Convergence in Games with Continua of Equilibria

Sebastian Bervoets[*] and Mathieu Faure[†]

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Abstract

In game theory, the question of convergence of dynamical systems to the set of Nash equilibria has often been tackled. When the game admits a continuum of Nash equilibria, however, a natural and challenging question is whether convergence to the set of Nash equilibria implies convergence to a Nash equilibrium. In this paper we introduce a technique developed in Bhat and Bernstein (2003) as a useful way to answer this question. We illustrate it with the best-response dynamics in the local public good game played on a network, where continua of Nash equilibria often appear.

Keywords: Convergence; Continua of Nash Equilibria; Best-Response Dynamics.

JEL Codes: C62, C65

1 Introduction

The question of convergence of dynamical systems to some Nash equilibrium has often been explored in economics. Usually, convergence is discussed in contexts where the set of Nash equilibria is finite (see, for instance, the series of papers about convergence to the Cournot solution - Theocharis (1960), Fisher (1961), Hahn (1962), Seade (1980) among many others. In other games, see i.a. Arrow and Hurwicz (1960) or Rosen (1965)). Yet continua of Nash equilibria may appear in several economic situations. As Seade (1980) points out when discussing convergence of dynamical systems, "Things would

[*]Corresponding author. Aix-Marseille University (Aix-Marseille School of Economics), CNRS and EHESS. Email address: sebastian.bervoets@univ-amu.fr. Postal address: Aix Marseille School of Economics - 5 Boulevard Maurice Bourdet - 13205 Marseille Cedex 01 - France.

[†]Aix-Marseille University (Aix-Marseille School of Economics), CNRS and EHESS. Email address: mathieu.faure@univ-amu.fr.

get trickier (...) if equilibria happened not to be regular, that is not even locally unique, isolated. This, one can dismiss as a non-generic, 'unlikely' occurrence, although that is often a risky stand to take." Proving convergence in that case becomes problematic. In fact, to the best of our knowledge, no paper in economics addresses this issue.

When Nash equilibria are isolated, proving convergence amounts to showing that the distance between any solution curve and the set of Nash equilibria goes to zero. This is also necessary, but no longer sufficient, when equilibria are not isolated. Actually, the solution curve could very well approach the set of Nash equilibria, without ever converging to one specific element of that set. Convergence to an equilibrium when continua of equilibria exist has been explored in the dynamical systems literature (see for instance the book by Aulbach (2006) devoted to this problem). However, these techniques generally require strong regularity assumptions which fail to hold in most economic situations. In particular, they assume that the state space is an open set, while economic variables (such as prices, time allocation, efforts, quantities.) are typically defined on non-open sets[1]. This makes these convergence results inapplicable[2].

Another technique, which we call the non-tangency technique, has been developed recently by Bhat and Bernstein (2003) without regularity assumptions. In this paper, we adapt their technique and illustrate how it works by analyzing a standard dynamical system - continuous-time best-response dynamics- in a game which has received considerable attention over recent years: the local public good game introduced in Bramoullé and Kranton (2007).

In this game, players are placed on a network and interact only with their neighbors. The game has linear best responses and strategic substitutes, where individuals' payoffs depend on the sum of their neighbors' actions. It has been extensively studied in the recent literature, both for its great simplicity and for its rich structure of Nash equilibria. Of course the structure of the set of equilibria critically depends on the structure of the network, and in fact, Bervoets and Faure (2018) show in a companion paper that a substantial fraction of networks have continua of Nash equilibria. These ingredients combine to make this game the perfect candidate for our analysis of convergence.

The dynamical system that we consider is continuous-time best-response dynamics. We choose to focus on this specific dynamical system for various reasons. First, it is widely used in economics (see for instance the papers mentioned earlier about conver-

[1]Usually, agents' actions would be defined on $[0, +\infty[$ or on some compact subset.

[2]For instance, the techniques in Aulbach (2006) rely on the analysis of the Jacobian matrix of the dynamical system at every point of the manifold of equilibria. Obviously, when the state space is not an open set, this Jacobian matrix is not defined everywhere.

gence to the Cournot solution). Second, it is related to many other dynamical systems. And third, it is simple enough to allow for a clear exposition of the non-tangency technique, when a more complex system would necessarily interfere with the understanding of the proof.

In the next section we present the local public good game and describe the structure of the set of Nash equilibria. In section 3 we define the continuous-time best-response dynamics and state our main result about convergence. We also provide an intuitive sketch of the proof, while the formal proof is in the appendix.

2 The local public good game

Consider a game $\mathcal{G} = (\mathcal{N}, X, u)$, where $\mathcal{N} = \{1, \ldots, N\}$ is the set of players, $X = \times_{i=1,\ldots,N} X_i$ where $X_i = [0, +\infty[$ is the action space, and $u = (u_i)_{i=1,\ldots,N}$ is the vector of payoff functions.

Agents are placed on a network represented by an undirected graph $\mathbf{G}$. By convention, we also denote by $\mathbf{G}$ the adjacency matrix of the graph, where $\mathbf{G}_{ij} = \mathbf{G}_{ji} = 1$ if players i and j are linked, and $\mathbf{G}_{ij} = 0$ otherwise. The set of neighbors of player i is $N_i(\mathbf{G}) := \{j \in \mathcal{N}, \mathbf{G}_{ij} = 1\}$. The game we focus on is the local public good game introduced in Bramoullé and Kranton (2007), where the payoff function is

$$u_i(x) = b\left(x_i + \sum_{j \in N_i(\mathbf{G})} x_j\right) - cx_i \tag{1}$$

where $c > 0$ is the marginal cost of effort and $b(.)$ is a differentiable, strictly increasing concave function. In order to rule out trivial cases, we assume that $\lim_{x\to+\infty} b'(x) < c < b'(0)$ and normalize so that $b'(1) = c$.

As can be seen by equation (1), a disconnected player i (i.e. $\mathbf{G}_{ij} = 0$ for all $j \in \mathcal{N}$) will choose to play $x_i^* = 1$. When connected, either the neighbors of i provide less than 1 and i aims to fill the gap to reach 1, or else neighbors provide more than 1 and i enjoys the benefits without exerting any effort.

Thus agents have a unique best response, given by:

$$\forall i \in \mathcal{N},\ \ Br_i(x_{-i}) = \max\left\{1 - \sum_{j \in N_i(\mathbf{G})} x_j, 0\right\}. \tag{2}$$

The set of Nash equilibria is therefore the set of all profiles x^* such that every player for which $x_i^* > 0$ is such that $x_i^* = 1 - \sum_{j \in N_i(\mathbf{G})} x_j^*$, while every player for which $x_i^* = 0$ is such that $\sum_{j \in N_i(\mathbf{G})} x_j^* \geq 1$.

This game is a *best-response potential game*, as introduced in Voorneveld (2000), i.e. there is a function P such that $Argmax_{x_i \in X_i} P(x_i, x_{-i}) = Br_i(x_{-i})$ for all i. The best-response potential P is given by

$$P(x) = \sum_i x_i \left(1 - \frac{1}{2} x_i - \frac{1}{2} \sum_j \mathbf{G}_{ij} x_j\right) \tag{3}$$

The set of Nash equilibria can take complex forms. Importantly for our purposes here, this game often admits continua of equilibria. Let us illustrate this on the simplest possible network, the pair (Figure 1).

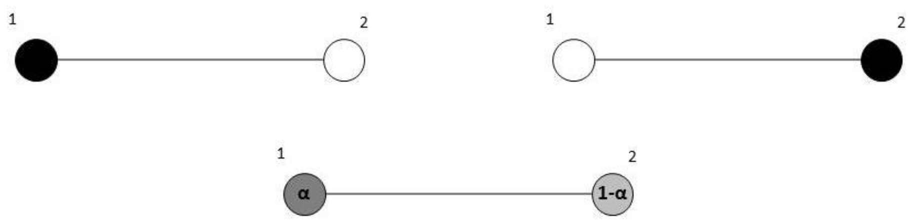

Figure 1: The pair network. Upper left and right panels: circles in black represent agents playing action 1 while white circles represent agents playing action 0; Lower panel: an action profile where agent 1 plays action α and agent 2 plays action $1-\alpha$. These three profiles belong to a continuum of Nash equilibria.

The profile $(1,0)$ where $x_1 = 1$ and $x_2 = 0$ is a Nash equilibrium. The profile $(0,1)$ is also a Nash equilibrium, together with, for instance, $(\frac{1}{3}, \frac{2}{3})$. In fact, any profile of the form $(\alpha, 1-\alpha)$, with $0 \leq \alpha \leq 1$, is a Nash equilibrium. We say that there is a *continuum* of equilibria[3], represented by the *connected component* $\Lambda = \{(\alpha, 1-\alpha) : \alpha \in [0,1]\}$.

Definition 1 *Let $NE(\mathbf{G})$ denote the set of all Nash equilibria of the public good game played on network $\mathbf{G}$. Then, Λ is a connected component of $NE(\mathbf{G})$ if and only if:*

1 - Λ is connected

[3] The reader can find a detailed description of the possible structures of this set in Bervoets and Faure (2018). In that paper, we also conjecture that the fraction of networks on which the game admits at least one continuum of Nash equilibria goes to 1 as N grows, and we stress that continua are generally complex objects of potentially high dimensions.

2 - $\Lambda \subset NE(\mathbf{G})$ *and*

3 - there exists an open neighborhood U *of* Λ *such that* $U \cap NE(\mathbf{G}) = \Lambda$.

The following result will be useful for our analysis.

Proposition 1 *Given any network* $\mathbf{G}$*, the set of Nash equilibria* $NE(\mathbf{G})$ *can be described as a union of connected components, i.e.* $NE(\mathbf{G}) = \cup_{i=1}^{L} \Lambda_i$*, where every* Λ_i *is a connected component.*

See Bervoets and Faure (2018) for a formal proof.

3 Convergence of the Best-Response Dynamics

As seen in (2), best responses are unique. Let

$$Br : X \to X,\ x \mapsto Br(x) := (Br_1(x_{-1}), ..., Br_n(x_{-n})).$$

The continuous-time best-response dynamics is given by the following dynamical system:

$$\dot{x}(t) = -x(t) + Br(x(t)) \tag{4}$$

The map $Br(.)$ being Lipschitz, the ordinary differential equation (4) has a unique solution curve for any initial condition in $\mathbb{R}^N$. Because we restrict attention to $X = \mathbb{R}^N_+$ instead of $\mathbb{R}^N$ and to positive times ($t \geq 0$), we consider the semi-flow

$$\varphi : (x, t) \in X \times \mathbb{R}_+ \to \varphi(x, t) \in X, \tag{5}$$

where, for $t \geq 0$, $\varphi(x, t)$ is the unique solution of (4) at time t, with initial condition $x \in \mathbb{R}^N_+$.

We say that the system (4), starting from a point $x \in X$, *converges to a set* $S \subset X$ if

$$\lim_{t \to +\infty} d(\varphi(x, t), S) = 0$$

Equivalently, given $x \in X$, we call y an omega limit point of x for (4) if there exists a non-negative sequence $t_n \uparrow_n +\infty$ such that $\varphi(x, t_n) \to y$. The set of omega limit points of x is called the *omega limit set of* x and denoted $\omega(x)$. Starting from a point $x \in X$, saying that (4) converges to a set S is equivalent to saying that $\omega(x) \subset S$, as long as the trajectories of system (4) are bounded.
The following result is proved in Bervoets and Faure (2018).

Lemma 1 *Let the network* $\mathbf{G}$ *be fixed. The potential* P *defined in (3) is a strict Lyapunov function for* $\dot{x} = -x(t) + Br(x(t))$*, that is:*

- *for* $x \in NE(\mathbf{G})$ *the map* $t \mapsto P(\varphi(x,t))$ *is constant;*
- *for* $x \notin NE(\mathbf{G})$ *the map* $t \mapsto P(\varphi(x,t))$ *is strictly increasing.*

This lemma implies that only Nash equilibria can be local maxima of the Lyapunov function P. Thus, for all $x \in X$, $\omega(x) \subset NE(\mathbf{G})$, since the omega limit set of system (4) is included in the set of points that maximize the Lyapunov function (see the appendix for more details). However, convergence to the set of Nash equilibria does not imply convergence to a Nash equilibrium.

In order to prove global convergence, i.e. convergence to an equilibrium point from any initial condition, we use the non-tangency method developed by Bhat and Bernstein (2003). The idea is to show that the vector field of the dynamical system (4) is non-tangent to the set of Nash equilibria. In order to proceed, we will use a proposition which determines some characteristics of the components of Nash equilibria. Let $\hat{x}$ be a Nash equilibrium of the game. We define the following partition of $\mathcal{N}$:

$$I(\hat{x}) := \{i \in \mathcal{N}; \hat{x}_i = 0 \text{ and } \sum_{j \in N_i(\mathbf{G})} \hat{x}_j = 1\}$$

$$SI(\hat{x}) := \{i \in \mathcal{N}; \hat{x}_i = 0 \text{ and } \sum_{j \in N_i(\mathbf{G})} \hat{x}_j > 1\}$$

$$A(\hat{x}) := \{i \in \mathcal{N}; \hat{x}_i > 0 \text{ and } \sum_{j \in N_i(\mathbf{G})} \hat{x}_j = 1 - \hat{x}_i\}$$

$I(\hat{x})$ is the set of inactive players, $SI(\hat{x})$ is the set of strictly inactive players, and $A(\hat{x})$ is the set of active players.

We have the following:

Lemma 2 *Let* $x_0 \in X$*. There exists* $\hat{x} \in \omega(x_0)$ *and* $\epsilon > 0$ *such that*

$$\forall y \in B(\hat{x}, \epsilon) \cap \omega(x_0), I(y) = I(\hat{x}).$$

One implication of this lemma is that it is always possible to find a point $\hat{x}$ in $\omega(x_0)$ for which the partition of agents into $I(\hat{x}), SI(\hat{x}), A(\hat{x})$ remains the same in a neighborhood.

We are now ready to state our result.

Theorem 1 *The continuous-time best-response dynamics defined in (4) converges to a Nash equilibrium from any initial condition:*

$$\forall x_0 \in X,\ \exists x^* \in NE(\mathbf{G}) \textit{ such that } lim_{+\infty}\varphi(x_0, t) = x^*.$$

In other words, for any initial conditions x_0, the solution curve starting from x_0 converges to a Nash equilibrium as t increases to infinity. While it was straightforward to show the convergence of system (4) to the set of Nash equilibria, establishing this stronger result is much more laborious. We sketch out the proof of this result. The detailed proof is in the appendix.

Let us denote by f the vector field of the dynamical system: $f(x) := -x + Br(x)$. We know that for any initial condition x_0 we have $\omega(x_0) \subset \Lambda$, where Λ is a connected component of Nash equilibria. Assume x is one point in the continuum of equilibria Λ, and $x \in \omega(x_0)$. We want to show that $\omega(x_0) = \{x\}$. One way of doing so is to look at how the vector field f behaves when the system approaches x. The way it behaves is captured by the possible directions the system can take when it approaches x. This is called the *direction cone* at point x, i.e. the convex cone generated by $f(U)$ for arbitrarily small open neighborhoods U of x.

Now we need to compare this set of possible directions with the set of directions that the system should take in order to stay in component Λ when close to x. These directions are given by the *tangent cone* at x. If the tangent cone and the direction cone have a nonempty intersection, then the system could keep moving along Λ, thus satisfying $\omega(x_0) \subset \Lambda$, without ever converging. If, on the contrary, the intersection between the tangent cone and the direction cone is the null vector, then we are guaranteed that the vector field f will be non-tangent to Λ at point x.

We illustrate how the proof works on a very simple case: consider the pair. In this case, as already stated above, there is one connected component of Nash equilibria $\Lambda = \{(\alpha, 1-\alpha), \alpha \in [0, 1]\}$. Assume that we start at x_0 with $x_0^1 + x_0^2 < 1$ and that there exists some $\hat{x} = (\hat{x}_1, \hat{x}_2) \in \omega(x_0)$, that is interior (i.e. $1 > \hat{x}_1, \hat{x}_2 > 0$). By definition, $\hat{x}$ is a Nash equilibrium so that $\hat{x}_1 + \hat{x}_2 = 1$. At some point, the solution curve will enter a small neighborhood of $\hat{x}$. In that neighborhood, along the trajectory of the BRD, we have $\dot{x}_1 = \dot{x}_2 = (1 - x_1 - x_2)$, and the direction cone is given by all vectors proportional to $(1, 1)$

Now the tangent cone is defined by all the directions compatible with staying in Λ. So if x is in the relative interior, the admissible deviations are of the form $(+\alpha, -\alpha)$. The tangent cone is thus given by all vectors proportional to $(1, -1)$. Obviously, the intersection of the tangent cone and the direction cone can only be the null vector. As a consequence, the vector field is non-tangent to x, which is what we need to prove convergence.

This example is relatively simple to deal with, especially since we use the simplest graph of interactions. In general however, the set of equilibria will possess corners, and finding the tangent cone is not easy. That is where lemma 2 comes in. Details are provided in the appendix; however the main intuition of what is going on is contained in the above example.

Remark 1 *The technique we use allows us to demonstrate that the system converges. The fact that it converges to some Nash equilibrium (and not elsewhere) is a consequence of the fact that the omega-limit set is included in the set of Nash equilibria, which is itself a consequence of the fact that the game is a best-response potential game. However, the technique does not rely on the existence of a potential function (and indeed it is nowhere used in the proof).*

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4 Proof of Theorem 1

Recall that $(x,t) \mapsto \varphi(x,t)$ is the flow associated with the best-response dynamics $\dot{x} = -x + Br(x)$. Then, for any $x_0 \in X$ and any $r > 1$, $\varphi(x_0,t) \in [0,r]^N$ for t large enough. Indeed, if $x_i \geq r$ then $\dot{x}_i \leq 1-r < 0$. This means that, for any $x_0 \in X$, $\omega(x_0)$ is nonempty and contained in $[0,1]^N$. For any $x \in \omega(x_0)$, we have $P(x) = \lim P(\varphi(x_0,t))$. Now $\omega(x_0)$ being invariant directly implies that $P(\varphi(x,t)) = P(x)$ for any t, which means that $x \in NE$ by lemma 1. Thus $\omega(x_0) \subset NE$.

We can actually go farther: recall that the set NE can be written as $NE = \cup_{i=1}^{L} \Lambda_i$. Then, since $\omega(x_0)$ is connected, it must necessarily be contained in a connected component of NE, i.e. $\omega(x_0) \subset \Lambda_l$ for some Λ_l. Yet this is not enough, since we wish to prove that $|\omega(x_0)| = 1$ for any $x_0 \in X$.

Recall that f denotes the vector field of the system (4): $f(x) := -x + Br(x)$. The next definition formally introduces the direction cone associated with f.

Definition 2 *Given $x \in X$, the* direction cone $\mathcal{F}_x$ *of f at x is defined by*

$$\mathcal{F}_x := \cap_{\epsilon>0} coco(f(B(x,\epsilon)) \setminus \{0\}),$$

where $coco(A)$ is the cone generated by the convex hull of A.

In practice, $v \in \mathcal{F}_x$ if and only if for any $\epsilon > 0$ there exists $\lambda > 0$ and u in the convex hull of $f(B(x,\epsilon) \setminus \{0\})$ such that $v = \lambda u$.

Definition 3 *Let $A \subset X$ and $x \in A$. The* tangent cone *to A in x is the set of directions $v \in \mathbb{R}^N$ such that there exists a sequence $(x_n)_n$ in A converging to x and $h_n \downarrow_n 0$ with the following property:*

$$v = \lim_{n \to +\infty} \frac{x_n - x}{h_n}.$$

We denote this set $T_x A$.

We now state the result of Bhat and Bernstein (2003), adapted to our context.

Proposition 2 (Proposition 5.2 in Bhat and Bernstein (2003)) *Let $x_0 \in X$ and $\hat{x} \in \omega(x_0)$. Then if $T_{\hat{x}}\omega(x_0) \cap \mathcal{F}_{\hat{x}} \subset \{0\}$ we have $\omega(x_0) = \{\hat{x}\}$.*

Let us determine the tangent cone. Let $\hat{x}$ be a Nash equilibrium and Λ be the connected component containing $\hat{x}$. Then $v \in T_{\hat{x}}\Lambda$ if and only if

$$\begin{aligned}
v_i &\geq 0 \;\; \forall i \in I(\hat{x}) && (6)\\
v_i &= 0 \;\; \forall i \in SI(\hat{x}) && (7)\\
((\mathbf{I_d} + \mathbf{G})v)_i &= 0 \;\; \forall i \in A(\hat{x}) && (8)\\
((\mathbf{I_d} + \mathbf{G})v)_i &= 0 \;\; \forall i \in I(\hat{x}) \text{ s.t. } v_i > 0, && (9)\\
((\mathbf{I_d} + \mathbf{G})v)_i &\geq 0 \;\; \forall i \in I(\hat{x}) && (10)
\end{aligned}$$

These conditions are the "marginal movements" which allow every agent to stay in Λ. Inactive agents can either remain inactive, become strictly inactive or become active. Condition (6) says that their efforts cannot become negative. Condition (9) says that if they become active, then the sum of efforts around them, own effort included, should remain constant. Condition (10) says that the sum of efforts around them, own effort included, can only increase. Condition (7) says that strictly inactive agents can only remain strictly inactive. Condition (8) says that active agents can move in any direction such that the sum of efforts around them, own effort included, remains constant. These conditions combined characterize the tangent cone at $\hat{x}$.

Now we need Lemma 2, which we prove here.

Proof of Lemma 2. Choose some $\hat{x} \in \omega(x_0)$ with minimal set of inactive players, i.e. for any $x \in \omega(x_0)$, $I(x)$ is not strictly contained in $I(\hat{x})$. Since $\hat{x}_i > 0$ for all $i \in A(\hat{x})$ and $((\mathbf{I_d} + \mathbf{G})\hat{x})_i > 1$ for all $i \in SI(\hat{x})$, there exist $\epsilon > 0$ such that, for any $y \in B(\hat{x}, \epsilon)$, by continuity we have $y_i > 0$ for all $i \in A(\hat{x})$ and $((\mathbf{I_d} + \mathbf{G})y)_i > 1$ for all $i \in SI(\hat{x})$. As a consequence $I(y) \subset I(\hat{x})$ $\forall y \in B(\hat{x}, \epsilon)$, and since $\hat{x}$ is minimal, it directly implies that $I(y) = I(\hat{x})$. ■

One useful corollary is the following:

Corollary 1 *Let $\hat{x}$ be as in Lemma 2. Then for any $v \in T_{\hat{x}}\omega(x_0)$, $v_i = 0$ and $((\mathbf{I_d} + \mathbf{G})v)_i = 0 \quad \forall i \in I(\hat{x})$.*

Proof of Corollary 1. Let $v \in T_{\hat{x}}\omega(x_0)$ and $i \in I(\hat{x})$. There exists some sequence x^n converging to $\hat{x}$ and h_n going to zero such that

$$v_i = \lim_n \frac{x_i^n - \hat{x}_i}{h_n}.$$

For large enough n we then have $x_i^n = 0$, which implies that $v_i = 0$. Also

$$((\mathbf{I_d} + \mathbf{G})v)_i = \lim_n \frac{1}{h_n} \left(((\mathbf{I_d} + \mathbf{G})x^n)_i - ((\mathbf{I_d} + \mathbf{G})\hat{x})_i \right) = 0$$

because $i \in I(x^n)$ for large enough n. ■

We can now prove our main result.

Proof of Theorem 1. We must prove that $|\omega(x_0)| = 1$ for any $x_0 \in X$. Let $x_0 \in X$ and $\hat{x} \in \omega(x_0)$ be as in Lemma 2. We assume without loss of generality that $T_{\hat{x}}\omega(x_0) \cap \mathcal{F}_{\hat{x}}$ is nonempty (otherwise there is nothing to prove). Pick $v \in T_{\hat{x}}\omega(x_0) \cap \mathcal{F}_{\hat{x}}$. We must prove that $v = 0$. According to the last corollary, we have $v_i = 0$ and $((\mathbf{I_d} + \mathbf{G})v)_i = 0$ $\forall i \in I(\hat{x})$.

Now let $i \in A(\hat{x})$. There exists $\epsilon > 0$ such that $f_i(x) = 1 - ((\mathbf{I_d} + \mathbf{G})x)_i = -((\mathbf{I_d} + \mathbf{G})(x - \hat{x}))_i$ for any $x \in U := B(\hat{x}, \epsilon)$. Also $f_i(x) = -x_i$ for $i \in SI(\hat{x})$. As a consequence $f(U) \subset C(U)$, where

$$C(U) := \{w \in \mathbb{R}^N : \exists x \in U \text{ s.t. } w_i = ((\mathbf{I_d} + \mathbf{G})(\hat{x} - x))_i \ \forall i \in A(\hat{x}),\ \ w_i = -x_i \ \forall i \in SI(\hat{x})\},$$

which is convex. By definition of $co(f(U))$, we therefore have $co(f(U)) \subset C(U)$. Now since $v \in coco(f(U))$, we have $v = \lambda w$, with $\lambda > 0$ (if $\lambda = 0$ there is nothing to prove) and $w \in C(U)$ associated with some $x \in U$. As $v \in T_{\hat{x}}\omega(x_0)$, we have $x_i = -w_i = -\frac{v_i}{\lambda} = 0$ for $i \in SI(\hat{x})$.

Let us prove that $v = 0$. We already know that $v_i = 0$ for all $i \in I(\hat{x})$ and for all $i \in SI(\hat{x})$. Thus $\|v\|^2 = \sum_{i \in A(\hat{x})} v_i^2$. Note that $v_i = \lambda((\mathbf{I_d} + \mathbf{G})(\hat{x} - x))_i$ for $i \in A(\hat{x})$. Thus,

$$\begin{aligned} \|v\|^2 &= \lambda \sum_{i \in A(\hat{x})} v_i \times ((\mathbf{I_d} + \mathbf{G})(\hat{x} - x))_i && (11) \\ &= \lambda \sum_{i \in A(\hat{x})} \sum_{j \in N} v_i (\mathbf{I_d} + \mathbf{G})_{i,j} (\hat{x}_j - x_j) && (12) \\ &= \lambda \sum_{j \in N} (\hat{x}_j - x_j) \sum_{i \in A(\hat{x})} (\mathbf{I_d} + \mathbf{G})_{j,i} v_i && (13) \\ &= \lambda \sum_{j \notin SI(\hat{x})} (\hat{x}_j - x_j)((\mathbf{I_d} + \mathbf{G})v)_j && (14) \\ &= 0 && (15) \end{aligned}$$

where we used the fact that $\mathbf{I_d}+\mathbf{G}$ is symmetric to obtain (13), that $x_j = 0$ for $j \in SI(\hat{x})$ to obtain (14), and (15) comes from the fact that $((\mathbf{I_d} + \mathbf{G})v)_j = 0$ for any $j \notin SI(\hat{x})$. The proof is complete. ■

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