Freedom of Choice in a Social Context : Comparing Game Forms *

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This version, September 2006.

Abstract

In this paper the set of outcomes of game forms is introduced as the relevant attribute for evaluating freedom of choice. These sets are defined as the cartesian product of every individual's set of available options. It is argued that doing so is one way of taking into account social interactions when evaluating individual freedom. A set of axioms is introduced that convey some intuitions about how interactions affect freedom of choice, axioms by the mean of which two criteria, the *Max* and the *MaxMin*, are characterized for comparing game forms in terms of the freedom of choice they offer. These criteria are based respectively on the comparison of the best and the worst outcome the individual can reach in the game form.

1 Introduction

This paper introduces the problem of comparison of game forms, as a way of measuring individual's freedom of choice. Freedom of choice has been the subject of a large and stimulating debate. Since fifteen years it has appeared to be a main concern to social choice theorists who believe that economic policies should no longer be judged only in terms of the consequences they have on individuals' well being, as they are in the traditional welfarist approach, but rather that the amount of freedom of choice available to the individuals should also be taken into account in the evaluation of these policies (see e.g. Rawls (1971), Sen (1985, 1988, 1991) for more detailed discussions).

*For helpful comments, I am grateful to Nicolas Gravel, Prasanta Pattanaik, Jean-François Laslier, Marc Fleurbaey, Michel Lubrano, Mamoru Kaneko, seminar participants at the GREQAM and two anonymous referees. Remaining deficiencies are mine. Financial support from the Spanish Ministry of Education and Science through grants SEJ2005-01481/ECON, PTA-2003-02-00005 and FEDER, as well as from the Generalitat de Catalunya through grant 2005SGR00454 is acknowledged.

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In a concern to formalise the concept of freedom of choice, the literature considers that individuals are facing opportunity sets, defined as the sets of all the options available to them and among which they will make their choice (see e.g. Arrow (1995), Bossert, Pattanaik and Xu (1994), Foster (1993), Gravel, Laslier and Trannoy (1998), Klemisch-Ahlert (1993), Pattanaik and Xu (1990, 1998), Puppe (1996), Sugden (1998), Suppes (1996), Van Hees (1997) for representative contributions. See also Barbera, Bossert and Pattanaik (2005) and Foster (2005) for surveys). The concerns one can find throughout the literature, although complex and subtle, could schematically be divided into two categories. On the one hand, some authors have explored the pure intrinsic value of freedom, evaluated by a quantitative measure of the choices available to individuals. The more possible choices there are, the more freedom individuals are enjoying, regardless of the description of the options. On the other hand, some authors consider the instrumental value of freedom according to which freedom is desirable only because it allows individuals to potentially increase their well being by choosing a better option according to their preference relation(s).

Both these views suffer from the fact that they overlook the existence of the interactions that take place between agents that form the society. This is rather problematic when attempting to define freedom of choice, especially when one has in mind the famous maxim according to which "one's freedom begins where the freedom of others ends". This maxim carries the idea that social interactions can be negative because the freedom of one individual restrains everyone else's freedom. Others could defend the position that "one's freedom begins where the freedom of others begins", as does Bakunin when he says (in Bakounine (1960), pp 310, translation is mine):

"*I only feel totally free when I am in the company of other human beings, men and women, who are also free. The freedom of others, far from being a limitation or a negation of my own freedom, is on the contrary its necessary condition and its confirmation. I only become really free with the others' freedom, so that the more there are free people there are around me and the deeper and the broader their freedom is, the deeper and the broader does mine become*".

In both views, it seems that the very concept itself of freedom is difficult to define if it is not considered in a social context where interactions play a role. The two following examples illustrate this point:

Assume that an individual enjoys having a walk in the forest every week-end as a way of evacuating the stress he accumulated during the week. Assume now that his neighbours, after a law has been voted, see their sets of options increased by the possibility of hunting in that same forest. Fearing to be shot by a hunter, the individual might no longer go for a walk in the forest as he used to, given that the consequences of the same action on his part, that of going into the forest for a walk, could be totally different in the second situation than in the first. This individual might then have the feeling that by increasing his neighbours' set of options, the law has reduced his freedom of choice. However, this reduction is not the result of a withdrawal of one of his options, it is the result of giving the others new options.

On the other hand, assume our individual is living in a country in which the exercise of modern medicine is forbidden, and patients are exclusively allowed to be treated by "traditional" methods. Assume now that a consitutional change gives to all the doctors the right to exercise freely any of both types of medicine. With this change, the individual might be better treated when ill, and this can be considered as an increase of the individual's freedom, as he might live a better and longer life. This increase, however, is only made possible because the set of options available to doctors has increased. It is not the result of giving our individual more options, it is again the consequence of giving others more options. In this example, the individual's freedom has been enhanced without his set of options being directly modified.

While the first example shows interactions can be harmful, the second suggests that they can also be beneficial. There are, of course, instances in which interactions can generate both negative and positive consequences. It is the case, for instance, with freedom of speech. If freedom of speech is given to the other individuals, it could have positive consequences on one given individual, as it gives him the opportunity to improve his knowledge through exchanges and debates, and thereby to increase his freedom (by giving him more autonomy in his choices, more insight on important issues etc...). It could also have negative consequences, if individuals decide to use this freedom of speech to slander and insult. Freedom of speech is thus a delicate issue, as it involves both positive and negative interactions.

Considering these simple examples, it appears that the definition itself of freedom should account for interactions, as in a world of interactions, individuals are not only confronted with their own choices but with the other players' one. Therefore, the "standard" opportunity sets framework does not seem to be suited for this issue, as it considers only individuals' options, regardless of any social context. It is usually assumed that individuals are facing sets of mutually exclusive options interpreted as perfect descriptions of the state of the world.

Of course, one could argue that interactions are already indirectly taken into account, as a part of the different options available in the opportunity set. Two different options could accordingly reflect a change in the other individuals' situations only. However, this is misleading as if it were the case that two options could differ uniquely in the other individuals' situation, there would be no reason to assume that an agent could choose freely between these two options. This would imply that the agent has the power of choosing not only his own actions, but also his neighbours' actions, which is of course an assumption one cannot make.

It thus seems more natural to assume that the options available to one individual only represent one feature of the state of the world. One can then consider the state of the world as an *outcome*, an outcome being the result of the combination of all the individuals' choices.

There are two ways (at least) to tackle this problem. The first one would be to redefine the opportunity set of an individual as a set of options where every option is associated with a set of possible outcomes, depending on the other individuals' choices. An opportunity set would then be a set of sets. Although it might be a promising direction to follow, a second route is explored in this paper, considering that the opportunity sets, even if modified, should no longer be seen as the accurate attribute one should focus on to evaluate freedom of choice. Intuitively, if one wishes to take into account the set of options available to others in addition to the set available to the individual making the judgement about his freedom of choice, *game forms*, as they were introduced by Gardenförs (1981) in the context of *right systems*, seem to represent a natural framework.

Game forms are defined as a set of individuals, a set of strategies and an outcome function associating strategies with outcomes, without preferences. They have been the object of attention in social choice theory, especially since Sen's seminal work (Sen (1970)) following which a large literature has been devoted to the problem of defining rights in a social context. After three decades of intensive debate, a disputed (see e.g. Sen (1983)) majority seems to have reached a consensus that rights should be modelled using game forms rather than the traditional social choice framework (see e.g. Deb (1994,2004), Deb, Pattanaik and Razzolini (1997), Gaertner, Pattanaik and Suzumura (1992), Gärdenfors (1981), Gibbard (1974), Peleg (1984, 1998)). Many issues concerning the interactions between individuals have therefore been tackled using the game form framework. However, to the best of my knowledge, this literature has not produced yet methods for comparing alternative game forms on the basis of the freedom of choice enjoyed by individuals.

The framework considered herein assumes that every individual knows what options are available to the others but ignores the others' preferences, although the individual who's freedom is evaluated is himself equipped with a given preference relation over the set of outcomes of the game forms. Hence preferences are made private knowledge and therefore, no strategic behaviour will be implemented in what follows. Considering game forms instead of games seems natural in the context of freedom of choice, because it is more realistic than assuming complete information about others' preferences. Indeed, although individuals might have some ideas about them, it seems too demanding to assume they know precisely and exhaustively what every neighbours' preferences are.

Along with this empirical argument, considering preferences as private knowledge can be justified by the fact that a change in one's neighbours' preferences should not have any consequence on the evaluation of one's freedom of choice. If both the sets of options available to an individual and to his neighbours remain exactly the same once the latter's preferences have changed, why the appraisal of the freedom of choice enjoyed by the former should change accordingly does not seem obvious. Arguably, it could be the case that when knowing the others' preferences, the outcome of the game (the game form hence would become a game) is different before and after such a change. However, this does not imply that the appraisal of one individual's freedom of choice should be affected, even though the action he takes might change.

This stresses a distinction between what could be called *ex ante* freedom and *ex post* freedom. Ex ante freedom is what is under consideration in this paper, it refers to judgements made by the individual about the set of possible outcomes of the game form, without any consideration of the other individuals' preferences. In turn, ex post freedom refers to the set of possible outcomes of the *games*, once preferences are known. Ex post freedom relies, for instance, on the set of possible equilibria of the game, given a particular compelling notion of equilibrium.

As an illustration, consider the case of freedom of speech in a society composed by two individuals. If each individual has the choice between using freedom of speech in a "positive" or a "negative" way, the set of all outcomes is formed of four elements. The ex-ante freedom of every individual should then be evaluated on the basis of these four outcomes. Assume next that the equilibrium notion and the preferences are such that there are two equilibria in the game, given by the outcomes corresponding first to the situation in which both individuals use freedom of speech in order to have constructive dialogues and second, to the situation in which both insult each other. Ex post freedom could then be evaluated on the basis of these two equilibria. Again, it is the ex ante type of freedom that is under consideration in this paper.

It is nevertheless assumed that the individual making the judgement is himself equipped with a given preference relation over the set of possible outcomes of the game form. It can be argued that when assessing one's freedom, it is better not to take into account the others' preferences, in order to judge the freedom offered by the "structure" of the society, the structure being here represented by the set of actions the neighbours can take. However, it is also of importance to take into account the evaluation this individual makes of the outcomes, as Sen says in (1991, p.22): *"The evaluation of the freedom I enjoy from a certain menu must depend to a crucial extent on how I value the elements included in that menu. Any plausible axiomatic structure in the comparison of the extent of freedom would have to take some note of the person's preferences."* In that respect, the choice is made to account for the preferences of the individual making the judgement.

In this paper two criteria are suggested and characterised to compare game forms, each of which embodies the idea of positive and negative interactions respectively. The first one, called the Max criterion, states that two game forms should be ranked on the basis of the best outcome the individual can reach in each game. Hence, when comparing two game forms, this criterion picks the best outcome for the individual in the set of all possible outcomes of each game form and compares them according to that individual's preferences. This is thus a crude expression of the idea according to which interactions can be positive.

The second criterion discussed in this paper, the $MaxMin$, compares game forms on the basis of their $MaxMin$ outcome. $MaxMin$ outcomes correspond to the outcomes obtained by individuals playing their MaxMin strategy, which have been the subject of a large attention in early game theory (see e.g. Luce and Raiffa (1957), Von Neumann (1928) and Von Neumann and Morgenstern (1953) for an introduction to and an analysis of MaxMin strategies). The $MaxMin$ criterion is almost the reverse of the Max as in this case the individual compares between the worst outcomes he is guaranteed in both game forms. It is not exactly, however, the opposite of the Max, because it is not the last outcome in the individual's ranking that will be selected, but the best outcome in the set of all worst outcomes of the game form. This criterion is, in turn, the crude expression of the fact that interactions can be negative.

The rest of the paper is organised as follows. In the next section the notations and definitions are introduced. Section 3 is devoted to the introduction of a set of axioms, which, combined together, provide characterisations of both Max and $MaxMin$ as proved in section 4. Section 5 concludes.

2 Notations and Definitions

Let $N = \{1, ..., n\}$ be the finite set of all individuals forming the society and let $\mathcal{A}_i$ be a fixed finite set of all conceivable options individual i can face. As the objects considered are game forms, options could be understood as strategies, thus from now on the terms "options" or "strategies" will be used indifferently. $\mathcal{A}^n$ denotes the cartesian product of every individual's conceivable set $\mathcal{A}_i$, $\mathcal{A}^n = \Pi_{i \in N} \mathcal{A}_i$. A generic element of $\mathcal{A}^n$ will be for instance $(a_1, ..., a_i, ..., a_n)$ where a_1 denotes individual 1's option, ..., a_n denotes individual n's option. The set of all non empty subsets of $\mathcal{A}_i$ will be denoted by $\mathcal{S}_i$ with generic elements A_i, B_i.... These subsets contain all the actions available to individual i.[1]

Game forms are generally defined in such a way that outcomes are given by an outcome function $g : \mathcal{A}^n \longrightarrow X$ where X is a set of all social outcomes. A game form is thus given by a pair $(A_1 \times ... \times A_n, g)$. Let $\mathcal{G}$ be the set of all game forms, and $\mathcal{G}_{inj} \subset \mathcal{G}$ be the set of game forms such that the function g is *injective.* Throughout this paper, only $\mathcal{G}_{inj}$ will be considered, but this domain restriction is made for notational convenience only. Hence, and so long as there is no ambiguity, the outcome function g is dropped so that outcomes are given by the n-tuples in $\mathcal{A}^n$. The set of outcomes of the game form will be denoted as $A_1 \times ... \times A_n \in \mathcal{S}^n$ and the question here is about how to compare two game forms $A_1 \times .. \times A_n$ and $B_1 \times .. \times B_m$ in terms of the freedom they offer to an individual i.

Note that what is looked for is a relation allowing to compare game forms involving not necessarily the same number of individuals. Indeed, there is no particular reason why we should only compare game forms played by the same number of individuals, as the question is "in which game form (society) individual i enjoys more freedom of choice ?" This question should be answered regardless of the number of individuals i lives with. Furthermore, as the game forms will be compared from individual i's point of view, the possibility for the other players to be different from one game form to the next is not excluded. For instance, comparisons between game forms played by individuals i, j and k on one side and game forms played by i, k, l and m on the other side should be allowed.

In order to perform these comparisons, the binary relation $\succeq_i$ over game forms is defined, with $\succ_i$ and $\sim_i$ its asymmetric and symmetric parts, as the relation that ranks two game forms from i's point of view. Thus $A_1 \times ... \times A_n \succeq_i B_1 \times ... \times B_m$

[1] Recall that in the framework considered here, an action does not determine uniquely the social state that will result. Hence it would be misleading to interpret A_i as an opportunity set.

is to be interpreted as "individual i enjoys at least as much freedom of choice in the game form $A_1 \times ... \times A_n$ than he does in the game form $B_1 \times ... \times B_m$". Formally, $\succeq_i$ is defined on the set $\cup_{n \in N} \mathcal{S}^n$ and this relation is required to be reflexive and transitive.

Finally, it is assumed throughout that individual i has a fixed linear preference ordering given by the reflexive, transitive and complete binary relation R_i over the set of all n-tuples in $\mathcal{A}^n$, for any n. Formally, R_i is defined on $\cup_{n \in N} \mathcal{A}^n$. For two outcomes, $(a_1, ..., a_i, ..., a_n) \in \mathcal{A}^n$, $(b_1, ..., b_i, ..., b_m) \in \mathcal{A}^m$, $(a_1, ..., a_i, ..., a_n)$ R_i $(b_1, ..., b_i, ..., b_m)$ will be interpreted as "the situation in which individual 1 has chosen option a_1, ..., individual i has chosen option a_i, ..., and individual n has chosen option a_n is preferred, by individual i, to the situation in which individual 1 has chosen option b_1, ..., individual i has chosen option b_i, ..., and individual m has chosen option b_m". Here again, the preference relation operates on outcomes formed by potentially different number of agents.

R_i being linear, we introduce the relation P_i as:
$\forall (a_1, ..a_i, ..a_j, ..a_n) \in \mathcal{A}^n$, $a_j^{'} \in \mathcal{A}_j$, $(a_1, ..a_i, ..a_j, ..a_n)$ P_i $(a_1, ..a_i, ..a_j^{'}, ..a_n)$ or $(a_1, ..a_i.., a_j^{'}, ..a_n)$ P_i $(a_1, ..a_i, ..a_j, ..a_n) \iff a_j \neq a_j^{'}$. In other words, P_i is the usual antisymmetric part of R_i, while indifference only holds if the options are identical. The consequence of linearity is that other individuals always have an influence on individual i, as any slight change in their choice causes a strict change in the evaluation of the outcome. However, the results presented in section 4 would be unaffected with non linear preferences, by assuming minor changes in the axioms but implying major changes in the notations, making the presentation less clear.

These preference relations are assumed to be given to individuals but as mentionned in the introduction, it is assumed that R_i is private knowledge.

3 Axioms

The following axioms capture different (and possibly contradictory) intuitions about the meaning of freedom of choice in a social context.

Axiom 3.1 *Extension Rule from Outcomes to Game Forms (EROG)*

$$\forall n, m \in N,\ \forall a_1, ...a_i, ...a_n \in \mathcal{A}^n, b_1, ..., b_i, ...b_m \in \mathcal{A}^m,$$

$$(a_1, ..a_i.., a_n) R_i (b_1, ..b_i.., b_m) \iff \{a_1\} \times ..\{a_i\} \times ..\ \{a_n\} \succeq_i \{b_1\} \times ..\{b_i\} \times ..\ \{b_m\}$$

This axiom seems very natural as it just requires the relative ranking of any two singleton sets according to $\succeq_i$ to be the same as the relative ranking of the corresponding outcomes according to R_i. This axiom is a natural generalisation of the very common "Extension Rule" axiom, widely used in the literature about extensions of an ordering over a set to the power set (see Barberà, Bossert and Pattanaik (2005) for a discussion and further references concerning this Extension Rule axiom). This seems to be the weakest way in which the preference relation over outcomes and the relation over game forms can be related. It will be the only link assumed herein between R_i and $\succeq_i$.

Axiom 3.2 *Weak Positive Interactions with respect to Set Inclusion (WPI)*

$$\forall n \in N,\ \forall A_1, ...A_i, ...A_n \in \mathcal{S}^n, \forall B_j \in \mathcal{S}_j,$$

$$B_j \subseteq A_j \Longrightarrow A_1 \times ..A_i \times ..A_j \times ..A_n \succeq_i A_1 \times ..A_i \times ..B_j \times ..A_n$$

This axiom says that if individual i's set doesn't change when j's set of strategies is increased to a superset, then i's freedom is weakly increased. This view would be consistent with the idea that "the freedom of one begins where the freedom of others begin" expressed by Bakunin, and illustrated in the introduction with the example about the freedom to exercise modern medicine. Giving to doctors the freedom to resort to a medicine different from the traditional one will surely increase the doctors' freedom, but that enhancement induces a positive externality on individual i's own freedom, as he might live longer and healthier. His evaluation of the freedom of choice when others have that option will be different from the situation in which they do not have it, eventhough his set of strategies has not been changed.

Although the concept of benefiting from neighbours' freedom that is vehiculated by WPI can be defended, yet it is a strong axiom in that it requires interactions to be always positive, regardless of the description of the options that are added to individual j's set. It is true, though, that a more intuitive conception of freedom with interactions should allow for more mitigated judgements. The exercise of other's freedom should be beneficial in some instances, while judged harmful in others, just as suggested with the example concerning freedom of speech. WPI does not allow for such considerations. However, WPI is in the weak form so that an addition to j's set will not necessarily strictly improve i's freedom, rather it will never strictly worsen it.

Axiom 3.3 *Weak Negative Interactions with respect to Set Inclusion (WNI)*

$$\forall n \in N,\ \forall A_1, ...A_i, ...A_n \in \mathcal{S}^n, \forall B_j \in \mathcal{S}_j,$$

$$B_j \subseteq A_j \Longrightarrow A_1 \times ..A_i \times ..B_j \times ..A_n \succeq_i A_1 \times ..A_i \times ..A_j \times ..A_n$$

This axiom is the counterpart of axiom WPI. In this case, individual i dislikes when his neighbours enjoy too many options. It could be interpreted in two ways. First, options for i's neighbours could generate negative interactions for him, being consistent with the maxim according to which "the freedom of one begins where the freedom of others end", as illustrated with the example concerning hunters and strollers. One can easily find other examples illustrating how an individual can be penalised by an increase in his neighbours' set of options. Second, we can interpret this axiom by saying that individual i dislikes uncertainty, independently from the preferences he has over the outcomes. Increasing j's freedom increases at the same time i's uncertainty about the final outcome that will be realized, what he might judge as a degraded situation, hence preferring when his neighbours have the smaller possible set of options.

In both these interpretations individual i prefers reducing others' set of strategies, conversely to what axiom WPI says. Although WNI seems more intuitive than its counterpart, it shares with WPI the same flaw that it requires interactions to be systematically negative, regardless of the description of the options that are added to individual j's set. Here neither, intermediate judgements are not allowed.

Remark 3.1 *Both axioms WPI and WNI, being contradictory, cannot coexist except in the case where all game forms, with i's set of strategies remaining unchanged, are considered indifferent by him, thereby stating that interactions do not affect individuals' freedom. In this case, the set A_i itself is the only relevant information for the judgment of individual i's freedom of choice, just as implicitely assumed in the traditional literature about opportunity sets.*

Axiom 3.4 *Indifference to Addition of Singletons for i (IAS)*

$$\forall n \in N,\ \forall A_1, ..A_i, ...A_n \in \mathcal{S}^n, \forall x_i \in \mathcal{A}_i \setminus A_i$$

$$A_1 \times ..A_i \times ..A_n \succ_i A_1 \times ..\{x_i\} \times ..A_n \Longrightarrow A_1 \times ..A_i \cup \{x_i\} \times ..A_n \sim_i A_1 \times ..A_i \times ..A_n$$

This axiom says that when others' sets remain unchanged, if individual i strictly prefers having the set A_i to the singleton $\{x_i\}$ then adding that singleton to the initial set should leave individual i indifferent. This axiom could be divided into two parts to make interpretation clearer. First it says that the initial situation cannot be strictly better than the one where an option was added. This requirement seems very natural because giving more options to an individual cannot strictly reduce her freedom of choice. As discussed earlier with axiom WNI, it could eventually reduce others' freedom, but definitely not i's. On the other hand, the axiom says that if the initial set is judged strictly better than the singleton, then it may be the case that when it comes to choice, individual i would not choose x_i from the set $A_i \cup \{x_i\}$. Therefore, adding this particular singleton to the initial set cannot strictly increase individual i's freedom. If freedom is neither strictly reduced nor strictly increased the situation will be judged indifferent.

Although adapted to the social context framework, this axiom carries the same spirit as Kreps' (1979) characterising condition of indirect utility[2]: $A \succeq B \Longrightarrow A \sim A \cup B$.

Axiom 3.5 *Addition of Bad Singletons to j's Strategy Set (ABS)*

$$\forall n \in N,\ \forall A_1, ..., A_n \in \mathcal{S}^n, \forall x_j \in \mathcal{A}_j,$$

$$A_1 \times ..A_i \times ..A_j \times ..A_n \succ_i A_1 \times ..A_i \times ..\{x_j\} \times ..A_n$$

$$\Longrightarrow$$

$$A_1 \times ..A_i \times ..A_j \times ..A_n \succeq_i A_1 \times ..A_i \times ..A_j \cup \{x_j\} \times ..A_n$$

[2]This was pointed out to me by an anonymous referee.

This axiom is weaker than axiom WNI, in the sense that if WNI is satisfied then ABS is. The reverse is not true. When the game form $A_1 \times ..A_i \times ..A_n$ is evaluated as being strictly better, from i's point of view, than the game form in which all sets remain unchanged except for one player j who only has the singleton $\{x_j\}$ for a set, then ABS states that the addition of this singleton to the initial set of j must leave the first game form at least as good for i as the enlarged one. Said differently, this axiom implies that if one particular singleton for another player j is not weakly better in terms of freedom for individual i than the entire initial set, the addition of this singleton should not strictly reverse the existing ranking. ABS is actually very close to IAS but concerns player j.

Axiom 3.6 *Addition of Good Singletons to j's Strategy Set (AGS)*

$$\forall n \in N,\ \forall A_1, ..., A_n \in \mathcal{S}^n, \forall x_j \in \mathcal{A}_j,$$

$$A_1 \times ..A_i \times ..\{x_j\} \times A_n \succ_i A_1 \times ..A_i \times ..A_j \times ..A_n$$

$$\Longrightarrow$$

$$A_1 \times ..A_i \times ..A_j \cup \{x_j\} \times ..A_n \succeq_i A_1 \times ..A_i \times ..A_j \times ..A_n$$

This axiom is a counterpart of axiom ABS. It says that if a game form in which individual i faces a set A_i and one other individual j faces a singleton is strictly better, from the point of view of individual i, than the game form $A_1 \times ..A_i \times ..A_j \times ..A_n$ where every players' set remain unchanged except for that of the individual having the singleton, then adding this particular singleton to the set of that individual should not strictly reduce i's freedom. Said differently, this axiom implies that if a particular singleton for one player is better in terms of freedom for individual i than his initial set, ceteris paribus, then the addition of this singleton should not be considered as a decrease in terms of freedom for individual i.

4 Characterisation results

In this section the different axioms are combined in order to characterise two criteria, the Max and the $MaxMin$ respectively. Before stating the results, the two criteria are defined.

Definition 4.1 *Let* $Max_{R_i}(A_1 \times ..A_i \times ..A_n) = (a_1^*, ..a_i^*.., a_n^*)$ *where*

$$(a_1^*, ..a_i^*.., a_n^*)\, R_i (a_1, ..a_i.., a_n) \quad \forall (a_1, ..a_i.., a_n) \in A_1 \times ...A_i \times ...A_n$$

$(a_1^*, ..a_i^*.., a_n^*)$ is the best outcome for individual i in the game form according to his preference relation.

Definition 4.2 *Let the* $\succeq_i^{\max}$ *relation be defined as*

$$A_1 \times ...A_i \times ...A_n \succeq_i^{\max} B_1 \times ...B_i \times ...B_m \Longleftrightarrow (a_1^*, ..a_i^*.., a_n^*)\, R_i\, (b_1^*, ..b_i^*.., b_m^*)$$

where $(a_1^*, ..a_i^*.., a_n^*) = Max_{R_i}(A_1 \times ..A_i \times ..A_n)$
and $(b_1^*, ..b_i^*.., b_m^*) = Max_{R_i}(B_1 \times ..B_i \times ..B_m)$

The Max relation thus compares two game forms in terms of the best outcome for individual i, so that i prefers playing a game in which he could end up with a better outcome, although the fact that the Max element will occur is not guaranteed as this depends on other individuals' choices. This criterion is thus, as mentioned in the introduction, a rather crude expression of the view that interactions can be positive. Eventhough the Max is too naive as to measure properly the freedom of choice individuals enjoy, its characterisation is presented in what follows as an illustration of a method for comparing game forms. Furthermore, this axiomatic characterisation provides us with the underlying principles we might disagree with when measuring freedom of choice in such a way.

The next definitions present the $MaxMin$ criterion, which is probably a more appropriate way of comparing game forms in terms of freedom of choice.

Definition 4.3 *Let* $Min_{R_i}(A_1 \times .. \{a_i\} \times ..A_n) = (\underline{a}_1, ..a_i.., \underline{a}_n)$ *where*

$$(a_1, ..a_i.., a_n)R_i(\underline{a}_1, ..a_i.., \underline{a}_n)\forall(a_1, ..a_i.., a_n) \in A_1 \times ... \{a_i\} \times ...A_n$$

$(\underline{a}_1, ..a_i.., \underline{a}_n)$ is the worst outcome in the game form for player i when he plays a_i.

Definition 4.4 *Let* $(\hat{a}_1, ..\hat{a}_i.., \hat{a}_n) = Max_{R_i}\{Min_{R_i}(A_1 \times ..\{a_i\} \times ..A_n), \forall a_i \in A_i\}$

$(\hat{a}_1, ..\hat{a}_i.., \hat{a}_n)$ is the R_i-maximal outcome between all the R_i-minimal outcomes in the game form. The $MaxMin$ element of the game form will be denoted as $MaxMin_{R_i}(A_1 \times ..A_i \times ..A_n)$.

Definition 4.5 *Let the* $\succeq_i^{\max\min}$ *relation be defined as*

$$A_1 \times ...A_i \times ...A_n \succeq_i^{\max\min} B_1 \times ...B_i \times ...B_m \iff (\hat{a}_1, ..\hat{a}_i.., \hat{a}_n)R_i(\hat{b}_1, ..\hat{b}_i.., \hat{b}_m)$$

where $(\hat{a}_1, ..\hat{a}_i.., \hat{a}_n) = MaxMin_{R_i}(A_1 \times ..A_i \times ..A_n)$,
and $(\hat{b}_1, ..\hat{b}_i.., \hat{b}_m) = MaxMin_{R_i}(B_1 \times ..B_i \times ..B_m)$

This relation thus compares two game forms in terms of individual i's $MaxMin$ outcomes (see e.g. Luce and Raiffa (1957), Von Neumann (1928) and Von Neumann and Morgenstern (1953) on MaxMin strategies). The reason why the $MaxMin$ element is more interesting than the worst element of the game form is that if individual i wishes, he can avoid ending up with the worst outcome just by not playing his strategy associated to that minimal outcome. The best she is guaranteed to obtain is thus the $MaxMin$ outcome.

While the Max criterion carries an idea of optimism, as individuals judge game forms according to the best possible outcome, the $MaxMin$ is on the contrary rather pessimistic in its judgement about interactions, considering them as always harmful. As we will see in the characterisation results, this opposition is the result of the two antithetic axioms WPI and WNI, used respectively for Max and $MaxMin$.

As to understand better how both criteria apply and differ, let us suggest the following illustration example: society is formed of individuals 1 and 2, both of them having no other choice for eating than going to the restaurant. In the first case, the only restaurant R_1 is exclusively non smoker, while in the second case, the only restaurant R_2 has a smoking area. In the first game form $A_1 \times A_2$ (called GF_1), the individuals have only one option **a**: eating in the non smoking restaurant R_1. In the second game form $B_1 \times B_2$ (called GF_2), both individuals have three strategies available: **ns**: eating in the non smoking zone of R_2, **sl**: eating and smoking reasonably in the smoking zone of R_2 and **sh**: eating and smoking a lot in the smoking zone of R_2

We represent the game forms as follows:

GF_1 :

	a
a	x

GF_2 :

	ns	**sl**	**sh**
ns	x	y	z
sl	w_1	w_2	w_3
sh	w_4	w_5	w_6

Smoking zones in restaurants are enjoyable for smokers, and do not bother non smokers so long as the quantity of smoke is not so high that the whole restaurant is filled with it. Individual 1, being an altruistic non smoker, appreciates the fact that smokers can smoke if they wish so long as it stays reasonable, but highly dislikes when smokers smoke a lot, as he can no longer enjoy his meal. His preferences are thus given by $y \; P_1 \; x \; P_1 \; z \; P_1 \; w_i \; \forall i$. Hence, we get

$Max_{R_1}(A_1 \times A_2) = x$ and $MaxMin_{R_1}(A_1 \times A_2) = x$
$Max_{R_1}(B_1 \times B_2) = y$ and $MaxMin_{R_1}(B_1 \times B_2) = z$

We then get $GF_2 \succ_1^{Max} GF_1$ while $GF_1 \succ_1^{MaxMin} GF_2$.

According to the Max criterion, individual 1 would rather live in a world in which restaurants have a smoking area, despite the fact that he could be thereby penalised, whereas according to the $MaxMin$ criterion, it is the first situation that is better as individual 1 has no risk of being bothered by smokers during his meal, eventhough he would enjoy it if his neighbour could smoke reasonably. We clearly see here why individual 1 is optimistic about others' behaviour when he uses the Max, while he is pessimistic when using the $MaxMin$.

We now turn to the main characterisation results of this section:

Theorem 4.1 *A transitive binary relation $\succeq_i$ over game forms satisfies axioms EROG, IAS, WPI and ABS if and only if* $\succeq_i = \succeq_i^{\max}$

Proof : See Appendix

Proposition 4.2 *Axioms EROG, IAS, WPI and ABS are independent.*

Proof : See Appendix

Theorem 4.3 *A transitive binary relation $\succeq_i$ over game forms satisfies axioms EROG, IAS, WNI and AGS if and only if* $\succeq_i = \succeq_i^{\max\min}$

Proof : See Appendix

Proposition 4.4 *Axioms EROG, IAS, WNI and AGS are independent.*

Proof : See Appendix

As mentioned earlier, axiom EROG is the only one in which the preference relation R_i appears in the characterisation of both $\succeq_i^{\max}$ and $\succeq_i^{\max\min}$. This is appealing because it appears in a very natural and weak form so that no unreasonable assumptions are made about preferences.

Remark 4.6 *Both* $\succeq_i^{\max}$ *and* $\succeq_i^{\max\min}$ *are complete criteria. This was not a specific requirement, it follows for the combination of the axioms used in the characterisation.*

Remark 4.7 *The axioms presented here have a general form in the sense that they apply for any value of n, the number of players. In particular they apply in a trivial way when $n = 1$. In that case, only axioms EROG and IAS remain. Combined, these two axioms characterise the usual Max criterion in the non interactive case:* $A \succeq B \iff Max_R(A)\ R\ Max_R(B)$. *Concerning the $MaxMin$, it is easy to see that it coincides with the $MaxMin$ in the case of only one individual. Interestingly the case with interaction can thus be considered as a generalisation of the case without interactions.*

Remark 4.8 *Both criteria have the particularity of being based on the comparison of only one outcome of the respective game forms. Moreover it is an outcome associated to a "best" strategy of individual i, the best one associated to the best possible choices of other individuals in the case of the Max, the best one associated to the worst possible choices of the other individuals in the case of $MaxMin$. This suggests that individual i will consider his own set A_i as equivalent to one of its "best" elements, applying thereby a sort of indirect utility criterion to his own set. This is the result of using axiom IAS, a close version of Kreps' (1979) characterisation condition of indirect utility.*

Remark 4.9 *As mentionned in section 3, WPI is a rather demanding axiom as it requires that no interaction should be judged strictly negative. However, it is combined in the characterisation of Max with axiom ABS which goes in the opposite direction. The combination of both axioms can be interpreted as a way of softening the extreme ethic carried by WPI. The same remark can be made for the characterisation of $MaxMin$ regarding axiom WNI which is combined with axiom AGS as a trade off with the radicality of WNI.*

Although no strategical concern is present here, one could argue that a criterion used for comparing game forms should be consistent with minimal game theoretic requirements since the individual whose freedom is evaluated is assumed to have well-defined preferences over outcomes. Two basic features in game theory are that dominated strategies, if any, should not be chosen by any player and that dominant strategies, if any, should always be chosen by players. These concepts of dominated and dominant strategies being probably the more consensual in game theory, it may seem desirable that the criteria respect the two following conditions: little importance should be given to dominated strategies and conversely, dominant strategies should be prevalent.

It is worth noticing that both $\succeq_i^{\max}$ and $\succeq_i^{\max\min}$ fulfil these requirements. Adding (resp. removing) a dominated strategy to (resp. from) an individual's set of options will neither strictly reduce nor increase the overall freedom of that individual. These strategies do not have any influence on the determination of the Max nor of the $MaxMin$. In the same way, if there is a dominant strategy in the set of options of an individual, then it is that one precisely that determines the Max and the $MaxMin$. Thus both criteria guarantee that a dominant strategy is preponderant in the evaluation of a game form.

Before concluding, it is of interest to make the following comment about axioms WPI and WNI. These axioms play an important role in the characterisation of $\succeq_i^{\max}$ and $\succeq_i^{\max\min}$ respectively. WPI and WNI say that interactions will be judged exclusively positively or exclusively negatively, so they cannot be used together (except in one particular instance, see remark 3.1). There is no possibility for interactions to be judged sometimes positively, sometimes negatively and this is why the criteria presented here give discriminating power to one particular outcome of the game form, the Max outcome in the first case, the $MaxMin$ in the second case.

Nevertheless, one can easily find examples for which interactions can be judged in both ways. It is the case for instance with freedom of speech. Giving the freedom of speech to others can increase one's freedom if that freedom is used as to share valuable ideas. It can also decrease it if it is used as a way of insulting people or as a way of incitating people to violence. This, amongst various examples one can imagine, illustrates the point that increasing individuals' sets of options is neither systematically positive nor negative for the individual under consideration. It seems obvious that further research should focus on criteria offering more trade off than the Max or $MaxMin$ do.

5 Conclusion

This paper was devoted to the question of interactions in the freedom of choice literature. Since the first contributions formalising the concept of freedom of choice, a large literature has been produced by economists, accounting for various issues, such as the quantity, the quality or more recently the diversity of choices (see Bervoets and Gravel (2006)). Issues concerning the way in which preferences should or should not intervene in the evaluation of freedom of choice have also been explored. In spite of the richness of these works, no particular attention has been devoted to the problem of interactions, as an important feature of freedom of choice. This is all the more surprising, especially when considering how intuitive it seems that freedom is a highly interactive notion.

This problem has been introduced in this paper, first by arguing that opportunity sets might be replaced by game forms in order to proceed to the construction of a criteria measuring freedom of choice, then by suggesting some ideas for such comparisons. Two criteria have been axiomatically characterised, the Max which compares two game forms on the basis of the best elements in the respective game forms, expressing the positive character of interactions, and the $MaxMin$ comparing game forms on the basis of the best element one is guaranteed by playing his MaxMin strategy, expressing the reverse idea.

While this investigation has been proved successful, the two criteria presented are far from perfect. Indeed, the first criterion seems difficult to defend as an appropriate measure of freedom of choice. The second is more acceptable, providing a measure of freedom as the minimal level one can reach. However they both share two basic weaknesses that are very much related one with the other. The first is that a discriminating power is given to one particular outcome of the game form, preventing any contribution of the other outcomes to the evaluation of the game form. The second is that for both criteria, interactions are either exclusively positive or exclusively negative. This is rather unsatisfactory as it reflects a categorical opinion about interactions between individuals.

These two limitations seem to be due to the use of WPI and WNI. Hence it would be nice to relax these axioms in order to obtain more flexible criteria. An interesting line of research would consist of "smoother" rankings of game forms, such as additive criteria, which would view the freedom offered by a game form as the sum of values that would be assigned to each outcome by a function that represents, in the sense of Debreu (1954), the preference relation of the individual whose freedom is evaluated. Such criteria would then allow for interactions to be judged both positive and negative at the same time, depending on the strategy they are associated with. Furthermore every outcome of the game form would be taken into account.

Beyond the research of more flexible criteria than those characterised herein, there is more work to do on the approach itself to the problem. In particular, it has been assumed throughout that individuals are equipped with a given preference relation. Actually, the reasons why a unique preference relation should be considered are in debate within the traditional freedom of choice literature, and multi-preferences approaches have been suggested as to justify the introduction of any evaluation of the outcomes[3] in the freedom of choice measurement (see e.g. Foster (1993), Kreps (1979), Pattanaik and Xu (1998), Sugden (1998)). Considering this, it becomes an interesting issue to try to adapt the framework presented here in order to account

[3] This discussion is in reality about the evaluation of *options*, defined as perfect descriptions of the state of the world.

for multiple preferences.

While mentioning the possible problems raised by the introduction of preferences in this approach, it seems that another avenue worth exploring for future research would be to introduce the preferences of the other players explicitely in the analysis and, therefore, allow for strategic interactions. It could certainly be interesting to consider some game theoretic notion in the analysis and see whether they can provide insights in understanding the interactive aspects of the notion of individual freedom of choice.

As a last comment, notice that some important issues raised by the presence of interactions are not considered within the approach provided here. Assume for instance that an individual is invalid to the point he cannot walk. He therefore is deprived from the freedom of movement. Assume now that another individual has the freedom to go to the medicine school and develop freely her skills as a surgeon. A consequence of that could be, depending on the choice of action of the individuals, that the first individual is operated by the second and recovers his capacity to walk. In that case it is not only the first individual's set of outcomes that has been enlarged, it is the set of options itself. An option given to one individual can thus lead to a positive change in the set of options of other individuals. This is not embraced in our framework.

On the other hand, assume the first individual is valid and drives his car everyday. If another individual also has the freedom of movement, thereby the freedom of driving a car, then an accident could occur and the first individual would be deprived of his freedom of movement if injured. That would be the result of the second individual having the freedom of movement (and incidently, of driving dangerously). Thus in this case, the first individual's set of options is directly reduced by the action of another individual. Here again, it is not only the set of outcomes that has been reduced, but directly the set of options available to the individual.

In both cases, one specific option given to the second individual can potentially change the first individual's set of options, in an increasing or a decreasing way. This notion is not embodied in the approach presented here, because options added to other individuals affect the set of outcomes but not the set of options of the concerned individual. It seems that cases such as those just presented can better be modelled by considering game forms in their extensive form, allowing then for a dynamic of the sets of options. This set could be defined at each node of the extensive game form. It might be the case that normal game forms such as implicitly assumed here are not sufficient as to describe all kind of interactions (see Deb (2004) in particular for a discussion on the equivalence between normal and extensive game forms).

Appendix

Theorem 4.1 A transitive binary relation $\succeq_i$ over game forms satisfies axioms EROG, IAS, WPI and ABS if and only if $\succeq_i = \succeq_i^{\max}$

Proof :
$\succeq_i^{\max}$ satisfies axioms EROG, IAS, WPI and ABS and is transitive, so necessity is proved. Let's now turn to suficiency.

Assume that $\succeq_i$ satisfies axioms EROG, IAS, WPI and ABS and is transitive. Assume furthermore that $A_1 \times ..A_i \times ..A_n \succeq_i^{\max} B_1 \times ..B_i \times ..B_m$. This, by definition, means that $(a_1^*, ..a_i^*.., a_n^*)\, R_i\, (b_1^*, ..b_i^*.., b_m^*)$ where $(a_1^*, ..a_i^*.., a_n^*) = Max_{R_i}(A_1 \times ..A_i \times ..A_n)$ and $(b_1^*, ..b_i^*.., b_m^*) = Max_{R_i}(B_1 \times ..B_i \times ..B_m)$. Using axiom EROG, one gets $(a_1^*, ..a_i^*.., a_n^*)\, R_i\, (b_1^*, ..b_i^*.., b_m^*) \iff \{a_1^*\} \times ..\{a_i^*\} \times ..\{a_n^*\} \succeq_i \{b_1^*\} \times ..\{b_i^*\} \times ..\{b_m^*\}$. Hence, if it is shown that for any game form, $A_1 \times ..A_i \times ..A_n \sim_i \{a_1^*\} \times ..\{a_i^*\} \times ..\{a_n^*\}$, the theorem will be proved. This equivalence between a game form $A_1 \times ..A_i \times ..A_n$ and the game form in which every individuals' set of options is the singleton yielding the best possible outcome, $\{a_1^*\} \times ..\{a_i^*\} \times ..\{a_n^*\}$, is shown in what follows in three steps, each step being led by an iterative method.

First step : In this first step we show that if $Max_{R_i}(\{a_1\} \times ..\{a_i\} \times ..\{a_{n-1}\} \times A_n) = (a_1, ..a_i.., a_{n-1}, a_n)$ then $\{a_1\} \times .. \{a_{n-1}\} \times \{a_n\} \sim_i \{a_1\} \times .. \{a_{n-1}\} \times A_n$.
$\forall a_n^{'} \in A_n \setminus \{a_n\}$, $(a_1, ..a_{n-1}, a_n)\ P_i\ (a_1, ..a_{n-1}, a_n^{'})$ so by EROG, $\{a_1\} \times .. \{a_{n-1}\} \times \{a_n\} \succ_i \{a_1\} \times .. \{a_{n-1}\} \times \left\{a_n^{'}\right\}$. Using ABS we thus have $\{a_1\} \times .. \{a_{n-1}\} \times \{a_n\} \succeq_i \{a_1\} \times .. \{a_{n-1}\} \times \left\{a_n, a_n^{'}\right\}$ and by using WPI we obtain $\{a_1\} \times .. \{a_{n-1}\} \times \{a_n\} \sim_i \{a_1\} \times .. \{a_{n-1}\} \times \left\{a_n, a_n^{'}\right\}$. Of course this is also true for $a_n^{''} \in A_n$ so $\{a_1\} \times .. \{a_{n-1}\} \times \{a_n\} \succ_i \{a_1\} \times .. \{a_{n-1}\} \times \left\{a_n^{''}\right\}$, hence by transitivity, $\{a_1\} \times .. \{a_{n-1}\} \times \left\{a_n, a_n^{'}\right\} \succ_i \{a_1\} \times .. \{a_{n-1}\} \times \left\{a_n^{''}\right\}$ and by using ABS and WPI, we obtain that $\{a_1\} \times .. \{a_{n-1}\} \times \{a_n\} \sim_i \{a_1\} \times .. \{a_{n-1}\} \times \left\{a_n, a_n^{'}, a_n^{''}\right\}$. By repeating this procedure as many times as required we can add on every element in A_n until obtaining $\{a_1\} \times .. \{a_{n-1}\} \times \{a_n\} \sim_i \{a_1\} \times .. \{a_{n-1}\} \times A_n$.

Second step : We will now add on all options of individual $n-1, n-1 \neq i$.
What we have just proved holds also for the Max element of the game form, that is $\{a_1^*\} \times ..\{a_{n-2}^*\} \times \{a_{n-1}^*\} \times \{a_n^*\} \sim_i \{a_1^*\} \times ..\{a_{n-2}^*\} \times \{a_{n-1}^*\} \times A_n$. However, according to step 1, it is true that $\{a_1^*\} \times ..\{a_{n-2}^*\} \times \{a_{n-1}\} \times \{a_n\} \sim_i \{a_1^*\} \times ..\{a_{n-2}^*\} \times \{a_{n-1}\} \times A_n$ where $\left(a_1^*, ..a_{n-2}^*, a_{n-1}, a_n\right) = Max_{R_i}(\{a_1^*\} \times ..\{a_{n-2}^*\} \times \{a_{n-1}\} \times A_n)$. By transitivity, we have $\{a_1^*\} \times .. \left\{a_{n-2}^*\right\} \times \left\{a_{n-1}^*\right\} \times A_n \succ_i \{a_1^*\} \times .. \left\{a_{n-2}^*\right\} \times \{a_{n-1}\} \times A_n$ and using ABS, WPI and transitivity leads us to $\{a_1^*\} \times .. \left\{a_{n-2}^*\right\} \times \left\{a_{n-1}^*\right\} \times \{a_n^*\} \sim_i \{a_1^*\} \times .. \left\{a_{n-2}^*\right\} \times \left\{a_{n-1}^*, a_{n-1}\right\} \times A_n$. Applying the same procedure allows us to add on all elements of A_{n-1} until we reach $\{a_1^*\} \times .. \left\{a_{n-2}^*\right\} \times \left\{a_{n-1}^*\right\} \times \{a_n^*\} \sim_i \{a_1^*\} \times .. \left\{a_{n-2}^*\right\} \times A_{n-1} \times A_n$. Again, this reasoning can be applied to all other sets A_j, except to A_i. Doing so leads us to the following : $\{a_1^*\} \times .. \{a_i^*\} \times .. \{a_n^*\} \sim_i$

$A_1 \times ..A_{i-1} \times \{a_i^*\} \times A_{i+1} \times ..A_n$.

Third step : Finally, we will add on all options of individual i. Let $Max_{R_i}(A_1 \times ..A_{i-1} \times \{a_i\} \times A_{i+1} \times ..A_n) = (\overline{a}_1, ..a_i.., \overline{a}_n)$. According to steps 1 and 2, $\{\overline{a}_1\} \times .. \{a_i\} \times .. \{\overline{a}_n\} \sim_i A_1 \times ..A_{i-1} \times \{a_i\} \times A_{i+1} \times ..A_n$. But $(a_1^*, ..a_i^*.., a_n^*)$ is the Max element of the game, so $(a_1^*, ..a_i^*.., a_n^*)\ P_i\ (\overline{a}_1, ..a_i.., \overline{a}_n)$, so by EROG and transitivity, $A_1 \times ..A_{i-1} \times \{a_i^*\} \times A_{i+1} \times ..A_n \succ_i A_1 \times ..A_{i-1} \times \{a_i\} \times A_{i+1} \times ..A_n$. We can use IAS to obtain $A_1 \times ..A_{i-1} \times \{a_i^*, a_i\} \times A_{i+1} \times ..A_n \sim_i A_1 \times ..A_{i-1} \times \{a_i\} \times A_{i+1} \times ..A_n$. The reasoning we just made remains valid with any $a_i^{'}$ in A_i, so we can repeat it, adding on every option in A_i until obtaining: $A_1 \times ..A_i \times ..A_n \sim_i \{a_1^*\} \times .. \{a_i^*\} \times .. \{a_n^*\}$. ■

Proposition 4.2 Axioms EROG, IAS, WPI and ABS are independent.

Proof :
- Let $\succeq_i^{ind1}$ be defined as the Max relation except for singletons, which are all judged indifferent, independently from the preference relation R_i. Formally, if $card(A_j) = 1$ $\forall j$, and $card(B_k) = 1$ $\forall k$, then $A_1 \times ..A_i \times ..A_n \sim_i^{ind1} B_1 \times ..B_i \times ..B_m$. Otherwise, $A_1 \times ..A_i \times ..A_n \succeq_i^{ind1} B_1 \times ..B_i \times ..B_m \Leftrightarrow A_1 \times ..A_i \times ..A_n \succeq_i^{\max} B_1 \times ..B_i \times ..B_m$. Trivially, this relation fails to satisfy EROG. However, for sets other than singletons, $\succeq_i^{ind1}$ coincides with $\succeq_i^{\max}$ so the axioms IAS, WPI and ABS are satisfied.
- Let $\succeq_i^{cardmax}$ be defined as follows. $A_1 \times ..A_i \times ..A_n \succ_i^{cardmax} B_1 \times ..B_i \times ..B_m \Leftrightarrow card(A_i) > card(B_i)$ or $card(A_i) = card(B_i)$ and $A_1 \times ..A_i \times ..A_n \succ_i^{\max} B_1 \times ..B_i \times ..B_m$. Otherwise, $A_1 \times ..A_i \times ..A_n \sim_i^{cardmax} B_1 \times ..B_i \times ..B_m \Leftrightarrow card(A_i) = card(B_i)$ and $A_1 \times ..A_i \times ..A_n \sim_i^{\max} B_1 \times ..B_i \times ..B_m$. $\succeq_i^{cardmax}$ ranks game forms in a lexicographic way, focusing first on the size of i's set, second on the Max outcome. This relation satisfies EROG. However, IAS is always violated. Besides, WPI and ABS are always satisfied.
- Let $\succeq_i^{card}$ be defined as follows. If $card(A_j) = 1$ $\forall j$, and $card(B_k) = 1$ $\forall k$, then $\succeq_i^{card} = R_i$. Otherwise, $A_1 \times ..A_i \times ..A_n \succeq_i^{card} B_1 \times ..B_i \times ..B_m \Leftrightarrow Max\{card(A_j), j \neq i\} \geq Max\{card(B_k), k \neq i\}$. $\succeq_i^{card}$ ranks singletons according to the Extension Rule, and therefore satisfies EROG. It ranks all other sets according to the greatest number of options in the sets A_j, $j \neq i$. WPI is satisfied as well as IAS. To see why IAS is satisfied, it is useful to notice that the left part of IAS can only be true in cases of singletons. However, the right part of the axiom is always true so the sets $A_1 \times ..A_i \times ..A_n$ and $A_1 \times .. \{x_i\} \times ..A_n$ are always judged indifferent by $\succeq_i^{card}$, except in cases of singletons where EROG applies. Then, IAS is satisfied. However, ABS is violated by $\succeq_i^{card}$.
- Let $\succeq_i^{cardinv}$ be defined as follows. If $card(A_j) = 1$ $\forall j$, and $card(B_k) = 1$ $\forall k$, then $\succeq_i^{cardinv} = R_i$. Otherwise, $A_1 \times ..A_i \times ..A_n \succeq_i^{cardinv} B_1 \times ..B_i \times ..B_m \Leftrightarrow Max\{card(B_j), j \neq i\} \geq Max\{card(A_k), k \neq i\}$. This relation is the exact inverse of $\succeq_i^{card}$ relation, except for singletons that are ranked in the same way. Then EROG is satisfied, IAS and ABS are also, unlike WPI which is violated. ■

Theorem 4.3 A transitive binary relation $\succeq_i$ over game forms satisfies axioms EROG, IAS, WNI and AGS if and only if $\succeq_i = \succeq_i^{\max\min}$

Proof :
$\succeq_i^{\max\min}$ satisfies axioms EROG, IAS, WNI and AGS and is transitive, so necessity is proved. Let's now turn to sufficiency.

This proof is similar to that of theorem 3.1: assume that $\succeq_i$ satisfies axioms EROG, IAS, WNI and AGS and is transitive. Assume furthermore that $A_1 \times ..A_i \times ..A_n \succeq_i^{\max\min} B_1 \times ..B_i \times ..B_m$. This, by definition, means that $(\hat{a}_1, .., \hat{a}_i, .., \hat{a}_n)$ R_i $(\hat{b}_1, .., \hat{b}_i, .., \hat{b}_m)$ where $(\hat{a}_1, ..\hat{a}_i.., \hat{a}_n) = MaxMin_{R_i}(A_1 \times ..A_i \times ..A_n)$ and $(\hat{b}_1, ..\hat{b}_i.., \hat{b}_m) = MaxMin_{R_i}(B_1 \times ..B_i \times ..B_m)$. Using axiom EROG, one gets $(\hat{a}_1, .., \hat{a}_i, .., \hat{a}_n)$ R_i $(\hat{b}_1, .., \hat{b}_i, .., \hat{b}_m) \Longleftrightarrow \{\hat{a}_1\} \times ..\{\hat{a}_i\} \times ..\{\hat{a}_n\} \succeq_i \{\hat{b}_1\} \times ..\{\hat{b}_i\} \times ..\{\hat{b}_m\}$. Hence, if it is shown that for any game form, $A_1 \times ..A_i \times ..A_n \sim_i \{\hat{a}_1\} \times ..\{\hat{a}_i\} \times ..\{\hat{a}_n\}$, the theorem will be proved. This will be done in the three following steps, the proof of which will be made by an iterative method.

First step : we show that if $Min_{R_i}(\{a_1\} \times ..\{a_i\} \times ..\{a_{n-1}\} \times A_n) = (a_1, ..a_i.., a_{n-1}, a_n)$ then $\{a_1\} \times ..\{a_{n-1}\} \times \{a_n\} \sim_i \{a_1\} \times ..\{a_{n-1}\} \times A_n$.
$\forall a_n^{'} \in A_n \setminus \{a_n\}$, $(a_1, .., a_{n-1}, a_n^{'})$ P_i $(a_1, .., a_{n-1}, a_n)$ so by EROG, $\{a_1\} \times ..\{a_{n-1}\} \times \{a_n^{'}\} \succ_i \{a_1\} \times ..\{a_{n-1}\} \times \{a_n\}$. Using AGS we thus have $\{a_1\} \times ..\{a_{n-1}\} \times \{a_n, a_n^{'}\} \succeq_i \{a_1\} \times ..\{a_{n-1}\} \times \{a_n\}$ and by using WNI we obtain $\{a_1\} \times ..\{a_{n-1}\} \times \{a_n\} \sim_i \{a_1\} \times ..\{a_{n-1}\} \times \{a_n, a_n^{'}\}$. Of course this is true for $a_n^{''} \in A_n$ so $\{a_1\} \times ..\{a_{n-1}\} \times \{a_n^{''}\} \succ_i \{a_1\} \times ..\{a_{n-1}\} \times \{a_n\}$, hence by transitivity, $\{a_1\} \times ..\{a_{n-1}\} \times \{a_n^{''}\} \succ_i \{a_1\} \times ..\{a_{n-1}\} \times \{a_n, a_n^{'}\}$ and by using AGS and WNI, we obtain that $\{a_1\} \times ..\{a_{n-1}\} \times \{a_n\} \sim_i \{a_1\} \times ..\{a_{n-1}\} \times \{a_n, a_n^{'}, a_n^{''}\}$. By repeating this procedure as many times as required we can add on every element in A_n until obtaining $\{a_1\} \times ..\{a_{n-1}\} \times \{a_n\} \sim_i \{a_1\} \times ..\{a_{n-1}\} \times A_n$.

Second step : We will now add on all options of individual $n-1$, $n-1 \neq i$. What we have just proved holds also for the $MaxMin$ element of the game form, that is $\{\hat{a}_1\} \times ..\{\hat{a}_{n-2}\} \times \{\hat{a}_{n-1}\} \times \{\hat{a}_n\} \sim_i \{\hat{a}_1\} \times ..\{\hat{a}_{n-2}\} \times \{\hat{a}_{n-1}\} \times A_n$. However, according to step 1, it is true that $\{\hat{a}_1\} \times ..\{\hat{a}_{n-2}\} \times \{a_{n-1}\} \times \{a_n\} \sim_i \{\hat{a}_1\} \times ..\{\hat{a}_{n-2}\} \times \{a_{n-1}\} \times A_n$ where $(\hat{a}_1, .., \hat{a}_{n-2}, a_{n-1}, a_n) = Min_{R_i}(\{\hat{a}_1\} \times ..\{\hat{a}_{n-2}\} \times \{a_{n-1}\} \times A_n)$. But $(\hat{a}_1, .., \hat{a}_{n-2}, \hat{a}_{n-1}, \hat{a}_n) = Min_{R_i}(A_1 \times ..\{\hat{a}_i\} \times ..A_n)$ so $(\hat{a}_1, .., \hat{a}_i, .., \hat{a}_{n-2}, a_{n-1}, a_n)$ P_i $(\hat{a}_1, .., \hat{a}_i, .., \hat{a}_{n-2}, \hat{a}_{n-1}, \hat{a}_n)$, and by EROG and transitivity, we have $\{\hat{a}_1\} \times ..\{\hat{a}_{n-2}\} \times \{a_{n-1}\} \times A_n \succ_i \{\hat{a}_1\} \times ..\{\hat{a}_{n-2}\} \times \{\hat{a}_{n-1}\} \times A_n$. Using AGS, WNI and transitivity leads us to $\{\hat{a}_1\} \times ..\{\hat{a}_{n-2}\} \times \{\hat{a}_{n-1}\} \times \{\hat{a}_n\} \sim_i \{\hat{a}_1\} \times ..\{\hat{a}_{n-2}\} \times \{\hat{a}_{n-1}, a_{n-1}\} \times A_n$. Applying the same procedure allows us to add on all elements of A_{n-1} until we reach $\{\hat{a}_1\} \times ..\{\hat{a}_{n-2}\} \times \{\hat{a}_{n-1}\} \times \{\hat{a}_n\} \sim_i \{\hat{a}_1\} \times ..\{\hat{a}_{n-2}\} \times A_{n-1} \times A_n$. Again, this reasoning can be applied to all other sets A_j, except to A_i. Doing so leads us to the following : $\{\hat{a}_1\} \times ..\{\hat{a}_i\} \times ..\{\hat{a}_n\} \sim_i A_1 \times ..A_{i-1} \times \{\hat{a}_i\} \times A_{i+1} \times ..A_n$.

Third step : Finally we will add on all options of individual i. Let $Min_{R_i}(A_1 \times$

$..A_{i-1} \times \{a_i\} \times A_{i+1} \times ..A_n) = (\underline{a}_1 \times ..a_i \times ..\underline{a}_n)$. According to steps 1 and 2, we obtain $\{\underline{a}_1\} \times .. \{a_i\} \times .. \{\underline{a}_n\} \sim_i A_1 \times ..A_{i-1} \times \{a_i\} \times A_{i+1} \times ..A_n$. But as $(\hat{a}_1, ..., \hat{a}_i, ..., \hat{a}_n)$ is the $MaxMin$ we have by definition $(\hat{a}_1, ..., \hat{a}_i, ..., \hat{a}_n)\ P_i\ (\underline{a}_1, ..., a_i, ..., \underline{a}_n)$. By EROG and transitivity we have $A_1 \times ..A_{i-1} \times \{\hat{a}_i\} \times A_{i+1} \times ..A_n \succ_i A_1 \times ..A_{i-1} \times \{a_i\} \times A_{i+1} \times ..A_n$. We can then use IAS to obtain $A_1 \times ..A_{i-1} \times \{\hat{a}_i, a_i\} \times A_{i+1} \times ..A_n \sim_i A_1 \times ..A_{i-1} \times \{\hat{a}_i\} \times A_{i+1} \times ..A_n$. Finally, the reasoning we just made remains valid with any $a_i^{'}$ in A_i, so we can repeat it, adding on every option in A_i until obtaining the final result : $A_1 \times ..A_i \times ..A_n \sim_i \{\hat{a}_1\} \times .. \{\hat{a}_i\} \times .. \{\hat{a}_n\}$. ■

Proposition 4.4 Axioms EROG, IAS, WNI and AGS are independent.

Proof :
- Let $\succeq_i^{ind2}$ be defined as the $MaxMin$ relation except for singletons, which are all judged indifferent, independently from the preference relation R_i. Formally, if $card(A_j) = 1\ \forall j$, and $card(B_k) = 1\ \forall k$, then $A_1 \times ..A_i \times ..A_n \sim_i^{ind2} B_1 \times ..B_i \times ..B_m$. Otherwise, $A_1 \times ..A_i \times ..A_n \succeq_i^{ind2} B_1 \times ..B_i \times ..B_m \Leftrightarrow A_1 \times ..A_i \times ..A_n \succeq_i^{\max\min} B_1 \times ..B_i \times ..B_m$. Trivially, this relation fails to satisfy EROG. However, for sets other than singletons, $\succeq_i^{ind2}$ relation coincides with $\succeq_i^{\max\min}$ so the axioms IAS, WNI and AGS are satisfied.
- Let $\succeq_i^{cardmaxmin}$ be defined as follows. $A_1 \times ..A_i \times ..A_n \succ_i^{cardmaxmin} B_1 \times ..B_i \times ..B_m \Leftrightarrow card(A_i) > card(B_i)$ or $card(A_i) = card(B_i)$ and $A_1 \times ..A_i \times ..A_n \succ_i^{\max\min} B_1 \times ..B_i \times ..B_m$. Otherwise, $A_1 \times ..A_i \times ..A_n \sim_i^{cardmaxmin} B_1 \times ..B_i \times ..B_m \Leftrightarrow card(A_i) = card(B_i)$ and $A_1 \times ..A_i \times ..A_n \sim_i^{\max\min} B_1 \times ..B_i \times ..B_m$. $\succeq_i^{cardmaxmin}$ ranks sets in a lexicographic way, focusing first on the size of i's set, second on the $MaxMin$ outcome. This relation satisfies EROG, WNI and AGS. However, IAS is violated.
- The relation $\succeq_i^{card}$ satisfies EROG, IAS and AGS but fails to satisfy WNI.
- The relation $\succeq_i^{cardinv}$ satisfies EROG, IAS and WNI but fails to satisfy AGS. ■

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