An Axiomatic Approach to Predictability of Outcomes in an Interactive Setting *

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Abstract

This paper is an axiomatic approach to the problem of ranking game forms in terms of the predictability they offer to individuals. Two criteria are proposed and characterized, the CardMin and the CardMax. Both compare game forms on the basis of the number of distinct outcomes that can result from the choice of a CardMin (resp. CardMax) strategy. The CardMin (resp. CardMax) strategy is defined as a strategy leading to the smallest (resp. highest) number of different outcomes. In both cases, the lower these numbers the better the game form.

1 Introduction

This paper examines the problem of ranking game forms in terms of the *predictability* they offer to individuals. Decision theory has provided an extensive axiomatic literature on the ranking of sets of objects. These sets generally refer to individual attributes in the sense that the elements forming the set concern one individual only (such as opportunity sets, sets of all possible consequences etc...)[1]. Yet, there are many instances in which individuals are involved in interactive activities and do not face sets of options but *game forms* in which every player faces a set of options or strategies.

In an individual setting, options and outcomes are identical, whereas in social activities, outcomes are the result of the combination of every individuals' chosen option. An option no longer leads to an outcome but to a set of potential outcomes, the determination of which depends on the choices of every individual. Following the tradition of

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[1]See Barbera et al. (2004) for an extensive survey of this literature

ranking individuals' sets, normative judgments could also be expressed over alternative game forms despite this difference[2]. This is what this paper tries to do for the specific issue of predictability of outcomes in game forms.

For several reasons, one may judge negatively the absence of predictability of outcomes. Obviously, the knowledge of what outcome an individual will face might be valued *per se*. In addition, predictability over outcomes might be valued for instrumental reasons by individuals. For instance, a stream of literature suggests that rather than evaluating situations on the basis of the individual's well-being, one should focus on the *freedom of choice* this individual enjoys, usually measured by the size of the set of options available to him (see among others Dowding and Van Hees (2008) or Gravel (2008) for surveys of this literature). When considering freedom of choice in a social context, individuals might value having some prediction over the outcomes. Indeed, if individuals cannot influence whatsoever the outcome of the game form they might consider their freedom of choice to be small even if they have a large set of options available to choose from. In this perspective predictability is important because it helps evaluating sets of options in terms of the freedom of choice they offer to individuals. Of course, freedom of choice should not be reduced to the notion of predictability but the latter should definitively enter the definition of the former in a social context.

Another instrumental value of predictability relates to the *uncertainty* individuals are facing in social contexts. Uncertainty has been the subject of considerable attention in decision theory and there have been many attempts of defining and measuring it (see e.g. Kreps (1988) or Machina (1987) for accessible surveys and Bossert (1997), Bossert, Pattanaik and Xu (2000), Nitzan and Pattanaik (1984), Pattanaik and Peleg (1984) for representative pieces of ranking sets in terms of *complete* uncertainty). Reasons for which individuals dislike uncertainty are numerous, two of them being that individuals wish to guarantee desirable outcomes and wish to avoid undesirable ones. Having a high predictability of the outcome when choosing a particular option helps reducing uncertainty and again, if it is not equivalent to uncertainty, the former should be taken into account when defining the later in an interactive setting.

Another stream of literature that bears some connection with what is attempted here is the one that followed Sen's Liberal Paretian Paradox (Sen (1970)) which concerns the modeling of individual rights. After three decades of intensive debate on this issue, a disputed (see e.g. Sen (1983)) majority seems to have reached a consensus that rights are better modeled using game forms rather than the traditional social choice framework (see e.g. Deb (1994,2004), Deb, Pattanaik and Razzolini (1997), Gaertner, Pattanaik and Suzumura (1992), Gärdenfors (1981), Peleg (1984, 1998)). In the game form framework, rights are often modeled by means of effectivity functions that specify the power given to individuals or groups to restrict the set of final outcomes to a special subset of the set of all initial possible outcomes (see e.g. Deb (1994), Moulin (1983),

[2]At the exception of Bervoets(2007) which suggests some rankings of game forms in terms of the freedom of choice they offer, there seem to be no other contribution on this issue.

Peleg (1984, 1998) on that particular issue). However, this literature is mainly devoted to the analysis of different notions of effectivity and their influence on the modeling of rights and, to the best of my knowledge, has not yet produced methods for comparing alternative game forms on the basis of the "effectivity" individuals enjoy.

This paper offers an axiomatic approach to the issue of ranking game forms in terms of the predictability they offer over outcomes. The approach developped herein departs from both the literature on uncertainty and on freedom of choice by the fact that no preferences are considered. Indeed, what is appraised here is a *pure* notion of predictability. A set of axioms is proposed, all of which suggest plausible rankings of very specific games forms. Combinations of these axioms will characterize two different criteria for ranking any game form in terms of predictability, both criteria being based on the cardinality of the different sets of outcomes an individual is facing when he chooses different options from his set. Indeed, every option leads to a set of possible outcomes, the realization of which depends on choices made by others.

By calling a CardMin strategy a strategy leading to the lowest number of *different* outcomes, we define the CardMin criterion that compares two game forms on the basis of the numbers associated to CardMin strategies. The smaller the number of different possible outcomes, the better the game form. Similarly we define the CardMax criterion associated to the CardMax strategy defined as a strategy leading to the highest number of different outcomes in the game forms. Again, the smaller the number the better the game form.

The former criterion focuses on the safest alternative in the game form, the one offering the best predictability of the final outcome. According to the CardMin ranking, individuals have more predictability if they can restrict the set of possible outcomes to a smaller set. On the contrary, the CardMax criterion ranks game forms according to the worst possible case, the one in which individuals face the highest number of different outcomes and thus the lowest possible predictability. It focuses on the largest possible set of outcomes the individuals can restrict to and declares a game form to be better than another if this set is smaller in the former than in the latter.

The next section introduces the formal framework. Section 3 presents the axioms used in the characterisation while section 4 proves the main results as well as the independance of the axioms. Section 5 concludes.

2 Notations

Let I be the finite set of individuals and $\mathcal{I}$ be the set of all non empty subsets of I, with generic elements $N = \{1, \dots, n\}$ or $M = \{1, \dots, m\}$ and $N, M \subseteq I$.
Let $\mathcal{A}_i$ be the infinite set of all conceivable strategies individual i can face and denote by $P(\mathcal{A}_i)$ the set of all finite subsets of $\mathcal{A}_i$, typical elements of $P(\mathcal{A}_i)$ are strategy sets A_i, B_i.... Let $Y = \bigcup_{N \in \mathcal{I}} \prod_{i \in N} P(\mathcal{A}_i)$ be the set of all cartesian products of individual strategy sets. Specific elements of Y are A, B... and for a given set N of individuals,

A_{-M} will stand for the cartesian product $\prod_{i\in N\setminus M} A_i$ with $M \subset N$.

X is the set of all *social outcomes* generated by the strategies in Y. An outcome function $g : Y \longrightarrow X$ is a function that associates a unique social outcome to any combination of individual strategies. An outcome is thus given by $g(a_1, \ldots, a_n) = x$ where $a_1 \in \mathcal{A}_1, \ldots a_n \in \mathcal{A}_n$ and $x \in X$. The notation $g = g'$ on $A \in Y$ will mean that the functions g and g' coincide on the domain A (formally, $g(a_1, \ldots, a_n) = g'(a_1, \ldots, a_n)$ for all $a_1 \in A_1, \ldots a_n \in A_n$ and $A = \prod_{i\in N} A_i$). Let G be the set of all outcome functions g.

A *game form* is given by the pair (A, g) with $A \in Y$ and $g \in G$. Let $\mathcal{G} = (Y, G)$ be the set of all game forms.

Let $\succeq_i$ be, for all $i \in I$, a reflexive and transitive binary relation over $\mathcal{G}$. For all $(A, g), (B, g') \in \mathcal{G}$, $(A, g) \succeq_i (B, g')$ means that "the predictability over outcomes offered by the game form (B, g') is lower for individual i than that over outcomes offered by game form (A, g)" [3]. $\succ_i$ and $\sim_i$ are respectively the antisymmetric and symmetric parts of $\succeq_i$. The paper aims at characterizing such a relation by means of axioms. The cardinality of any set D will be denoted $\#D$.

For notational simplicity, and without loss of generality, the analysis will be conducted using individual 1's standpoint ($i = 1$).

We introduce the following definition.

Definition 1 *For any $i \in N\setminus_{\{1\}}$, a strategy $a_i^{'} \in A_i$ is $(\{a_1\} \times A_{-1}, g)$-neutral for individual 1 if there exists $a_i \in A_i$, $a_i \neq a_i'$ such that $g(a_1, \ldots, a_i, \ldots, a_n) = g(a_1, \ldots, a_i', \ldots, a_n)$ for any $a_2 \in A_2, \ldots, a_n \in A_n$*

This definition says that a strategy available to individual i is neutral for individual 1 if any outcome generated by that neutral strategy a_i' could be generated using another strategy a_i in A_i.

3 Axioms

We now introduce and discuss the axioms used in the characterizations.

Axiom 1 *For any g, g' and any $A = \prod_{j\in N}\{a_j\}$ and $B = \prod_{j\in M}\{b_j\}$, we have $(A, g) \sim_1 (B, g')$*

This axiom says that all game forms in which individuals have only one strategy are indifferent in terms of predictability. Basically, there is full predictability of what will occur. In its form, this axiom suggests that what is under scrutiny is a "pure" notion of predictability.

[3] Note that the binary relation is defined in such a way that comparisons between game forms with different numbers of individuals are allowed. However, doing this would introduce unnecessary notational complexity, we therefore stick to comparisons between n-player game forms, although the axiomatic structure does not require this condition.

Axiom 2 *For any g and for any $i \in N\backslash_{\{1\}}$, if $A = \prod_{j \in N}\{a_j\}$ and $b_i \in \mathcal{A}_i$ is not (A, g)-neutral then $(A, g) \succ_1 (B, g)$ where $B = A_{-i} \times \{a_i, b_i\}$*

Axiom 2 states that any set with two elements offers strictly less predictability than the singletons it contains. It is again very natural that individual 1 should have a strictly better evaluation of predictability over one unique possible outcome than over a set of two possible outcomes, including the one in the first game form. Indeed, in the second game form, the choice of the final outcome depends entirely on the other player's decision. Axiom 2 is inspired by the second axiom used in Pattanaik and Xu (1990), although it is used reversely.

Axiom 3 *For any g and for any $i \in N\backslash_{\{1\}}$, if $A = \{a_1\} \times A_{-1}$ and $b_i \in \mathcal{A}_i$ is (A, g)-neutral, then $(A, g) \sim_1 (B, g)$ where $B = \{a_1\} \times A_{-\{1,i\}} \times A_i \cup \{b_i\}$*

This axiom says that whenever individual 1's set is a singleton, adding to another individual a strategy that generates no new outcome, leaves individual 1 indifferent. Axiom 3 makes this approach a set-based one rather than a vector-based one. Indeed, options in game forms can be described on the basis of the *vectors* of contingent outcomes that they generate. Alternatively options can be described on the basis of the *sets* of possible outcomes. This second approach leads to some loss of information. However in terms of predictability this loss of information should not influence our judgment since no probabilities are attached to outcomes. While one could easily admit that the set $\{x, y, z\}$ would offer less predictability than the set $\{x, y\}$, it seems natural to state that both vectors (x, y) and (x, y, y), yielding the same set $\{x, y\}$, offer the same predictability to individual 1. Axiom 3 says that the information lost with the choice of the set-based model does not play any role in the ranking of game forms. (A good discussion of the differences between set-based and vector-based models is provided by Pattanaik and Peleg (1984)).

Axiom 4 *For any $i \in N\backslash_{\{1\}}$, if $A = \prod_{j \in N\backslash_{\{i\}}}\{a_j\} \times A_i$ and $B = \prod_{j \in N\backslash_{\{i\}}}\{b_j\} \times B_i$ and if $s \in \mathcal{A}_i$ is not (A, g)-neutral nor (B, g')-neutral, then*

$$(A, g) \succeq_1 (B, g') \Longrightarrow (A', g) \succeq_1 (B', g')$$

with $A' = A_{-i} \times A_i \cup \{s\}$ and $B' = B_{-i} \times B_i \cup \{s\}$

This axiom applies in very specific cases only, as it concerns game forms in which all individuals except individual i have a singleton as a strategy set. Thus adding one neutral strategy s to i's set implies adding exactly one new outcome to the set of outcomes of the game form. It is then required that, when adding the same strategy to individual i's set in two different game forms, the preexisting ranking should not be reversed.

Axiom 5 *If* $(A_1 \times A_{-1}, g) \succeq_1 (B_1 \times A_{-1}, g)$ *then* $(A_1 \times A_{-1}, g) \sim_1 (A_1 \cup B_1 \times A_{-1}, g)$

Note that this axiom, contrary to the four previous ones, considers what happens for changes affecting individual 1's set of strategies, assuming individual 2's set to be fixed.

To make interpretation clearer, axiom 5 should be divided in two parts. On one hand, adding a set of strategies offering less predictability than the available strategies do, cannot increase the overall predictability in the game form. This axiom was first introduced by Kreps (1979) in a different framework, but the spirit remains the same: adding to a set another set which is judged to be worse than the initial one cannot improve its overall appraisal.

On the other hand, offering more strategies to an individual, whether they are better or worse, cannot decrease his evaluation of the predictability over the outcomes that he faces. Actually, if the individual is not interested in the new strategies, he is free not to choose them, making it difficult to accept that he could be penalised by the addition of strategies. His situation is thus not worse. If it is required that the addition does neither strictly reduce nor strictly increase the overall predictability, then both game forms are indifferent.

Axiom 6 *For any* $i \in N \backslash_{\{1\}}$, *if* $A = \{a_1\} \times A_{-\{1,i\}} \times A_i$ *with* $\#A_i > 1$, $B = A_{-\{i\}} \times A_i \backslash \{b_i, c_i\}$, *if* g, g' *are such that* $g = g'$ *on* B *and if there exists* $(a'_1, \ldots, a'_{i-1}, a'_{i+1}, \ldots a'_n)$ *in* $A_{-\{i\}}$ *such that*

$$g(a_1, \ldots b_i, \ldots a_n) = g'(a_1, \ldots b_i, \ldots a_n)$$

$$g(a_1, \ldots c_i, \ldots a_n) = g'(a_1, \ldots c_i, \ldots a_n)$$

for any $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n) \neq (a'_1, \ldots, a'_{i-1}, a'_{i+1}, \ldots a'_n)$ *and*

$$g(a'_1, \ldots b_i, \ldots a'_n) = g'(a'_1, \ldots c_i, \ldots a'_n)$$

$$g(a'_1, \ldots c_i, \ldots a'_n) = g'(a'_1, \ldots b_i, \ldots a'_n)$$

then

$$(A, g) \sim_1 (A, g')$$

This axiom applies only to game forms in which individual 1 is facing a singleton. Although its formalization is lengthy, it carries a simple idea. In the case of two individuals, it states that individual 1 enjoys as much predictability when he is facing the set of outcomes $\{a, b, x, y, c, d\}$ than when he is facing $\{a, b, y, x, c, d\}$. More precisely, axiom 6 has two main implications: first, if $b_i = c_i$, the axiom becomes $[(A, g) \sim_1 (A, g')$ if $g = g'$ on $A]$. It states that two game forms in which all strategy sets and outcome sets are identical are judged indifferent. In other words, if two outcome functions differ only on n-tuples which are not in the domain of the game forms, these differences should not be taken into account when comparing the game forms.

Second, if $b_i \neq c_i$, then axiom 6 requires that if all outcomes are exactly the same, but two of them have been interchanged in the sense that one n-tuple in the first game form leads to the outcome produced by the second n-tuple in the second game form and vice-versa, then both game forms should be judged indifferent. As an example, consider three individuals respectively facing strategies $\{a_1\}$, $\{b, c, d\}$ and $\{e, f, g\}$ and g and g' being such that

$$(A,g) = \begin{array}{cccc} & b & c & d \\ e & \mathbf{x} & \mathbf{y} & \mathbf{z} \\ f & \mathbf{s} & \mathbf{t} & \mathbf{u} \\ g & \mathbf{v} & \mathbf{w} & \mathbf{x} \end{array} \text{ and } (A, g') = \begin{array}{cccc} & b & c & d \\ e & \mathbf{x} & \mathbf{y} & \mathbf{z} \\ f & \mathbf{s} & \mathbf{w} & \mathbf{u} \\ g & \mathbf{v} & \mathbf{t} & \mathbf{x} \end{array}$$

These two game forms are identical except for outcomes t and w which have been interchanged. Axiom 6 says that both game forms (A, g) and (A, g') are indifferent.

The last axiom is a modification of axiom 5, needed for the characterization of the second criterion.

Axiom 7 *If* $(A_1 \times A_{-1}, g) \succeq_1 (B_1 \times A_{-1}, g)$ *then* $(B_1 \times A_{-1}, g) \sim_1 (A_1 \cup B_1 \times A_{-1}, g)$

Axiom 7, on the contrary of axiom 5, states that adding a set of options A_1 to B_1, when A_1 is better than B_1, does not affect the ranking of the game forms.

4 Characterization result

To illustrate the definitions below, consider the following example of a game form (A_{ex}, g) with $A_1 = \{a_1, a_2\}$, $A_2 = \{s, t, u\}$ and $A_3 = \{b, c, d\}$:

$$(A_{ex}, g) \qquad \text{With } a_1\text{:} \begin{array}{cccc} & b & c & d \\ s & \mathbf{y} & \mathbf{z} & \mathbf{z} \\ t & \mathbf{w} & \mathbf{x} & \mathbf{z} \\ u & \mathbf{y} & \mathbf{x} & \mathbf{x} \end{array} \qquad \text{With } a_2\text{:} \begin{array}{cccc} & b & c & d \\ s & \mathbf{w} & \mathbf{z} & \mathbf{z} \\ t & \mathbf{v} & \mathbf{w} & \mathbf{z} \\ u & \mathbf{w} & \mathbf{v} & \mathbf{v} \end{array}$$

Definition 2 *Let* $D(x, A, g) = \{g(x, a_2, \ldots, a_n); a_2 \in A_2, \ldots a_n \in A_n\}$

$D(x, A, g)$ is the set of outcomes associated to strategy x played by individual 1. The cardinality of $D(x, A, g)$ may be smaller than $\prod_{j \in N \setminus \{1\}} \# A_j$. For instance, $D(a_1, A_{ex}, g) = \{w, x, y, z\}$ and $D(a_2, A_{ex}, g) = \{v, w, z\}$.

Definition 3 *A strategy* $a_* \in A_1$ *is a CardMin strategy for individual* 1 *if* $\#D(a_*, A, g) \leq \#D(x, A, g)\ \forall x \in A_1$. *Furthermore,* $CM(A, g) = \#D(a_*, A, g)$.

The CardMin strategy is not uniquely defined, it is a strategy of individual 1 that generates the least possible outcomes in the game. Hence $\#D(a_1, A_{ex}, g) = 4$, $\#D(a_2, A_{ex}, g) = 3$. Therefore $a_2 \in A_1$ is a CardMin strategy, it is the only one and $CM(A_{ex}, g) = 3$.

We now introduce the CardMin criterion, $\succeq_1^{CM}$.

Definition 4 $(A,g) \succeq_1^{CM} (B,g') \Longleftrightarrow CM(A,g) \leq CM(B,g')$

The CardMin criterion compares game forms in terms of the smallest set of outcomes that can result from individual 1's choice of strategy. According to this criterion, individuals prefer facing very few outcomes rather than many, the most preferable case being when an individual has a strategy leading to one single outcome. In that case there is no doubt about which outcome will be realized, the predictability is maximal. The CardMin criterion focuses on the safest alternative in terms of predictability and compares game forms on the basis of these safest alternatives.

Again, no preferences enter the analysis so it is a pure notion of predictability we are considering. Therefore the set offering the least possible different outcomes serves as a relevant set for evaluating game forms.

Let us now turn to the characterization result.

Theorem 1 *A reflexive and transitive binary relation $\succeq_1$ satisfies Axioms 1 to 6 if and only if $\succeq_1 = \succeq_1^{CM}$*

Proof of Theorem 1:
The CardMin criterion $\succeq_1^{CM}$ is transitive and satisfies axioms 1 to 6. Now assume $\succeq_1$ is transitive and satisfies axioms 1 to 6.

The proof will proceed in three steps. First we show that any game form (A,g) in which individual 1 has a singleton as a strategy set is indifferent to another specific game form (A^*,g^*) with $A^* = \prod_{j\in N\setminus\{n\}}\{a_j\} \times A_n^*$, that is, a game form in which every player has a singleton as a strategy set except for one individual.

Next we show that two game forms in which every player has a singleton as a strategy set except for one individual, i.e. (A^*,g^*) where $A^* = \prod_{j\in N\setminus\{n\}}\{a_j\} \times A_n^*$ and (B^*,g'^*) where $B^* = \prod_{j\in N\setminus\{n\}}\{b_j\} \times B_n^*$ can be compared on the basis of the cardinality of the sets A_n^* and B_n^*. These cardinalities will happen to coincide with $\#D(a_1,A^*,g^*)$ and $\#D(b_1,B^*,g'^*)$. Finally we show that if a_* is a CardMin strategy for individual 1, then the games $(\{a_*\}\times A_{-1},g)$ and (A,g) are indifferent. Putting the three steps together completes the proof.

Consider any game form (A,g) with $A = \{a_1\}\times A_{-1}$ in which individual 1 has only one available strategy. To prove the first step, we will show that the game form (A,g) is indifferent to another game form (A',g') with $A' = \{a_1\}\times\{a_2\}\times A_{-\{1,2,n\}}\times A_n'$, that is a game form in which the set of individual 2's strategies is reduces to a singleton while the set of individual n has increased.

Let $A_2 = \{t_1,\dots t_p\}$ and $A_n = \{x_1,\dots x_m\}$ and consider $y_1,\dots,y_m \in \mathcal{A}_n \setminus A_n$, m strategies which are not available to individual n. Call $\tilde{A}$ the cartesian product with $\tilde{A}_j = A_j$ for $j < n$ and $\tilde{A}_n = A_n \cup \{y_1,\dots y_m\}$. Let g_1 be such that $g_1 = g$ on A and $g_1(a_1,a_2,\dots,y_j) = g(a_1,a_2,\dots x_j)$ for all $a_2 \in A_2,\dots a_{n-1}\in A_{n-1}$ and all $j \in \{1,\dots m\}$, so that all outcomes generated by strategies in A_n are duplicated. All strategies $y_1,\dots y_m$ are $(\tilde{A},g_1)$-neutral for individual 1, and by repeated use of axiom

3 we have $(\tilde{A}, g_1) \sim_1 (A, g_1)$. Furthermore, since $g = g_1$ on A, axiom 6 applied with $b_i = c_i$ guarantees that $(A, g_1) \sim_1 (A, g)$, so that $(A, g) \sim_1 (\tilde{A}, g_1)$ by transitivity.

Now consider g_2 such that $g_2 = g_1$ on $\tilde{A}_{-2} \times \tilde{A}_2 \backslash_{\{t_1,t_2\}}$, and

$$g_2(a_1, t_1, a_3, \ldots, a_{n-1}, x_j) = g_1(a_1, t_1, a_3, \ldots, a_{n-1}, x_j)$$

$$g_2(a_1, t_2, a_3, \ldots, a_{n-1}, x_j) = g_1(a_1, t_2, a_3, \ldots, a_{n-1}, x_j)$$

and

$$g_2(a_1, t_1, a_3, \ldots, a_{n-1}, y_j) = g_1(a_1, t_2, a_3, \ldots, a_{n-1}, y_j)$$

$$g_2(a_1, t_2, a_3, \ldots, a_{n-1}, y_j) = g_1(a_1, t_1, a_3, \ldots, a_{n-1}, y_j)$$

for all $a_3 \in \tilde{A}_3, \ldots a_{n-1} \in \tilde{A}_{n-1}$, $x_j \in A_n$ and $y_j \in \tilde{A}_n \setminus A_n$. Then by repeated use of axiom 6 and by transitivity we get $(\tilde{A}, g_1) \sim_1 (\tilde{A}, g_2)$. Because $g_1(a_1, a_2, \ldots, y_j) = g_1(a_1, a_2, \ldots x_j)$ for any $a_2 \in A_2$ by definition of strategies y_j, we have

$$g_2(a_1, t_1, \ldots, x_j) = g_2(a_1, t_2, \ldots, y_j)$$

and

$$g_2(a_1, t_1, \ldots, y_j) = g_2(a_1, t_2, \ldots, x_j)$$

so that every outcome generated by t_2 is also generated by t_1. Now consider g_3 such that

$$g_3(a_1, t_1, \ldots, x_j) = g_2(a_1, t_2, \ldots, y_j)$$

and

$$g_3(a_1, t_1, \ldots, y_j) = g_2(a_1, t_2, \ldots, x_j)$$

and $g_3 = g_2$ otherwise. Then using axiom 6 we get $(\tilde{A}, g_2) \sim_1 (\tilde{A}, g_3)$. Furthermore, with this construction, t_2 becomes $(\tilde{A}, g_3)$-neutral for individual 1. Hence axiom 3 leads us to $(\tilde{A}, g_3) \sim_1 (\tilde{A}_{-2} \times \tilde{A}_2 \backslash_{\{t_2\}}, g_3)$. We thus have constructed a game form in which one option of A_2 has been removed and several options have been added to A_n, this game form being indifferent to the original one.

Repeating the same procedure by adding options $\{z_1, \ldots z_{2m}\}$ to $\tilde{A}_n$ and using an appropriate function g_4 so as to duplicate the outcomes generated by the $2m$ options in $\tilde{A}_n$, and by swapping outcomes between t_1 and t_3, one can remove t_3 from $\tilde{A}_2 \backslash_{\{t_2\}}$, then t_4 etc... until reaching $A_2' = \{t_1\}$. Finally, $(\tilde{A}, g_1) \sim_1 (\tilde{A}, g_2) \sim_1 (\tilde{A}, g_3) \sim_1 (\tilde{A}_{-\{2,n\}} \times \tilde{A}_2 \backslash_{\{t_2\}} \times \tilde{A}_n, g_3) \sim_1 (\tilde{A}_{-\{2,n\}} \times \tilde{A}_2 \backslash_{\{t_2,t_3\}} \times \tilde{\tilde{A}}_n, g_4) \sim_1 \ldots \sim_1 (\tilde{A}_{-\{2,n\}} \times \{t_1\} \times A_n', g')$. By transitivity, we get

$$(A, g) = (\{a_1\} \times A_2 \times A_3 \ldots \times A_n, g) \sim_1 (\{a_1\} \times \{t_1\} \times A_3 \ldots \times A_n', g') = (A', g')$$

so we have constructed a game form in which individual 2 is facing a singleton and individual n has many new strategies while the strategy sets of all other players are unchanged, and this game form is indifferent to the initial one. The same reasoning can be applied with A_3, A_4 etc... until A_{n-1} so as to reach $(A, g) \sim_1 (A', g') \sim_1 \ldots \sim_1 (\bar{A}, g^*)$ with $\bar{A} = \{a_1\} \times \{a_2\} \ldots \times \{a_{n-1}\} \times \bar{A}_n$. We are now left with a unique row of outcomes, all of them depending on individual n's choice. By applying a last time axiom 3, we can remove all strategies in $\bar{A}_n$ which are neutral for individual 1 so as to obtain A_n^*, such that

$$g(a_1, t_1, \ldots, a_i^*) \neq g(a_1, t_1, \ldots, a_j^*)$$

for any $a_i^*, a_j^* \in A_n^*$. Hence, $(A, g) \sim_1 (A^*, g^*)$. Notice here that every outcome generated by any n-uplet of strategies in (A, g) is also generated by a n-uplet in (A^*, g^*). Furthermore no new outcome has been produced when (A^*, g^*) was constructed. This implies that $\#D(a_1, A, g) = \#D(a_1, A^*, g^*)$.

The proof of step 2 is very much inspired by the proof found in Pattanaik and Xu (1990) for the Cardinal criterion. For two games forms (A, g) where $A = \prod_{j \in N \setminus \{n\}} \{a_j\} \times A_n$ and (B, g') where $B = \prod_{j \in N \setminus \{n\}} \{b_j\} \times B_n$ we show that

$$\#A_n = \#B_n \Longrightarrow (A, g) \sim_1 (B, g')$$

and

$$\#A_n < \#B_n \Longrightarrow (A, g) \succ_1 (B, g')$$

The argument proceeds by induction on the cardinality of the sets A_n and B_n. Assume $\#A_n = B_n = 1$. According to axiom 1, $(A, g) \sim_1 (B, g')$. Now assume that for any game forms (A', g) and (B', g') such that $\#A'_n = \#B'_n = m-1$ we have $(A', g) \sim_1 (B', g')$ and assume $A_n = \{s_1, \ldots s_m\}$ and $B_n = \{t_1, \ldots t_m\}$ where no option in A_n is (A, g)-neutral for individual 1 and no option in B_n is (B, g')-neutral for individual 1. We know by hypothesis that $(A_{-n} \times \{s_1, \ldots, s_{m-1}\}, g) \sim_1 (B_{-n} \times \{t_1, \ldots, t_{m-1}\}, g')$. Consider the function $\tilde{g}$ such that $\tilde{g} = g'$ on B and such that s_m is $(B, \tilde{g})$-neutral for individual 1. Then, using axiom 4 we get $(A, g) = (A_{-n} \times \{s_1, \ldots, s_{m-1}, s_m\}, g) \sim_1 (B_{-n} \times \{t_1, \ldots, t_{m-1}, s_m\}, \tilde{g})$. But by the induction hypothesis, $(B_{-n} \times \{t_1, \ldots, t_{m-2}, s_m\}, \tilde{g}) \sim_1 (B_{-n} \times \{t_1, \ldots, t_{m-2}, t_m\}, \tilde{g})$ so using axiom 4 once more by adding option t_{m-1} on both sides yields $(B_{-n} \times \{t_1, \ldots, t_{m-2}, t_{m-1}, s_m\}, \tilde{g}) \sim_1 (B_{-n} \times \{t_1, \ldots, t_{m-2}, t_{m-1}, t_m\}, \tilde{g})$. Of course, as $g' = \tilde{g}$ on B, we also have $(B_{-n} \times \{t_1, \ldots, t_{m-2}, t_{m-1}, t_m\}, \tilde{g}) = (B, \tilde{g}) \sim_1 (B, g')$, thus by transitivity, $(A, g) \sim_1 (B, g')$.

Hence $\#A_n = \#B_n \Longrightarrow (A, g) \sim_1 (B, g')$. To show that $\#A_n < \#B_n \Longrightarrow (A, g) \succ_1 (B, g')$, we basically follow the same route as Pattanaik and Xu (1990) using axioms 2 and 4 and being careful to adapt the proof in the same way we just have for the first implication. Notice here that in game forms in which only individual n has more than one option, and no option is neutral, trivially $\#A_n = \#D(a_1, A, g)$. Thus steps 1 and 2 together show that any game form in which individual 1 faces a singleton can be compared on the basis of $\#D(a_1, A, g)$. In particular, if $A = \{a_1\} \times A_{-1}$ and $B = \{b_1\} \times B_{-1}$, then $(A, g) \succeq_1 (B, g) \Longleftrightarrow \#D(a_1, A, g) \leq \#D(b_1, B, g)$

For the third step, consider a_*, a CardMin strategy for individual 1. By definition,

$\#D(a_*, A, g) \leq \#D(x, A, g)\ \forall x \in A_1$, so $(\{a_*\} \times A_{-1}, g) \succeq_1 (\{x\} \times A_{-1}, g)$. Using axiom 5 we get $(\{a_*\} \times A_{-1}, g) \sim_1 (\{a_*, x\} \times A_{-1}, g)$. Again, a_* being a CardMin strategy, we have $\#D(a_*, A, g) \leq \#D(y, A, g)$, so $(\{a_*\} \times A_{-1}, g) \succeq_1 (\{y\} \times A_{-1}, g)$ and by transitivity, $(\{a_*, x\} \times A_{-1}, g) \succeq_1 (\{y\} \times A_{-1}, g)$. Axiom 5 again gives $(\{a_*\} \times A_{-1}, g) \sim_1 (\{a_*, x, y\} \times A_{-1}, g)$. Carrying on as many times as necesary one can add all options in A_1 until we finally reach $(\{a_*\} \times A_{-1}, g) \sim_1 (A_1 \times A_{-1}, g)$ which concludes the proof. ■

This criterion gives a discriminating power to one particular set of strategies in the game form, that is the set of all CardMin strategies. A weakness thus lies in the fact that no contribution of other strategies to the overall predictability is taken into account.

Next the independence of the axioms is proved.

Proposition 1 *Axioms 1 to 6 are independent*

Proof : In order to show that the axioms are independent, let us define some criteria such that in turn each axiom is violated although all the others are satisfied.

- Let $\succeq^1$ be defined exactly as $\succeq^{CM}$, except for the comparison of singletons. Singletons such as considered in axiom 1 will be compared on the basis of any predefined ranking (alphabetical order, any exogenous ranking...), allowing for strict comparisons. Then $\succeq^1$ satisfies axioms 2 to 6, but can eventually violate axiom 1.

- Let $\succeq^2$ be defined as the reverse of $\succeq^{CM}$, that is $A \succeq^2 B \Longleftrightarrow CM(A) \geq CM(B)$. Then axioms 1 and 3 to 6 are satisfied, however, axiom 2 is violated.

- Let $\succeq^3$ be defined as $A \succeq^3 B \Longleftrightarrow \sum_{i \in N \setminus \{1\}} \#A_i \leq \sum_{i \in N \setminus \{1\}} \#B_i$. $\succeq^3$ violates axiom 3 but satisfies all others.

- Let $\succeq^4$ be defined as $\succeq^{CM}$, except in the case in which we are comparing two game forms in which all individuals except individual i are facing singletons. In that particular case, let $\succeq^4$ be defined as follows: if the parity of $\#A_i$ is the same as the parity of $\#B_i$ (i.e. if they are both odd or both even), then $A \succeq^4 B \Longleftrightarrow A \succeq^{CM} B$. If $\#A_i$ is odd and $\#B_i$ is even, then $A \succ^4 B$. $\succeq^4$ satisfies all axioms except for axiom 4.

- Let $\succeq^5$ be defined as follows: $A \succ^5 B \Longleftrightarrow \#A_1 > \#B_1$ or $(\#A_1 = \#B_1$ and $CM(A) < CM(B))$; $A \sim^5 B \Longleftrightarrow \#A_1 = \#B_1$ and $CM(A) = CM(B)$. $\succeq^5$ violates axiom 5 but satisfies all others.

- let $\succeq^6$ be defined as $\succeq^{CM}$, except in the particular case of comparing two game forms in which individual 1 faces any set, 2 and 3 face two options, all other individuals face a singleton. In that specific case, let $C_{ij}(A, g) = \#\{c_j \in A_3; g(a_1, b_i, c_j, \ldots a_n) = x\}$

and let $C_j(A,g) = Max_i C_{ij}(A,g)$. Then $A \succeq^6 B \iff C_j(A,g) \geq C_j(B,g)$. By construction, the conditions of axioms 1 to 4 are such that $\succeq^6 = \succeq^{CM}$ so they are satisfied. Axiom 5 is also satisfied for both cases, whereas axiom 6 is violated. ■

We now show how a simple modification of axiom 5 can help characterize another criterion reversing the spirit of $\succeq^{CM}$.

Definition 5 *Let a CardMax strategy for individual 1 be defined as $\hat{a} \in A_1$ such that $\#D(\hat{a}, A, g) \geq \#D(x, A, g)\ \forall x \in A_1$. Furthermore, $CMax(A,g) = \#D(\hat{a}, A, g)$.*

Definition 6 *$\succeq_1^{CMax}$ is a transitive binary relation comparing game forms defined as:*

$$(A,g) \succeq_1^{CMax} (B,g') \iff CMax(A,g) \leq CMax(B,g')$$

When $\succeq^{CM}$ compares two game forms in terms of the smallest set of possible outcomes, $\succeq^{CMax}$ compares them in terms of the largest set. The criterion $\succeq^{CMax}$ ensures that the worse case in game form (A,g) is not worse than the worse case in game form (B,g') for individual 1. When the spirit of $\succeq^{CM}$ is of maximizing the best, that of $\succeq^{CMax}$ is of minimizing the worse.

Theorem 2 *A transitive binary relation $\succeq_1$ satisfies Axioms 1 to 4, 6 and 7 if and only if $\succeq_1 = \succeq_1^{CMax}$*

Proof of Theorem 2:
The proof is straightforward using the proof of theorem 1. Only the third step changes by using axiom 7 instead of axiom 5.

In the same way we can easily show that axioms 1 to 4, 6 and 7 are independent.

As for theorem 1, the criterion characterized in theorem 2 is based on a particular subset of strategies of individual 1 and therefore on the cardinality of one particular subset of outcomes of the game form. No other strategies than the CardMax ones contribute to the overall predictability as evaluated by individual 1. Another important criticism that applies to both criteria is the absence of judgment about the value of the outcomes. As we are interested in measuring a pure notion of predictability we did not introduce preferences into the framework. This could be a shortcoming for several reasons, the main one being the following: assume a society of two individuals in which individual 1 has only one strategy and faces the set $\{x, y\}$ of possible outcomes in game form 1 while he faces the set $\{s, t, u, v, w\}$ in game form 2. Then both criteria CardMin and CardMax would agree on ranking game form 1 above game form 2 in terms of predictability. But if x consists of winning 10 dollars and y of winning 11 dollars while s, t, u, v, w respectively consist of winning 100, 110, 120, 130 or 140 dollars, then individual 1 might prefer playing game form 2. This is probably because individuals prefer having no prediction about good outcomes than being able to predict bad outcomes. In fact, in the second game form individuals can predict with total certainty that they will end up with a better outcome than in the first game form. This important shortcoming is acknowledged but it cannot be tackled unless we introduce preferences into the framework.

Another important criticism one could address to this approach concerns the *diversity* of outcomes. Indeed, assume that individual 1 is compelled to follow his partner's choice for a restaurant and assume that game form 1 leads to consequences {*chinese restaurant, french restaurant*} while game form 2 leads to {*italian restaurant 1, italian restaurant 2, italian restaurant 3, italian restaurant 4, italian restaurant 5*}. Again, CardMin and CardMax would both agree to say that game form 1 offers more predictability than game form 2. However, here this judgment seems counterintuitive. This is because the diversity of consequences in the first game form is much higher than the diversity of consequences in the second game form[4].

5 Conclusion

The purpose of this paper was to introduce the notion of predictability over outcomes in a social context. Although a characterisation result has been reached, the *CardMin* criterion, as much as its counterpart the *CardMax*, carries some weaknesses. There are at least two interesting lines of research that should be explored in order to push the analysis a bit further. First, by using the same framework, more subtle criteria, allowing for every strategy to contribute to the overall predictability, merit to be studied. A criterion that ranks game forms according to the mean value of the cardinality of the set of outcomes associated to every strategy of individual 1 could be an example of such a criterion. Second, it could be of interest to explore the reasons why individuals might want to have predictions over outcomes, thereby bringing preferences into the framework. Finally, introducing the notion of predictability such as defined herein into the literature on freedom of choice in social contexts would be interesting. One way of doing this could be to provide a lexicographic ordering taking into account both the number of strategies available to an individual and the predictability these strategies offer.

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