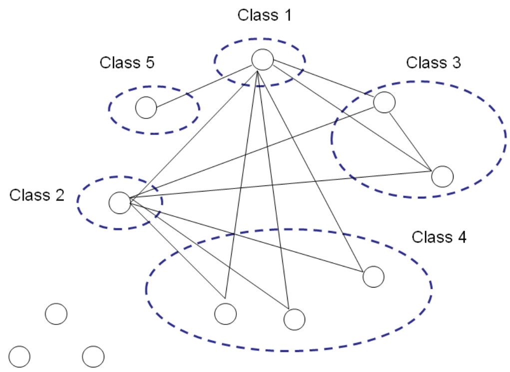

Figure 1: $n = 11, l = 13$. NSG with 5 classes and 3 isolated agents

We now present some members of the class of NSGs that will play a role in the rest of the analysis. In addition to the standard complete network (a single class of agents with no isolated agents) or the star network (two classes, the central agent in class 1 and the peripherals in class 2), the NSG class contains other prominent networks. We present four subclasses of interest.

The *Core-Periphery networks* are a generalization of the star network with several central players. They are defined and explored in Galeotti and Goyal (2010) among others. We provide the following equivalent definition:

Definition 2 (Core-Periphery networks)**.** *A network is a Core-Periphery if and only if it is an NSG with at most two classes of agents and no isolated agents.*

Another prominent subclass of NSGs is the *Dominant Group Architecture*, introduced by Goyal and Joshi (2003). These consist of a complete component and isolated agents. We provide the following definition in terms of classes:

[4]By convention we assume that isolated agents do not form a class of the NSG, as they will play a limited role in our analysis.

Definition 3 (Dominant Group Architecture)**.** *A network is a Dominant Group Architecture if and only if it is an NSG with one class of agents and possibly some isolated agents.*

In order to define the next member of the class, let K_p denote a complete subgraph with p agents.

Definition 4 (Quasi-Complete graph)**.** *A graph $G \in \mathcal{G}^n(l)$ is called a Quasi-Complete graph, noted $QC(l)$, if it contains the complete subgraph K_p with $\frac{p(p-1)}{2} \leq l < \frac{p(p+1)}{2}$, and the remaining $l - \frac{p(p-1)}{2}$ links are set between one other agent and agents in K_p.*

A Quasi-Complete graph is an NSG either with a single class of agents when $l = \frac{p(p-1)}{2}$, or with three. Figure 2 presents some Quasi-Complete graphs. Note that $QC(3) = K_3$, $QC(6) = K_4$ and $QC(10) = K_5$.

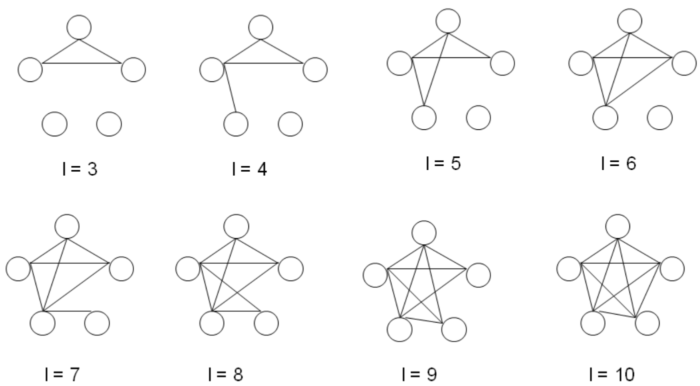

Figure 2: Quasi-Complete graphs with $n = 5$.

Definition 5 (Quasi-Star graph)**.** *A graph $G \in \mathcal{G}^n(l)$ is called a Quasi-Star graph, noted $QS(l)$, if it has a set of p central agents with $n-1$ links, and the remaining $l - p(n-1)$ links are set so as to construct another central agent.*

A Quasi-Star graph contains one class of agents if it is the complete network; otherwise it contains either two, three or four classes of agents. Figure 3 presents some Quasi-Star graphs, where for instance $QS(4)$ contains two classes, $QS(5)$ contains three classes and $QS(6)$ contains four classes.

3 Efficient Networks

First we identify a specific procedure of link reallocation that increases the sum of utilities without changing the number of links. This operation, possibly combined with some appropriate permutation of agents, will help us characterize efficient networks.

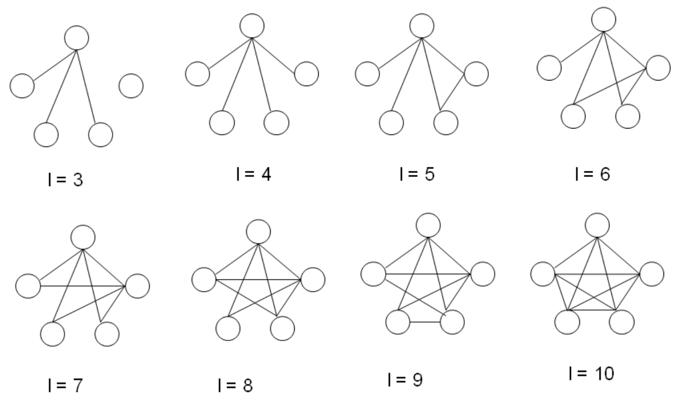

Figure 3: Quasi-star graphs with $n = 5$

A natural way to tackle our problem would be to delete an existing link between a pair of agents i and j in the network and replace it by a link between agent i and another agent k that has a higher Bonacich centrality than agent j. Unfortunately, these intuitive single link reallocations may not be enough to increase social welfare. This is illustrated in Figure 4, where we present a network that is never the efficient network, whatever the cost function, but where such a link reallocation would result in a strict decrease of social welfare. In this example, $b_k(G,\delta) - b_j(G,\delta) = 0.005$ for $\delta = 0.03$. A single link reallocation from ij to ik decreases social welfare. This is due to the fact that after the reallocation, although all the agents in the star are better off, all the agents in the complete component are less well off. Despite the increased sum of efforts after the reallocation, in this example the losses in terms of utility are not compensated by the gains.

We thus investigate specific multiple link reallocations: Consider agents j and k and define $N_{j\backslash k}(G)$ as the set of neighbors of j who are not neighbors of k: $N_{j\backslash k}(G) = \{i \in N; g_{ij} = 1 \text{ and } g_{ik} = 0\}$. We define a *Neighborhood-switch from j to k*, called hereafter an $N_{(j,k)}$-switch, as a reallocation of the links between j and $N_{j\backslash k}(G)$ to links between k and $N_{j\backslash k}(G)$.

Let $A_l^{j\backslash k} = (a_{im})_{i,m\in(1,n)\times(1,n)}$ be such that $a_{im} = a_{mi} = 1$ if $i \in N_{j\backslash k}(G)$ and $m = l$, $a_{im} = 0$ otherwise. This matrix contains a 1 between agent l and all of agent j's neighbors who are not neighbors of agent k, and it contains 0 otherwise.

Definition 6 ($N_{(j,k)}$-switch)**.** *Consider a network G. An $N_{(j,k)}$-switch is a multiple link reallocation leading to the network G' where $G' = G + A_k^{j\backslash k} - A_j^{j\backslash k}$.*

Figure 5 illustrates an $N_{(j,k)}$-switch. Note that agent j could be isolated after the switch. An N-switch has interesting properties in terms of aggregate utilities:

Lemma 1. *Consider a network G with agents j and k such that $b_j(G,\delta) \leq b_k(G,\delta)$ and $N_{j\backslash k} \neq \emptyset$. Let $G' = G + A_k^{j\backslash k} - A_j^{j\backslash k}$. Then for any $\frac{1}{n-1} > \delta > 0$, $\sum\limits_{i\in N} u_i(G',\delta) > \sum\limits_{i\in N} u_i(G,\delta)$.*

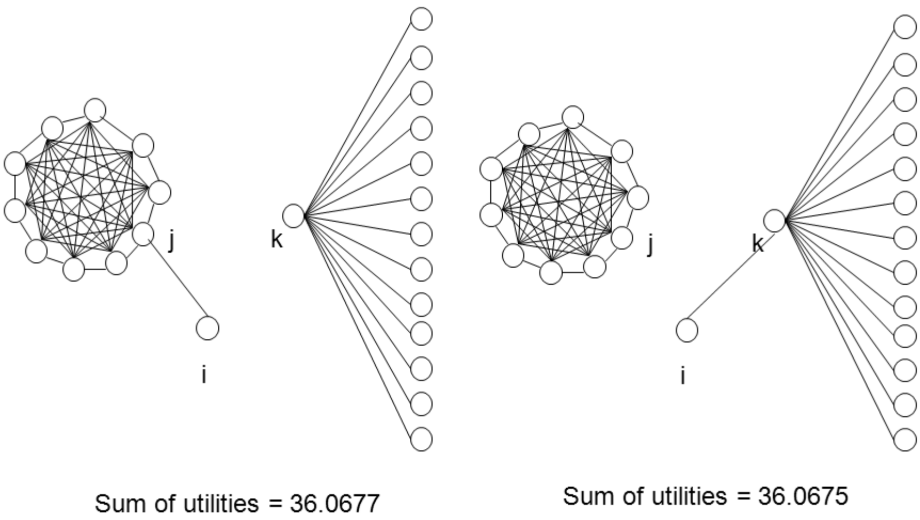

Figure 4: $n = 25$, $l = 59$ and $\delta = 0.03$.

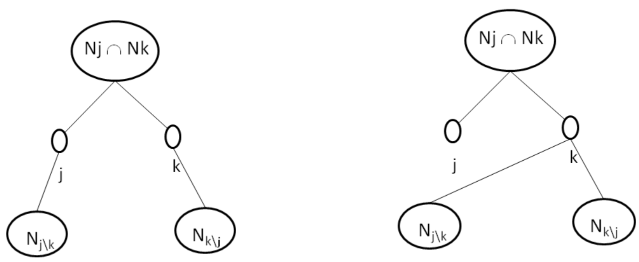

Figure 5: An $N_{(j,k)}$-switch

In order to prove lemma 1, we first determine the equilibrium efforts in the network before the $N_{(j,k)}$-switch. Then we implement the $N_{(j,k)}$-switch and we track how equilibrium efforts are modified. The special feature of the type of reallocation we consider is that the only agent who may suffer from the reallocation is j, while all others benefit from k's higher centrality. This is its main difference from the single link reallocation illustrated above, where all of j's neighbors may also suffer from j's decreased centrality. The use of a Simultaneous Best-Response Algorithm (SBRA) then allows us to conclude that the aggregate gains of others are higher than j's loss.[5]

Lemma 1 identifies an operation that increases the sum of utilities. At the same time, the $N_{(j,k)}$-switch also changes the network cost. If $c_k \leq c_j$, the switch decreases the network cost, while it could increase it if $c_j < c_k$. In that case, we resort to a permutation of agents j and k at the same time as we implement the $N_{(j,k)} - switch$, in order to guarantee that the network cost does not increase. We can now state our main result:

Theorem 1. *An efficient network G is a Nested-Split Graph. Moreover, if $c_i < c_j$ then $deg(i, G) \geq deg(j, G)$.*

The proof of this theorem heavily relies on lemma 1 and on the very fact that the only networks with no N-switches are the Nested-Split Graphs.

Complementarities, together with the convexity of equilibrium utilities, encourage accumulation around a subset of agents. This is what NSGs do, because agents in the first class of an NSG are connected to everyone else in the single component, turning them into (partially) central agents. This accumulation leads to very high centrality for these agents, and to short distances which will guarantee strong feedback effects on other agents. Furthermore, the agents with the lowest individual linking types are more central because they have more links. Indeed, in NSGs, centrality and degree coincide.

Remark 2. *Because N-switches potentially exclude agents from the network, they might bring to mind key player policies (see Ballester et al., 2006). In our setting, there are two possible analogies with the key player. First, the key player could be the agent contributing most to aggregate efforts, who should receive the reallocations. Alternatively, we could define the "inverse key player" as the player contributing least to aggregate efforts and consider that this agent should be the one excluded from the network.*

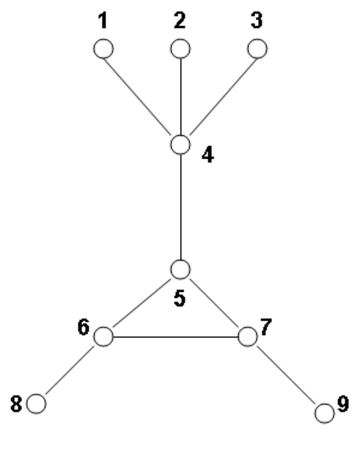

Figure 6

[5]Note that the proof also holds if we consider the sum of efforts instead of the sum of utilities. As a consequence, all the implications of lemma 1 regarding the maximization of social welfare also hold if the social planner wishes to maximize aggregate efforts instead.

However, we show with the counterexample in Figure 6 that neither of these analogies is correct. This network has 9 *agents and* 9 *links. For* $\delta = 0.19$*, the player contributing most (the key player) is agent* 5 *while the players contributing least (the inverse key players) are agents* 8 *and* 9*. However, the best N-switch to be implemented on this network is the* $N_{(5,4)}$*-switch (consisting of reallocating links* 56 *and* 57 *to* 46 *and* 47*). Thus, on the one hand it is better to maintain the agents contributing least in the network (agents* 8 *and* 9*) rather than disconnecting them, while on the other hand it is better to partly disconnect the key player (agent* 5*) despite his high contribution to aggregate outcomes.*

The problem of discriminating between different NSGs in order to find the efficient network is difficult to tackle analytically, principally because in an NSG, no link reallocations are possible from an agent with low centrality to another with higher centrality. This implies that any improvement on a given NSG is obtained by reallocating links toward agents with lower centralities. However, with some specific cost functions, we are able to refine our results and discriminate among NSGs.

4 Specific cost functions

In this section we examine some cost function specifications that include standard cases. Before proceeding, we present a process of adding links that guarantees that the gains in aggregate utility are increasing as links are added.

Consider an arbitrary network G. Let G^- be the network in which links from k to $N_{k\setminus j}(G)$ are severed, together with the links from j to $N_{j\setminus k}(G)$: $G^- = G - A_k^{k\setminus j} - A_j^{j\setminus k}$; let G^+ be the network in which links from j to $N_{k\setminus j}(G)$ are added, together with the links from k to $N_{j\setminus k}(G)$: $G^+ = G + A_j^{k\setminus j} + A_k^{j\setminus k}$. The three different networks are illustrated in Figure 7.

We have the following:

Lemma 2. *If* $N_{k\setminus j}(G) \neq \emptyset$*, then* $\sum_{i\in N}\Big(u_i(G^+,\delta) - u_i(G,\delta)\Big) > \sum_{i\in N}\Big(u_i(G,\delta) - u_i(G^-,\delta)\Big)$

To prove our result, we decompose the effect of both link additions (from G^- to G and from G to G^+) on agents j and k on the one hand and on the other agents on the other hand. The effect on the other agents is unambiguous: the second link addition increases their utilities more than the first link addition does. For agents j and k the effects are not straightforward. We show that the effect on agent k's utility (respectively j's utility) of the second link addition is greater than the effect of the first link addition on agent j's utility (respectively k's utility).[6]

4.1 Homogeneous individual linking types

Assume $c_{ij} = c$ for any pair of agents i and j. We start by examining the case of low levels of interaction intensity δ and show that only specific NSGs can be efficient, whatever the form

[6]The process just considered consists of adding several links at once. This is a necessary condition. Indeed, we have constructed an example in which the addition of any single link increases aggregate utility less than the contribution of any existing link. The example is available upon request.

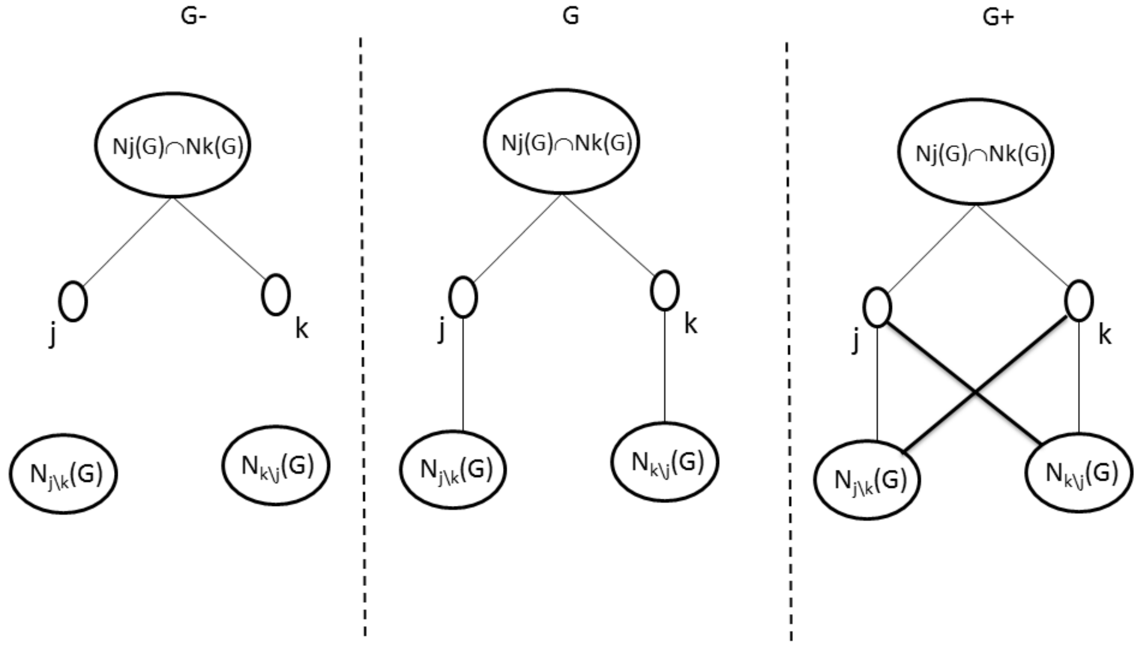

Figure 7: Networks G^- and G^+ are respectively left and right of the initial network G.

of the network cost function $\Phi(.)$. We then turn to general levels of interaction intensity δ and distinguish two cases: when the network cost function $\Phi(.)$ is concave or linear, and when it is convex.

- *Low levels of interaction*

When interactions go to 0, the effects of complementarities vanish faster as they transit along longer paths. Short paths contribute more to centralities and the efficient networks are those maximizing the number of short paths.

Proposition 1. *Assume $c_{ij} = c$ for every i and j. When δ tends to 0, the network maximizing social welfare is either empty, a Quasi-Star or a Quasi-Complete network.*

The sketch of the proof goes as follow. First, we note that when δ goes to zero, the problems of maximizing the sum of the Bonacich centralities or the sum of their squares are equivalent. Then, we decompose the sum of Bonacich centralities into the weighted sum of paths of length 0, 1 and 2 and the weighted sum of all paths of length greater than 3. We first show that when δ goes to 0, this last sum is negligible compared to the first sum. We are then left with three terms, of which the first two are shown to be constant over the whole set of networks with n agents and l links. Finding the network maximizing the sum of Bonacich centralities thus boils down to finding the network which maximizes the number of paths of length 2. Observing that the number of such paths is equal to the sum of squares of the degrees in a network, we rely on Abrego et al. (2009) who identify $QS(l)$ and $QC(l)$ as well as other networks identified for a few very specific values of n and l. All the networks belong to the NSG class. Where there is equality between different networks, we turn to paths of length 3 in order to discriminate. All the networks they identify are beaten at the third order either by $QS(l)$ or $QC(l)$.

Exploiting the results of Abrego et al. (2009), we can go a little further: if $l \leq \frac{1}{2}\binom{n}{2} - \frac{n}{2}$, the network with l links that maximizes social welfare is $QS(l)$, while if $l \geq \frac{1}{2}\binom{n}{2} + \frac{n}{2}$, it is $QC(l)$. This implies that the efficient network's structure is not unique, as it depends on the optimal number of links. When this number is low, $QS(l)$ performs better, whereas $QC(l)$ performs better when this number is high. This condition reveals the trade-off between having a hub linking a large number of agents, all of whom transit through the hub to generate many short paths with few links, or building many triangles with a complete subgraph.

Interestingly, the principles behind accumulation of links on a subset of agents is still what drives the result, but we clearly see here that there are two typical ways of accumulating. Accumulation can either be built around the largest possible subset of agents (as in $QC(l)$), or it can be achieved by increasing the centrality of one central agent as much as possible before trying to include another central agent (as in $QS(l)$). Which type of accumulation is best depends on the number of links in the network.

- *General levels of interaction*

For general levels of interaction, which network is efficient depends on whether $\Phi(.)$ is concave, linear or convex. We start with the concave and linear cases.

Proposition 2. *Assume $c_{ij} = c$ for every i and j and $\Phi(.)$ is concave or linear. If $\frac{n}{2(1-\delta(n-1))^2} > \Phi(\frac{n(n-1)}{2}.c)$ then the efficient network is the complete network, otherwise it is the empty network.*

When the cost function is concave or linear, it can be shown that the link contributing most to social welfare is the last link that is added when the complete network is formed. This is no longer true if $\Phi(.)$ is not concave. In that case, accumulating links until the complete network is reached may not be efficient because the last links could be too costly. In order to find the efficient network for a given level of interactions, the number of links has to be fixed and the corresponding network maximizing aggregate utilities found. Then the social welfare for every possible number of links can be compared, identifying the best number of links l^*, together with the corresponding network structure.

However, the problem of finding, for a fixed number of links, the network maximizing the sum of utilities as a function of δ remains unsolved. We ran simulations to compare Nested Split Graphs and check whether any pattern emerged. We tested for values of n between 3 and 24, and for every n we varied l from 3 to $\frac{n(n-1)}{2}$. For every n and l pair, we simulated 10^4 values of δ. Once n, l and δ were fixed, we computed the sum of utilities in every possible Nested-Split Graph in order to find the best one[7].

[7] The generation of all NSGs is made possible by a mapping between the set of NSGs with n agents and the integers between 0 and 2^{n-1}, according to the following rule: every agent is a bit in a binary number, with state either 0 or 1. Every agent in state 1 is linked to all his predecessors while agents in state 0 are

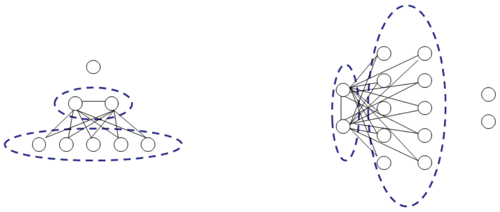

Figure 8: QS-like

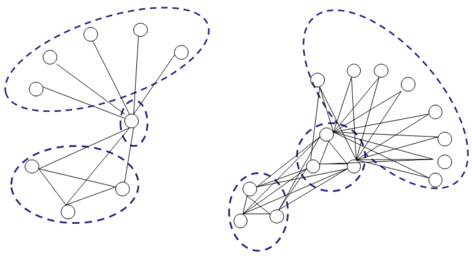

Figure 9: Hybrids

Our simulations reveal two main features. First, for a low density of links, only two networks appear to be candidates for efficiency: $QC(l)$ and $QS(l)$. The Quasi-Star emerges as best for low values of δ, while the Quasi-Complete is best for high values. In turn, when the density of links is high, the $QC(l)$ appears to be the best, whatever the value of δ.

Second, for intermediate densities, other complex NSGs can be efficient. We illustrate in Figures 8 to 10 some typical structures that might emerge. Figure 8 shows two examples of "QS-like" networks formed of isolated agents and a Quasi-Star with the remaining agents.

not. The last agent, having no predecessor, does not count. For instance, with $n = 4$, all NSGs are mapped by numbers from 0 to 7, with respective binary sequences 000 to 111. Sequence 000 is the empty network, sequence 111 is the complete network, sequence 001 is the star, and sequence 101 for instance is the kite. The reader interested in the computational details of constructing NSGs can refer to Hagberg, Swart and Schult (2006).

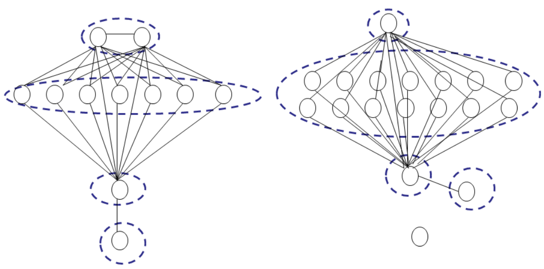

Figure 10: Others

The one on the left ($n = 8$) is efficient for $\delta \in [.001, .025]$ when the network cost function $\Phi(.)$ is such that the optimal number of links l^* is 11. The one on the right is efficient for $n = 14$ and $l^* = 21$ and $\delta \in [.029, .038]$. Figure 9 shows structures that are hybrids of QC and QS, for $n = 9$ and $l^* = 11$ (efficient for $\delta \in [.001, .04]$) as well as for $n = 14$ and $l^* = 39$ (efficient for $\delta \in [.001, .011]$). We also found other structures that do not fit into the first two categories. In figure 10 we illustrate two examples, for $n = 11$ and $l^* = 25$ (efficient for $\delta \in [.001, .012]$) and for $n = 18$ and $l^* = 30$ (efficient for $\delta \in [.021, .053]$).

4.2 Heterogeneous individual linking types with linear network cost function

In this part, we assume that $\Phi(x) = x$ and allow for heterogeneity across individual linking types. We start by examining two polar cases where $c_{ij} \equiv Min\{c_i; c_j\}$ and $c_{ij} \equiv Max\{c_i; c_j\}$. Next, we briefly show that other functions can lead to complex structures.

- $c_{ij} \equiv Min\{c_i; c_j\}$

If the cost of a link is determined by the pair's lowest individual linking type, we get the following:

Proposition 3. *Suppose that $\Phi(x) = x$ and $c_{ij} \equiv Min\{c_i; c_j\}$. The efficient network is a Core-Periphery network.*

This result drastically reduces the class of efficient networks to NSGs with one or two classes and no isolated agents. The intuition of this result relies on the observation that as soon as one agent is connected to another, any other agent should be connected to the least costly of these two agents. By lemma 2, social welfare will increase in this case.

- $c_{ij} \equiv Max\{c_i; c_j\}$

Here again, we have a drastic reduction of the class of efficient networks:

Proposition 4. *Suppose that $\Phi(x) = x$ and $c_{ij} \equiv Max\{c_i; c_j\}$. The efficient network is a Dominant Group Architecture.*

Although we again use Theorem 1 and lemma 2 as in the previous result, here we show that an efficient network can only have one class of agents. The difference from the *Min* function is the following: When a pair is formed, every agent with an individual linking type lower than the most costly of the two agents in the pair should be connected to him. But he should also be connected to the other member of the pair, who has a lower linking type. This is why the component is complete and the NSG only contains one class. Contrary to the previous case, we can end up with some isolated agents, those with too high an individual linking type.

Remark 3. *Assume that $\Phi(x) = x$ and $c_{ij} = \alpha c_i + \beta c_j$ (where $\alpha = \beta = \frac{1}{2}$ is a particular case). Propositions 3 and 4 combined give us some bounds on the efficient network: it contains the Dominant Group Architecture maximizing social welfare when the cost function is $c_{ij} = (\alpha + \beta)Max\{c_i; c_j\}$ and it is contained by the Core-periphery network maximizing social welfare when the cost function is $c_{ij} = (\alpha + \beta)Min\{c_i; c_j\}$.*

- *Other functions*

With other cost functions, NSGs with more than two classes can be efficient, for the same reasons as with a convex function $\Phi(.)$: some links may be worth creating while others may not. Indeed, assume for instance that $c_{ij} = c_i.c_j$ and take as an example the complex network on the left side of Figure 10, with 11 agents and four classes of agent. By setting $c_1 = 0.001$, $c_2 = c_3 = 0.1$, $c_4 = ... = c_{10} = 0.3$ and $c_{11} = 10$, the four-class NSG is efficient. Accordingly, the same network would be efficient if the linking type function were $c_{ij} = c_i + c_j$ and we set $c_1 = 0.001$, $c_2 = c_3 = 0.015$, $c_4 = ... = c_{10} = 0.03$ and $c_{11} = 0.045$.

5 Conclusion

We have examined the problem of finding the efficient structure in a network game where interactions are linear, neighbors' efforts are pure strategic complements and network formation is costly. We focus on the role played by the class of Nested-Split Graphs and some specific members of this class. This work complements König et al. (2014), who examine strategic network formation in a dynamic setting, and show that stable networks are NSGs.

This analysis could be extended to more general utility functions. Indeed, one can show that the N-switch increases the sum of utilities as soon as utilities are convex in Bonacich centralities. However, not much is known for more general utility functions. It would therefore be interesting to characterise the class of network games for which efficient networks are NSGs.

6 Proofs

Proof of lemma 1. We show that, following an $N_{(j,k)}$-switch, a sequence of simultaneous myopic individual best-responses leads to an increase of the sum of utilities. Consider an initial network G with equilibrium efforts $X^*(G)$. There are two cases.

. Case 1: $jk \notin G$.
We modify network G by implementing an $N_{(j,k)}$-switch so as to obtain the network $G' = G + A_k^{j\backslash k} - A_j^{j\backslash k}$ and initiate a simultaneous best-response algorithm (SBRA) on the modified network G'. We denote by $X^{(t)}$ the vector of efforts of agents at the end of period t. We start with initial conditions $X^{(0)} = X^*(G)$ that satisfy:

$$\begin{cases} x_j^{(0)} = 1 + \delta \sum\limits_{c \in N_j(G) \cap N_k(G)} x_c^{(0)} + \delta \sum\limits_{s \in N_{j\backslash k}(G)} x_s^{(0)} \\ x_k^{(0)} = 1 + \delta \sum\limits_{c \in N_j(G) \cap N_k(G)} x_c^{(0)} + \delta \sum\limits_{p \in N_{k\backslash j}(G)} x_p^{(0)} \end{cases}$$

and agent i's best-response updating process at period $t+1$ is given by $x_i^{(t+1)} = 1 + \delta \sum\limits_{j \in N_i(G)} x_j^{(t)}$.

This SBRA will converge to the unique effort equilibrium on the network G' by standard contraction properties (see for instance Milgrom and Roberts, 1990, for convergence in games with strategic complementarities).

At step 1 of the algorithm we get:

$$\begin{cases} x_j^{(1)} = x_j^{(0)} - \delta \sum\limits_{s \in N_{j\setminus k}(G)} x_s^{(0)} \\ x_k^{(1)} = x_k^{(0)} + \delta \sum\limits_{s \in N_{j\setminus k}(G)} x_s^{(0)} \\ x_s^{(1)} = x_s^{(0)} + \delta(x_k^{(0)} - x_j^{(0)}) \text{ for all } s \in N_{j\setminus k}(G) \\ x_q^{(1)} = x_q^{(0)} \text{ for all } q \neq j, k;\ q \notin N_{j\setminus k}(G) \end{cases}$$

Since, by hypothesis, $x_k^{(0)} \geq x_j^{(0)}$, all efforts weakly increase except that of agent j. But agent k's increase compensates agent j' s decrease. Further, utilities being quadratic in effort, the increase in utility of agent k is larger than the decrease in utility of agent j by convexity (given that $x_k^{(0)} \geq x_j^{(0)}$). It follows that the sum of utilities at $X^{(1)}$ is greater than at $X^{(0)}$. However, agent j's loss could have feedback effects on other agents in future steps of the SBRA. We examine step 2.

$$\begin{cases} x_j^{(2)} = x_j^{(1)} \\ x_k^{(2)} = x_k^{(1)} + \delta \sum\limits_{s \in N_{j\setminus k}(G)} (x_s^{(1)} - x_s^{(0)}) \\ x_s^{(2)} = x_s^{(1)} + \delta(x_k^{(1)} - x_k^{(0)}) \\ x_q^{(2)} \geq x_q^{(1)} \text{ for all } q \neq j, k;\ q \notin N_{j\setminus k}(G) \end{cases}$$

with $x_q^{(2)} \geq x_q^{(1)}$ because of complementarities. Therefore $X^{(2)} \geq X^{(1)}$. By complementarities, from step 2 onwards, the efforts will increase at each step of the SBRA and converge to $X^{(\infty)} = X^*(G') \geq X^{(1)}$.

. Case 2: $jk \in G$.
The process needs to be decomposed into two sequential SBRA on network G'. First, we take $X^{(0)} = X^*(G)$ as the initial efforts on G' and we restrict the SBRA to agents j and k, keeping all other efforts fixed. This process converges to $X^{(\infty)}$, where only agents j and k have changed their effort. Second, we take $Y^{(0)} = X^{(\infty)}$ as the initial efforts and apply a SBRA to all agents. This process will converge to $Y^{(\infty)} = X^*(G')$, which is the equilibrium in the modified network.

As $jk \in G$, $X^{(0)}$ now satisfies:

$$\begin{cases} x_j^{(0)} = 1 + \delta \sum\limits_{c \in N_j(G) \cap N_k(G)} x_c^{(0)} + \delta \sum\limits_{s \in N_{j\setminus k}(G)} x_s^{(0)} + \delta x_k^{(0)} \\ x_k^{(0)} = 1 + \delta \sum\limits_{c \in N_j(G) \cap N_k(G)} x_c^{(0)} + \delta \sum\limits_{p \in N_{k\setminus j}(G)} x_p^{(0)} + \delta x_j^{(0)} \end{cases} \quad (5)$$

Thus we get:

$$\begin{cases} y_j^{(0)} = 1 + \delta \sum_{c \in N_j(G) \cap N_k(G)} x_c^{(0)} + \delta y_k^{(0)} \\ y_k^{(0)} = 1 + \delta \sum_{c \in N_j(G) \cap N_k(G)} x_c^{(0)} + \delta \sum_{s \in N_{j \setminus k}(G)} x_s^{(0)} + \delta \sum_{p \in N_{k \setminus j}(G)} x_p^{(0)} + \delta y_j^{(0)} \end{cases} \tag{6}$$

Using (5) and (6), we find:

$$x_j^{(0)} + x_k^{(0)} = y_j^{(0)} + y_k^{(0)} \tag{7}$$

Noticing by (5) and (6) that $y_k^{(0)} - x_k^{(0)} = \delta \sum_{s \in N_{j \setminus k}(G)} x_s^{(0)} + \delta(y_j^{(0)} - x_j^{(0)})$ and by (7), that $y_k^{(0)} - x_k^{(0)} = x_j^{(0)} - y_j^{(0)}$, we get

$$y_k^{(0)} - x_k^{(0)} = \frac{\delta}{1+\delta} \sum_{s \in N_{j \setminus k}(G)} x_s^{(0)} > 0$$

Altogether, the first SBRA, restricted to agents j and k, leads to:

$$\begin{cases} y_j^{(0)} + y_k^{(0)} = x_j^{(0)} + x_k^{(0)} \\ y_k^{(0)} > x_k^{(0)} \end{cases} \tag{8}$$

Now apply a second SBRA on network G', with $Y^{(0)}$ as initial efforts. Let us describe the modified efforts after the first step for every type of agent. Agents in $N_{j \setminus k}(G)$ increase their effort because $y_k^{(0)} > x_j^{(0)}$. Agents in $N_{k \setminus j}(G)$ also increase their effort because $y_k^{(0)} > x_k^{(0)}$. Agents in $N_j(G) \cap N_k(G)$ do not modify their effort because $x_j^{(0)} + x_k^{(0)} = y_j^{(0)} + y_k^{(0)}$. Finally, agents j and k do not modify their effort because they are at the equilibrium effort level of the first SBRA.

At the end of the first step, $Y^{(1)} \geq Y^{(0)}$ and by complementarities, $X^*(G') = Y^{(\infty)} \geq Y^{(0)}$. In profile $Y^{(0)}$, the sum of utilities exceeds the sum of utilities in profile $X^{(0)}$, therefore our conclusion holds. ■

Proof of Theorem 1. We first need to prove a lemma that uses the following definition:

Definition 7. *Let $G_{(j,k)} = G + A_k^{j \setminus k} - A_j^{j \setminus k} + A_j^{k \setminus j} - A_k^{k \setminus j}$ denote the network G in which agents j and k are permuted.*

Lemma 3. *Consider a network G and agents j and k such that $b_j(G, \delta) \leq b_k(G, \delta)$ and $N_{j \setminus k} \neq \emptyset$. If $c_k \leq c_j$, the $N_{(j,k)}$-switch strictly increases social welfare. If $c_j < c_k$, then permuting agents j and k and implementing the $N_{(k,j)}$-switch on network $G_{(j,k)}$ strictly increases social welfare.*

Note that when $c_j < c_k$, the N-switch and the agent permutation have to be implemented together. Indeed, either the permutation alone or the switch alone might result in increased the network cost.

Proof of lemma 3. We examine the two cases and use the following observation: because $f(.,.)$ is increasing in both arguments, we have $[c_i < c_j \Rightarrow f(c_i, c_k) \leq f(c_j, c_k)]$ for all $k \in N \setminus \{i, j\}$.

. **Case 1:** If $c_k \leq c_j$ then every link that is reallocated from agent j to agent k is weakly less costly. As the number of links in the network is constant with the reallocation, lemma 1 guarantees that social welfare strictly increases after the switch.

. **Case 2:** If $c_j < c_k$, consider network $G_{(j,k)}$ in which agents j and k exchange their position. Because this permutation does not affect the network structure, we have $b_j(G, \delta) = b_k(G_{(j,k)}, \delta) \leq b_j(G_{(j,k)}, \delta) = b_k(G, \delta)$. We now implement an $N_{(k,j)}$-switch on network $G_{(j,k)}$ and we denote by G' the resulting network.

In G', the cost of every link that does not involve j nor k is the same as the cost in the initial network G. This is also true for every link in $N_j(G') \cap N_k(G')$ as well as for every link that has been reallocated (because agents j and k exchanged positions before the reallocation). Finally, the links between k and $N_{k\setminus j}(G)$ in the original network G have become links between j and $N_{k\setminus j}(G)$. Therefore their cost is lower in G' than in G. ■

We can now proceed with the proof of Theorem 1. We show that a network which is not an NSG always offers the possibility of an N-switch, i.e. there is a pair j, k such that $N_{j\setminus k}(G) \neq \emptyset$ and $b_k(G, \delta) \geq b_j(G, \delta)$. Assume G is not an NSG. Then, by definition, there are three agents i, j and k such that $ij \in G$, $deg(k, G) \geq deg(j, G)$ and $ik \notin G$. Therefore, $i \in N_{j\setminus k}(G)$. Assume now $b_k(G, \delta) < b_j(G, \delta)$. If $N_{k\setminus j} \neq \emptyset$, then we can implement the $N_{(k,j)}$-switch, a contradiction. Hence, $N_{k\setminus j} = \emptyset$, and therefore, $N_k(G) \subseteq N_j(G)$. Together with the condition that $deg(k, G) \geq deg(j, G)$, we obtain $deg(k, G) = deg(j, G)$ which is a contradiction to the fact that $i \in N_{j\setminus k}(G)$. An efficient network is necessarily an NSG.

Next, assume G is an NSG in which $deg(j, G) > deg(k, G)$ and $c_j > c_k$. Then we can increase social welfare by permuting agents j and k. ■

Proof of lemma 2. If, for a fixed δ, we denote by X (resp. Y and Z) the vector of equilibrium efforts in network G (resp. G^- and G^+), we have

$$\sum_{i \in N} \big(u_i(G^+, \delta) - u_i(G, \delta)\big) > \sum_{i \in N} \big(u_i(G, \delta) - u_i(G^-, \delta)\big)$$

$$\Longleftrightarrow$$

$$\sum_{i \neq j,k} (z_i^2 - x_i^2) + (z_j^2 - x_j^2) + (z_k^2 - x_k^2) > \sum_{i \neq j,k} (x_i^2 - y_i^2) + (x_j^2 - y_j^2) + (x_k^2 - y_k^2)$$

Let $A^+ = G^+ - G$ and $A^- = G - G^-$. Then using the fact that $(I - \delta G)X = (I - \delta G^-)Y = (I - \delta G^+)Z = \mathbf{1}$, we get

$$\begin{cases} X - Y = \delta M^- A^- X \\ Z - X = \delta M A^+ Z \end{cases} \tag{9}$$

where $M = (I - \delta G)^{-1}$ and $M^- = (I - \delta G^-)^{-1}$. Hence,

$$\begin{cases} x_i - y_i = \left(\sum_{l \in N_{k \setminus j}} \delta m^-_{il} \right) x_k + \delta m^-_{ik} \left(\sum_{l \in N_{k \setminus j}} x_l \right) \\ z_i - x_i = \left(\sum_{l \in N_{k \setminus j}} \delta m_{il} \right) z_j + \delta m_{ij} \left(\sum_{l \in N_{k \setminus j}} z_l \right) \end{cases} \tag{10}$$

STEP 1: We show that for all $i \neq j, k$,

$$z_i^2 - x_i^2 > x_i^2 - y_i^2 \tag{11}$$

Note that $G^- \subsetneq G \subsetneq G^+$ implies $Y < X < Z$, as well as $M^- \leq M$ (this inequality holds term by term). Next, agents j and k have symmetric positions in G^- and G^+, so $z_j = z_k$ and $y_j = y_k$ and for all l, $m^-_{lk} = m^-_{lj}$ and $m^+_{lk} = m^+_{lj}$. Putting these observations together, we get $z_i - x_i > x_i - y_i$. Because $z_i + x_i > x_i + y_i$, we get (11).

STEP 2: We show that

$$\begin{cases} z_k^2 - x_k^2 > x_j^2 - y_j^2 \\ z_j^2 - x_j^2 > x_k^2 - y_k^2 \end{cases}$$

Since $G^- \subset G \subset G^+$, we have $x_k > y_k$, and as $y_k = y_j$, we get $x_k > y_j$; we also have $z_j > x_j$, and as $z_k = z_j$, we get $z_k > x_j$. Altogether,

$$z_k + x_k > x_j + y_j \tag{12}$$

We now use (10), applied to j for the first equality and to k for the second equality. Because $m^-_{lj} = m^-_{lk}$ and $M^- \leq M$, we get $m^-_{lj} \leq m_{lk}$ for all $l \in N_{k \setminus j}$; we also have $x_k < z_k = z_j$ and $m^-_{jk} \leq m_{jk}$, so that

$$z_k - x_k > x_j - y_j \tag{13}$$

Putting (12) and (13) together, we obtain

$$z_k^2 - x_k^2 > x_j^2 - y_j^2$$

Reproducing the same steps, we also obtain

$$z_j^2 - x_j^2 > x_k^2 - y_k^2$$

and the desired conclusion follows. ■

Proof of Proposition 1. Let $P_k(G)$ be the number of paths of length k in network G (including loops). Then

$$\sum_i b_i(G, \delta) = n + \delta 2l + \delta^2 P_2(G) + \sum_{k=3}^{+\infty} \delta^k P_k(G)$$

The first two terms of the sum are constant across any network in $\mathcal{G}(l)$. Furthermore, if G^c is the complete network with n agents (i.e. the network with $\binom{n}{2}$ links), then

$$\sum_{k=3}^{+\infty} \delta^k P_k(G) \leq \sum_{k=3}^{+\infty} \delta^k P_k(G^c) = \sum_{k=3}^{+\infty} \delta^k n(n-1)^k$$

while

$$\delta^2 P_2(G) \geq \delta^2$$

Therefore,

$$\frac{\sum_{k=3}^{+\infty} \delta^k P_k(G)}{\delta^2 P_2(G)} \leq \frac{\delta^3 n(n-1)^3 [\sum_{j=0}^{+\infty} (\delta(n-1))^j]}{\delta^2}$$

As $\sum_{j=0}^{+\infty} (\delta(n-1))^j = \frac{1}{1-\delta(n-1)}$, we get

$$\frac{\sum_{k=3}^{+\infty} \delta^k P_k(G)}{\delta^2 P_2(G)} \leq \frac{\delta n(n-1)^3}{1-\delta(n-1)}$$

which implies

$$\lim_{\delta \to 0} \frac{\sum_{k=3}^{+\infty} \delta^k P_k(G)}{\delta^2 P_2(G)} = 0$$

Hence, when $\delta \to 0$, the network maximizing the sum of Bonacich centralities is the network maximizing the number of paths of length 2. Simple algebra leads to the conclusion that maximizing the sum of the squares of Bonacich centralities is also equivalent to maximizing the number of paths of length 2.

The number of paths of length 2 in a network G is given by the sum of all the elements of G^2: $P_2(G) = \mathbf{1}^T G^2 \mathbf{1}$. Because $G = G^T$, we get $P_2(G) = (G\mathbf{1})^T (G\mathbf{1})$, where $G\mathbf{1}$ is the vector of degrees in network G. Therefore, $P_2(G) = \sum_i d_i^2$. We then refer to Abrego et al. (2009) to conclude. ■

Proof of Proposition 2. Theorem 1 tells us that the efficient network G is an NSG. Assume that G is not empty and that it contains at least two classes of agents. Pick an agent j in the second class and another agent k in the first class. Then $N_j(G) \subset N_k(G)$ and lemma 2 ensures that adding the links between j and agents that are in $N_{k \setminus j}(G)$ will increase aggregate utilities more than the contribution to aggregate utilities of all existing links between k and $N_{k \setminus j}(G)$. Because $\Phi(.)$ is concave or linear, the cost of adding these new links to G is weakly lower than the cost of the existing links between k and $N_{k \setminus j}(G)$. This contradicts the fact that G is efficient and implies that an efficient NSG that is non-empty contains only one class.

Next, an efficient network has no isolated agents: if an agent i is isolated, we can apply the same reasoning as above by replacing j by i and get the same contradiction. The only NSG containing all agents in one class is the complete network.

Finally, the social welfare of the empty network being 0, the complete network becomes the efficient network once it induces positive social welfare. The Bonacich centrality of every agent in the complete network with n agents is $\frac{1}{1-\delta(n-1)}$, and the sum of utilities is $\frac{n}{2(1-\delta(n-1))^2}$. There are $\frac{n(n-1)}{2}$ links in the complete network, so the cost of the complete network is $\Phi(\frac{n(n-1)}{2}.c)$ ■

Proof of Proposition 3. Consider a non-empty NSG G and assume it is efficient. We first show that it has no isolated agents. Assume agent j is isolated, and pick any agent k in the non-trivial component of the NSG. Then $N_{k\backslash j}(G) = N_k(G)$ and $N_{j\backslash k}(G) = \emptyset$. By lemma 2, adding links between j and $N_k(G)$ is profitable in terms of aggregate utilities. Further, Theorem 1 tells us that $c_j \geq c_i$ for every agent i in the component. Therefore, $c_{ij} = c_i$ for any i in the component, and linking j to the network by forming links between j and $N_k(G)$ increases social welfare.

Next we show that a non-empty efficient network has at most two classes. Assume G has p classes of agents ($p > 2$). We consider two cases:

• $p \geq 4$. Then pick agent j in the class p (the last one) and k in class $p-1$. By definition of an NSG, j is connected to every agent of class 1, while k is connected to everyone in class 1 as well as to agents in class 2. By lemma 2,

$$\sum_{i\in N}\Big(u_i(G^+,\delta) - u_i(G,\delta)\Big) > \sum_{i\in N}\Big(u_i(G,\delta) - u_i(G^-,\delta)\Big) \tag{14}$$

where $G^+ = G + A_j^{k\backslash j}$ and $G^- = G - A_k^{k\backslash j}$. Moreover, by Theorem 1, $c_i \leq c_j$ for all $i \in N_{k\backslash j}(G)$, so that every link in $G^+ - G$ has a cost of $Min\{c_i; c_j\} = c_i$, which is exactly the same cost as for the links in $G - G^-$. Taking benefits and costs together, the addition of links from G to G^+ strictly increases social welfare.

• $p = 3$. Pick agent j in the last class (class 3) and agent k as the agent in class 2 with the highest individual linking type. By definition, j is linked to every agent in class 1 while k is also linked to agents of his own class. Therefore $N_{k\backslash j}(G)$ is the set of all agents in class 2, except k. Applying lemma 2, it is clear that aggregate utilities strictly increase.

Because k is the agent with the highest linking type in class 2, we have $Min\{c_i; c_k\} = c_i$ for all links in $G - G^-$. Also, $Min\{c_i; c_j\} = c_i$ for all new links in $G^+ - G$, so that $c_{ij} = c_{ik}$ for all i in class 2 other than k.

Again, taking benefits and costs together, the addition of links from G to G^+ strictly increases social welfare. ■

Proof of Proposition 4. Consider a non-empty NSG G, with at least two classes of agents and assume it is efficient. Pick agent k in the first class (the higher one) and pick the agent j with highest individual linking type c_j. By Theorem 1, agent j has to be in the last class (the lower one).

Lemma 2 says that:

$$\sum_{i\in N}\Big(u_i(G^+,\delta)-u_i(G,\delta)\Big) > \sum_{i\in N}\Big(u_i(G,\delta)-u_i(G^-,\delta)\Big)$$

where $G^+ = G + A_j^{k\backslash j}$ and $G^- = G - A_k^{k\backslash j}$.

On the other hand, because $c_j \geq c_i$ for all i, we have $Max\{c_j; c_k\} = c_j = Max\{c_j; c_l\}$ for all $l \in N_{k\backslash j}(G)$ so that

$$\sum_{i,j\in N} c_{ij}(g_{ij}^+ - g_{ij}) = \sum_{i,j\in N} c_{ij}(g_{ij} - g_{ij}^-)$$

Combining the two, we get

$$W(G^+,\delta,C) - W(G,\delta,C) > W(G,\delta,C) - W(G^-,\delta,C)$$

which contradicts the fact that G is efficient. ■

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