Appraising Diversity with an Ordinal Notion of Similarity: An axiomatic Approach*

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Abstract

This paper axiomatically characterizes a rule for comparing alternative sets of objects on the basis of the *diversity* that they offer. The framework considered assumes a finite universe of objects and an *a priori* given ordinal quaternary relation that compares alternative *pairs* of objects on the basis of their dissimilarity. The rule that we characterize is the *maxi-max* criterion. It considers that a set is more diverse than another if and only if the two most dissimilar objects in the former are weakly as dissimilar as the two most dissimilar objects in the later. Some connections with the issue of appraising freedom of choice are also provided.

1 Introduction

Would the killing of 50 000 flies of a specific species have the same impact on the reduction of biological diversity than that of 200 white rhinoceros ? Is the diversity of opinions expressed in the written press larger in France than in the US ? Is the choice of models of cars offered by a particular retailer more diverse than that of another ? These are examples of questions whose answers require a precise notion of *diversity*.

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Biologists have been probably the first scientists interested in developing and implementing numerical indices that aim at measuring the biological diversity offered by alternative ecosystems. One of the most widely used of these indices is a generalization of Shannon (1948)'s *entropy* measure proposed in biology by Good (1953) (see e.g. Baczkowski, Joanes and Shamia (1997), Baczkowski, Joanes and Shamia (1998) and Magurran (1998) for other refinements and discussions of this class of indices). This class of indices evaluates the diversity of any ecosystem by counting, for each species, the frequency of living individuals within the species relative to the total number of living individuals and calculates a weighted entropy over these relative frequencies. Yet, despite its wide use and computational convenience for applications, this index lacks sound justifications. Why after all should one use the specific entropy formula for appraising the impact of major changes on biodiversity ?

Answering questions like this is important in these days where many countries who have ratified the UN 1992 convention on biological diversity have adopted economically costly environmental regulations in order to prevent a deterioration of biological diversity caused by human activities. It is all the most important as the generalized entropy measure suffers from the drawback of paying no attention whatsoever to either inter-species *dissimilarities*, or to the possibility for two individuals of the same species to be more dissimilar than two individuals coming from different species. For instance, according to the generalized entropy formula, a world in which all living individuals are equally split between two species of fly is just as diverse as one in which the living individuals are split equally between chimpanzees and hippocampi.

Efforts, often due to economists, have been made in the last 15 years to develop criteria for appraising diversity that are sensitive to the dissimilarity that may exist between living individuals. At the origin of these efforts are the contributions of Weitzman (1992), Weitzman (1993) and Weitzman (1998) which assume the *a priori* given existence of a *cardinally meaningful* numerical distance between living creatures. Such a numerical distance enables one to say things such as "the biological distance between a chimpanzee and a bee is exactly twice that between a trout and a salmon". Using such a numerical distance, Weitzman (1992) proposes a sophisticated iterative lexicographic method for appraising the diversity offered by a set of living individuals. Using a somewhat different setting, Bossert, Pattanaik and Xu (2003) provide an axiomatic characterization of Weitzman's method by taking as given a cardinal numerical measure of distances between the objects.

An alternative approach to diversity appraisal has been proposed by Nehring and Puppe (2002) (see also Nehring (1997) and Nehring and Puppe (2003)) who suggest defining the diversity of a set as the sum of the values of the attributes realized by the elements in the set. Operational use of Nehring and Puppe (2002) approach requires the diversity appraiser to select a class of relevant attributes for the objects (for instance being a mammal) and of a (cardinally meaningful) function that weights the various attributes.[1]

Both Nehring and Puppe and Weitzman's approaches exhibit sensitivity to inter-species dissimilarities. On the other hand, it is not at all clear that the current state of knowledge in biology leads to such a precise cardinal measure of distance between living creatures, or of cardinally meaningful attribute weight, as what is required by these approaches. Much biologists would probably agree that a chimpanzee and a bee are more dissimilar than a trout and a salmon. Yet is seems unlikely that they would agree to say that the dissimilarity between a chimpanzee and a bee is *exactly* twice that between a trout and a salmon.

In the last fifteen years or so, interest in diversity measurement has also arisen in *non-welfarist* normative economics, in connection with the issue of comparing *opportunity sets* on the basis of their freedom of choice (see e.g. Barberà, Bossert and Pattanaik (2004) and Sugden (1998) for recent surveys of this literature). A major weakness of many rankings of opportunity sets examined in this literature is their insensitivity to the diversity of the options contained in opportunity sets. After all even if the fact of being forced (by lack of alternative) to drive a blue car to go to some destination can be considered equivalent, from a freedom point of view, to being forced to make the same trip by train, this does *not* imply that the possibility of getting to destination by driving either a blue or a red car offers the same freedom as the possibility of making the trip either by train or by a red car.[2] Yet many rankings of opportunity sets examined in the literature fail to make the distinction.

While the present paper is primarily concerned with the issue of appraising diversity, it does provide some indication as to how the evaluation of diversity may interfere with that of freedom of choice. Specifically, this paper's main contribution is to provide an axiomatic characterization of a ranking of sets on the sole basis of their diversity. As in Bossert et al. (2003), Pattanaik and Xu (2000), Weitzman (1992), Weitzman (1993), Weitzman (1998), the ranking characterized in this paper is based on an *a priori* notion of "proximity", or "dissimilarity", between the objects that is taken as given.

However, the notion of similarity considered herein requires less information than what is necessary to define a cardinally meaningful numerical distance function such as that used in these contributions. Rather, the prim-

[1] As recognized by Nehring and Puppe (2003) themselves, "the cardinal scale inherent in our concept of diversity is essential". A more detailed discussion of the literature on diversity,including the multi-attributes, approach is provided in Gravel (2006).

[2] Or, as Thomas Aquinas put it about nine hundred years ago, "An angel is more valuable than a stone. It does not follow, however, that two angels are more valuable than one angel and one stone" (quoted by Nehring and Puppe (2002) (p. 1155)).

itive notion of similarity on which we base our axiomatic construction is *ordinal*. That is, it requires the ability to perform statements like "the biological distance between a chimpanzee and a bee is larger than the biological distance between a trout and a salmon" but does *not* suppose the capacity of quantifying further these statements assumed in cardinal distance functions.

Pattanaik and Xu (2000) have also examined a diversity-based ranking of sets of objects that refers explicitly to an *a priori* given ordinal notion of similarity. However, the ordinal notion of similarity considered by these authors is rather crude, for it only allows objects to be either pairwise dissimilar or pairwise similar. No intermediate categories of similarities are allowed. With this "zero-one" notion of similarity, Pattanaik and Xu (2000) characterize a ranking of sets based on the number of elements contained in the smallest partition of the sets into subsets of similar objects. According to their ranking, set A offers at least as much diversity as set B if, and only if, the smallest partition of A into subsets of similar objects contains at least as many elements as the corresponding partition in B.

While very interesting as a first step in the process of building a diversity ranking of sets based on an ordinal notion of similarity, this result suffers from the paucity of the information conveyed by the "zero-one" notion of similarity used. To the best of our knowledge, the only other contribution that bases diversity appraisal on a primitive ordinal notion of dissimilarity is Nehring (1997). Yet this paper does so by modelling dissimilarity as a *ternary* relation. A ternary relation enables one to say things like "a trout is closer to a salmon than a bee is" but does not enable one to compare the dissimilarity between a salmon and a trout to that between, say, a bee and chimpanzee. While a ternary relation may be an appropriate way of expressing a notion of similarity between objects in a framework, such as that considered by Nehring (1997), where objects are grouped into various attributes ("a trout is closer to a salmon than a bee is if any attribute possessed by bees and salmons is also possessed by trouts"), we feel that it is not so natural in the context considered in this paper where such an attributes structure is *a priori* absent.

In this paper, we characterize axiomatically a diversity ranking of sets based on a primitive ordinal notion of similarity that is not restricted to be of the "zero-one" type considered by Pattanaik and Xu (2000). Moreover, the primitive notion of similarity that we consider is not a ternary relation but, instead, a *quaternary* relation (or a binary relation on the set of all pairs of objects) that is only restricted to satisfy mild properties (reflexivity, completeness, transitivity and a weak form of symmetry). Using this notion, we characterize by three simple axioms the *maxi-max* criterion that compares sets on the basis of the dissimilarity of their two most dissimilar objects.

While this ranking is of intrinsic interest for the problem at hand, we believe that the general methodology employed for obtaining a consistent method for assessing diversity on the basis of a primitive ordinal notion of similarity is more important than the ranking itself. We further illustrate this by showing how our approach to diversity measurement can shed light on some aspects of the problem of ranking opportunity sets on the basis of their freedom of choice.

For this sake, we adopt Pattanaik and Xu (1998)'s framework in which the freedom of choice offered by alternative opportunity sets is appraised by referring to a *set* of possible preference orderings that a reasonable person can have (see also Foster (1993), Nehring and Puppe (1999) or Puppe (1998) for other use and/or interpretation of this *multiple preferences* approach to freedom of choice). In this framework, the options of an opportunity set that are maximal with respect to some of the possible preference orderings are typically considered to be a sufficient information for appraising the freedom of choice offered by that opportunity set. Following this tradition, we provide in this paper an axiomatic characterization of a ranking of opportunity sets that compares their sets of maximal options with respect to some of the possible preference orderings by means of the *maxi-max* criterion mentioned above.

The rest of this paper is organized as follows. The next section presents the notation and the formal definitions of the axioms and the ranking characterized for the purpose of diversity measurement. Section 3 presents and briefly discusses the main characterization result that concern diversity measurement. Section 4 explores some of the connections between diversity and freedom of choice measurement and section 5 concludes.

2 Notations and definitions

2.1 Notations

Given any finite set A, we denote its cardinality by $|A|$. By a binary relation B on a set Ω, it is meant a subset of $\Omega \times \Omega$. Following common use we write $x\ B\ y$ instead of $(x, y) \in\ B$. For a binary relation B, its *asymmetric* factor B_A is defined by $x\ B_A\ y \Longleftrightarrow x\ B\ y \wedge \neg(y\ B\ x)$ and its *symmetric* factor B_S by $x\ B_S\ y \Longleftrightarrow (x\ B\ y) \wedge (y\ B\ x)$. A binary relation B on Ω is *reflexive* if $x\ B\ x$ for all $x \in \Omega$, is *complete* if $(x\ B\ y)$ or $(y\ B\ x)$ holds for every distinct x and $y \in \Omega$, is *transitive* if $x\ B\ z$ follows from $x\ B\ y$ and $y\ B\ z$ for any x, y and z in Ω. A reflexive, complete and transitive binary relation is called an *ordering*. Let X be a finite set of options (living individuals, type of means of transportations, opinions expressed in newspapers, etc.) and $\mathfrak{P}(X)$ be the set of all non empty subsets of X with generic elements A, B,....

2.2 Definitions

At the basis of our approach is a *quaternary* relation Q on X (alternatively, a binary relation on $X \times X$) with asymmetric and symmetric factors Q_A and Q_S respectively which reflects one's *ordinal a priori* knowledge about options dissimilarities. In this light, the statement $(w,\ z)\ Q\ (x,\ y)$ is interpreted as meaning "w is at least as dissimilar from z than x is from y". To motivate this interpretation, we assume throughout that, for every distinct objects x and $y \in X$, both $(x,\ y)\ Q\ (x, x)$ and $(x, x)\ Q_S\ (y, y)$ hold (that is, two distinct objects are always weakly more dissimilar than any of the two objects in isolation, and pairs of identical objects are just equally similar (or dissimilar)). We assume also that Q is *symmetric* in the sense that $(x, y)\ Q\ (y, x)$ holds for every objects x and y and, as a binary relation on $X \times X$, is *complete* and *transitive*. All these properties of Q would clearly hold true if, like Bossert et al. (2003), Pattanaik and Xu (2000) or Weitzman (1992), we would accept to go as far as measuring the dissimilarity by a (cardinally significant) distance function $d : X \times X \to \mathbb{R}$.

We also make the extra assumption that Q is such that $(x,\ y)\ Q_A\ (x, x)$ for every two distinct x and y (two distinct options are always strictly more dissimilar than one of the two options and itself). Although there exists distance functions that violate this property, we believe it to be fairly natural in the current context. After all, if two objects x and y are considered to be distinct for the sake of the analysis performed, they should be considered to have some degree of "dissimilarity".

We let $\mathbb{Q}$ denote the set of all quaternary relations that satisfy these properties. We record the obvious following fact (whose proof is omitted).

Fact 1 *If Q is a dissimilar quaternary relation in $\mathbb{Q}$, then, for all distinct x and $y \in X$, and for all $z \in X$, $(x,y)\ Q_A\ (z,z)$*

Given a dissimilarity quaternary relation Q in $\mathbb{Q}$ and a set $P \subseteq X \times X$ of pairs of objects of X, we denote by $O^Q(P)$ the arrangement of the pairs in P in *increasing order* of similarity. That is $O^Q(P) = \{a_{(1)}, ..., a_{(|P|)}\}$ where, for every $i = 1, ..., |P| - 1$, $a_{(i)} \in P$, $a_{(i+1)} \in P$ and $a_{(i)}\ Q\ a_{(i+1)}$. Since Q is symmetric, there is some arbitrariness in numbering the elements of $O^Q(P)$ as the order of appearance of any two symmetric pairs (x, y) and (y, x) is irrelevant. Moreover, for any set $A \in \mathfrak{P}(X)$, it can be noticed that the set $A \times A$ of all (ordered) pairs that can be formed with the elements of A (including the "pairs" made of the duplications of each element of A), has $|A|^2$ elements. Thank to fact 1, the $|A|$ last elements of the set $O^Q(A \times A)$ are precisely these duplicated pairs while the pair $a_{(|A|(|A|-1))} \in O^Q(A \times A)$ is the pair of distinct elements which are the most similar as per the quaternary relation Q.

Let now $\succeq$ (with asymmetric and symmetric factors $\succ$ and $\sim$ respectively) be a *transitive* binary relation on $\mathfrak{P}(X)$ that aims at comparing the *diversity* offered by alternative sets of objects in $\mathfrak{P}(X)$. We interpret accordingly the statement $A \succeq B$ as meaning "set A offers at least as much diversity as set B".

We wish to propose plausible properties (axioms) that $\succeq$ could satisfy in order to serve as a sensible method for appraising diversity, taking as given the ordinal notion of dissimilarity embodied in Q. The three following axioms are examined in this paper.

Axiom 1 $\forall\ w, x,\ y,\ z\ \in X,\ \ (w,\ z)\ Q\ (x,\ y) \Longleftrightarrow \{w,\ z\} \succeq \{x,\ y\}$.

Axiom 2 $\forall\ A,\ B \in \mathfrak{P}(X)$, *if* $A \supseteq B$, *then* $A \succeq B$.

Axiom 3 $\forall\ A,\ B, C$ *and* $D \in \mathfrak{P}(X)$ *such that* $B \cap C = B \cap D = C \cap D = \emptyset$, $[A \succeq B \cup C,\ A \succeq B \cup D$ *and* $A \succeq C \cup D] \Longrightarrow [A \succeq B \cup C \cup D]$ *and* $[A \succ B \cup C,\ A \succ B \cup D$ *and* $A \succ C \cup D] \Longrightarrow [A \succ B \cup C \cup D]$.

Axiom 1 just says that the ranking of sets made of two elements in terms of diversity must *coincide* with the ranking of the pairs in terms of dissimilarity as *per* the quaternary relation Q. It is difficult to imagine a diversity ranking of sets based on an *a priori* notion of dissimilarity between options that would violate this axiom. Notice carefully that the formal statement of axiom 1 does *not* require the options w, x, y and z to be distinct. Hence, when employed with a quaternary relation belonging to $\mathbb{Q}$, axiom 1 implies the widely discussed (at least in the freedom of choice literature) axiom of *indifference to no choice situations* first introduced by Pattanaik and Xu (1990).

Axiom 2 is also well-known in the freedom of choice literature and is very natural in that context. It seems also plausible in the context of diversity measurement although perhaps not as much. At first sight, it is indeed difficult to imagine a plausible conception of diversity that would consider that *adding* an object to a set could *reduce* its diversity. After all, if the added object is considered different, as an object, from those already contained in the set, how could its addition reduce the diversity of the world ?

Yet the weighted entropy indices used by biologists violate this axiom and, at second sight, one can see how a plausible "relativist" conception of diversity could, in some circumstances, contradict the partial ranking of sets provided by inclusion. Suppose a world in which the population of living individuals is equally split between species 1 and 2. Consider now adding to this population a large number of living individuals of species 1 in such a way that the *ratio* of individuals from species 2 over those of species 1 becomes negligible. A "relativist" conception of biological diversity according to which diversity is maximized when all living individuals are equally split-up among the different categories could plausibly consider such a change as a reduction in diversity. To that extent, the rankings characterized in this paper are not relativist as they both satisfy axiom 2. They share this property with all rankings which, like those of Pattanaik and Xu (2000), Weitzman (1992), Weitzman (1993) or Weitzman (1998), view the diversity of a set as the aggregation of the dissimilarity of the pairs of its elements as per an *a priori* given notion of dissimilarity.

Axiom 3 is, perhaps, more disputable than the two preceding ones but is not unreasonable. It requires roughly that the domination of a set by another be robust to the addition, in the dominated set, of options when the options added are themselves dominated in terms of diversity. Specifically, axiom 3 requires that if adding *separately* objects in sets C and D to a set B is insufficient to reverse the domination of this set B by some set A, then adding *jointly* the objects in C and D to B should also be insufficient to reverse the domination if the diversity offered by A is deemed larger than that offered by C and D.

A ranking of sets that satisfies this "robustness of domination" property is the maxi-max criterion that ranks sets according to the dissimilarity of their two most dissimilar objects. As revealed by theorem 1 below, the maxi-max criterion happens to be the only transitive ranking of sets satisfying axioms 1 and 2 which possesses this property. The maxi max criterion, denoted $\succeq_{\max}$ is formally defined as follows.

Definition 1 *For all A, $B \in \mathfrak{P}(X)$, $A \succeq_{\max} B \iff a_{(1)}\ Q\ b_{(1)}$ for $a_{(1)} \in O^Q(A \times A)$ and $b_{(1)} \in O^Q(B \times B)$.*

To illustrate, suppose that X is the set of conceivable transportations mode between two cities defined as $X = \{bike,\ car, foot,\ train\}$. Assume also that the ordinal notion of dissimilarities between these transportation modes is as in the following table.

rank	pair
1	$(foot, train)$
2	$(car, foot)$
3	$(bike, train)$
4	$(bike, car)$
5	$(car, train)$
6	$(bike, foot)$

In this case the maxi-max criterion would generate the following ranking of all conceivable sets of transportation modes (excluding singletons which are obviously ranked last thanks to fact 1).

rank	sets
1	$\{foot, train\}, \{car, foot, train\}, \{bike, foot, train\}, X$
2	$\{car, foot\}, \{bike, car, foot\}$,
3	$\{bike, train\}, \{bike, car, train\}$
4	$(bike, car)$
5	$\{car, train\}$
6	$\{bike, foot\}$

This ranking is clearly extreme. It ranks the set $\{foot, train\}$ (made of the two most dissimilar modes of transportation) as being weakly more diverse than any other set. Some of these verdicts are quite plausible. For instance, it is not unreasonnable to consider that $\{foot, train\}$ offers more diversity than, say, the set $\{bike, car, train\}$ since, according to the notion of dissimilarity Q, there is more dissimilarity between taking a train and walking than between any pair of alternatives contained in the set $\{bike, car, train\}$. However, it is somewhat counterintuitive to consider, as does the maxi-max criterion, that the set $\{train,\ car,\ bike,\ foot\}$ of all conceivable modes of transportation offers no more diversity than the pair $\{foot, train\}$. The biggest weakness of the maxi-max criterion is obviously to focus only on the two most dissimilar objects in the sets and to ignore completely the contribution to diversity made by the presence of more similar objects.

A partial way out of this problem would be to consider a *lexicographic extension* of the maxi-max criterion. Such an extension would rank sets on the basis of the dissimilarity of their most dissimilar objects, just like the maxi-max criterion, in the case of a strict ranking of these. However, in the case of a tie in the dissimilarity ranking of the two most dissimilar objects, the lexicographic extension would switch to the second most dissimilar pair objects and, if there is also a tie there, to the third most dissimilar two objects and so on. The proposed lexicographic extension would avoid some of the pitfalls of the maxi-max criterion by considering the set X to be strictly more diverse than the set $\{train, foot\}$ (and more generally, by considering any set to be strictly more diverse than any of its proper subset). Yet, as its maxi-max cousin, the lexicographic extension would have the somewhat unpleasant feature of giving a "veto-power" to the two most dissimilar options of a set. A large set whose two most dissimilar objects are not maximally dissimilar would be considered less diverse than the simple pair made of the two most dissimilar objects in the universe by either the maxi-max criterion or its lexicographic extension.

While this paper focuses on the axiomatic characterisation of the maxi-max criterion, a characterisation of its lexicographic extension can be found in Bervoets and Gravel (2004).

3 Results for diversity measurement

We provide the characterization of $\succeq_{\max}$ by means of axioms 1-3.

Theorem 1 *Let $\succeq$ be a transitive binary relation defined on $\mathfrak{P}(X)$ and let Q be an ordinal notion of similarity belonging to $\mathbb{Q}$. Then $\succeq$ satisfies Axioms 1 to 3 if and only if $\succeq \;=\; \succeq_{\max}$.*

Proof. *It is immediate to see that the transitive binary relation $\succeq_{\max}$ satisfies axioms 1 and 2. As for axiom 3, suppose that $A \succeq_{\max} B \cup C = E$, $A \succeq_{\max} B \cup D = F$ and $A \succeq_{\max} C \cup D = G$ and let $H = B \cup C \cup D$. Then $a_{(1)}\ Q\ e_{(1)}$, $a_{(1)}\ Q\ f_{(1)}$ and $a_{(1)}\ Q\ g_{(1)}$ for $a_{(1)}$, $e_{(1)}$, $f_{(1)}$ and $g_{(1)}$ denoting, respectively, the first element of the sets $O^Q(A \times A)$, $O^Q(E \times E)$, $O^Q(F \times F)$ and $O^Q(G \times G)$. We therefore have $a_{(1)}\ Q\ \max_Q(e_{(1)}, f_{(1)}, g_{(1)}) = \max_Q H \times H$ and, therefore, $A \succeq_{\max} H$.*
We now show that if $\succeq$ is transitive and satisfies axioms 1 to 3, then we have, for every A and $B \in \mathfrak{P}(X)$, $A \succeq B \Longrightarrow A \succeq_{\max} B$. Suppose $A \succeq B$ and let $a_{(1)} = (a_1, a_2)$ and $b_{(1)} = (b_1, b_2)$ denote the most dissimilar pairs of objects in A and B respectively. By axiom 2, we have $B \succeq \{b_1, b_2\}$ and by transitivity, $A \succeq \{b_1, b_2\}$.
In the trivial case where $|B| = |A| = 1$, we can write that $A = \{x\}$ and $B = \{y\}$ for some options x and y so that $a_{(1)} = (x, x)$ and $b_{(1)} = (y, y)$. Since $(x, x)\ Q_S\ (y, y)$ for every x, $y \in X$, we therefore have $a_{(1)}\ Q\ b_{(1)}$ and $A \succeq_{\max} B$ in this case. We can rule out the case where $|A| = 1$ and $|B| \geq 2$ which would imply that $\{x\} \succeq \{b_1, b_2\}$ for some distinct b_1 and $b_2 \in B$, in contradiction with axiom 1 and fact 1.
Assume now that $|A| = 2$ and, therefore, that $A = \{a_1, a_2\}$. Then $\{a_1, a_2\} \succeq \{b_1, b_2\}$ and, by axiom 1, $a_{(1)}\ Q\ b_{(1)}$, which implies $A \succeq_{\max} B$.
For the last case, assume that $|A| > 2$, write $A = \{a_1, ..., a_{|A|}\}$ and assume by contradiction that $a_{(1)}\ Q\ b_{(1)}$ is false. Since Q is complete, this amounts to assuming that $b_{(1)}\ Q_A\ a_{(1)}$ and, therefore, that $b_{(1)}\ Q_A\ (a_i, a_j)$ holds for all i, $j \in \{1, ..., |A|\}$. Pick any option a_1 in A. By axiom 1, one has $\{b_1, b_2\} \succ \{a_1, a_i\}$, $\{b_1, b_2\} \succ \{a_1, a_j\}$ and $\{b_1, b_2\} \succ \{a_i, a_j\}$ for all i, $j \in \{1, ..., |A|\}$. By axiom 3, we must have $\{b_1, b_2\} \succ \{a_1, a_i, a_j\}$. Redoing the same procedure while replacing the option a_j by some option $a_h \in A$, one obtains that $\{b_1, b_2\} \succ \{a_1, a_i, a_h\}$. Using axiom 3 again and the fact that $\{b_1, b_2\} \succ \{a_j, a_h\}$, one is led to the conclusion that $\{b_1, b_2\} \succ \{a_1, a_h, a_i, a_j\}$. Redoing the last procedure if necessary while replacing a_h by $a_g \in A$, one can analogously obtain the statement $\{b_1, b_2\} \succ \{a_1, a_g, a_i, a_j\}$ and combining the last two statements and the fact that $\{b_1, b_2\} \succ \{a_g, a_h\}$, one obtains again by axiom 3 that $\{b_1, b_2\} \succ \{a_1, a_g, a_h, a_i, a_j\}$. This procedure can clearly be repeated with as many options in A as needed to finally obtain, using transitivity of $\succeq$ and axiom 2, the required contradictory conclusion that $B \succeq \{b_1, b_2\} \succ A$.

We end the proof by showing that for every sets A and B in $\mathfrak{P}(X)$, $A \succeq_{\max} B$ implies $A \succeq B$ for every transitive binary relation $\succeq$ on $\mathfrak{P}(X)$ satisfying axioms 1 to 3. Suppose $A \succeq_{\max} B$. Then $a_{(1)}\ Q\ b_{(1)}$ where again $a_{(1)} = (a_1, a_2)$ and $b_{(1)} = (b_1, b_2)$ denote the most dissimilar pairs of objects in A and B respectively. Let $|B| = m$ and write $B = \{b_1, b_2, ..., b_m\}$. For the same reason as above, we can rule out from the start the case $m = 1$. If $m = 2$, then, by axiom 1, $\{a_1, a_2\} \succeq \{b_1, b_2\}$ and, by axiom 2, $A \succeq \{a_{(1)}\}$, so that, by transitivity, $A \succeq B$. For the other cases, we show the result by induction. For that purpose, we start with the case $m = 3$ and we write $B = \{b_1, b_2, b_3\}$. Because $a_{(1)}\ Q\ b_{(1)}$, we have $a_{(1)}$ Q (b_1, b_2), $a_{(1)}\ Q\ (b_1, b_3)$ and $a_{(1)}\ Q\ (b_2, b_3)$. Using axiom 1, we can write $\{a_1, a_2\} \succeq \{b_1\} \cup \{b_2\}$, $\{a_1, a_2\} \succeq \{b_1\} \cup \{b_3\}$ and $\{a_1, a_2\} \succeq \{b_2\} \cup \{b_3\}$. By axiom 3, it follows that $\{a_1, a_2\} \succeq \{b_1\} \cup \{b_2\} \cup \{b_3\} = B$ and, by axiom 2 and transitivity, that $A \succeq B$. The case $m = 3$ is then proved.
Suppose now that the result is true for any $m \in \{3, ..., |X| - 1\}$. That is, suppose that if A is a set in $\mathfrak{P}(X)$ and B is another set in $\mathfrak{P}(X)$ such that $|B| = m$, then $A \succeq_{\max} B \Longrightarrow A \succeq B$ and suppose $A \succeq_{\max} B'$ where $B' = B \cup \{b_{m+1}\}$ for some $b_{m+1} \in X \backslash B$. We wish to show that $A \succeq B'$ Let $b'_{(1)}$ denote the pair of two most dissimilar objects in B' and write $\overline{B} = \{b_1, b_2, ..., b_{m-1}\}$, $C = \{b_m\}$ and $D = \{b_{m+1}\}$. As $\succeq_{\max}$ is transitive and satisfies axiom 2, we have that $A \succeq_{\max} \overline{B} \cup C$ and, since $\left|\overline{B} \cup C\right| = m$, we have, by the induction hypothesis, that $A \succeq \overline{B} \cup C$. Because $a_{(1)}\ Q\ b'_{(1)}$, we have, by the transitivity of the quaternary relation Q, $a_{(1)}\ Q\ (b_m, b_{m+1})$ and, by virtue of axioms 1 and 2 and the transitivity of $\succeq$, $A \succeq C \cup D$. Finally, let $B'' = \overline{B} \cup D$. Then $b'_{(1)}\ Q\ b''_{(i)}$ and, by transitivity of Q, $a_{(1)}\ Q\ b''_{(i)}$ for all $b''_{(i)} \in O^Q(B'' \times B'')$. We therefore have $A \succeq_{\max} B''$. Yet $|B''| = m$ so that, by the induction hypothesis, we have $A \succeq B''$. By axiom 3, we have $A \succeq \overline{B} \cup C$, $A \succeq \overline{B} \cup D$ and $A \succeq C \cup D$, so that $A \succeq \overline{B} \cup C \cup D$, and this concludes the proof. ■

We first remark that we obtain the completeness of the ranking as a by-product of axioms 1 to 3. It is also worth noticing that this first characterization of $\succeq_{\max}$ is obtained from the (reasonably) intuitive axioms 1 to 3 that only uses properties of sets and *elements*. Only axiom 1 makes the connection between the ranking of pairs in terms of dissimilarity and the ranking of sets.

We now show that axioms 1, 2 and 3 used to characterize $\succeq_{\max}$ are independent.

Proposition 1 *For any $Q \in \mathbb{Q}$, Axioms 1 to 3 are independent.*

Proof. *Let $\succeq_*$ be defined by: $A \succeq_* B \Longleftrightarrow b_{(|B|(|B|-1))}\ Q\ a_{(|A|(|A|-1))}$ where $b_{(|B|(|B|-1))} \in O^Q(B \times B)$ and $a_{(|A|(|A|-1))} \in O^Q(A \times A)$. This tran-*

sitive and complete binary relation on $\mathfrak{P}(X)$ considers that set A offers at least as much diversity as B if and only if the two most similar distinct *objects in B are weakly more* dissimilar *than the two most similar distinct objects in A. It is certainly a peculiar criterion for comparing sets on the basis of their diversity. It is immediate to see that $\succeq_*$ violates axiom 1. To see that it satisfies axiom 2, consider A and B in $\mathfrak{P}(X)$ such that $A \supseteq B$. As the two most similar objects in B are also in A, we must have the two most similar objects in A are weakly more similar than the two most similar objects in B. Hence the two most similar objects in B are weakly more dissimilar than the two most similar objects in A and, for this reason, one has $A \succeq_* B$. To see that $\succeq_*$ satisfies axiom 3, assume that $A \succeq_* B \cup C$, $A \succeq_* B \cup D$ and $A \succeq_* C \cup D$. Write $E = B \cup C$, $F = B \cup D$ and $G = C \cup D$. One has, by definition of $\succeq_*$, that:*

$$e_{(|B \cup C|(|B \cup C|-1))} \ Q \ a_{(|A|(|A|-1))},$$

$$f_{(|B \cup D|(|B \cup D|-1))} \ Q \ a_{(|A|(|A|-1))}$$

and

$$g_{(|C \cup D|(|C \cup D|-1))} \ Q \ a_{(|A|(|A|-1))}$$

where $e_{(|B \cup C|(|B \cup C|-1))} \in O^Q(E \times E)$, $f_{(|B \cup D|(|B \cup D|-1))} \in O^Q(F \times F)$ and $g_{(|C \cup D|(|C \cup D|-1))} \in O^Q(G \times G)$. Let now $H = B \cup C \cup D$ and consider $h_{(|B \cup C \cup D|(|B \cup C \cup D|-1))} \in O^Q(H \times H)$. Clearly, since either

$$\begin{aligned} h_{(|B \cup C \cup D|(|B \cup C \cup D|-1))} &= e_{(|B \cup C|(|B \cup C|-1))}, \\ h_{(|B \cup C \cup D|(|B \cup C \cup D|-1))} &= f_{(|B \cup D|(|B \cup D|-1))}, \ \textit{or} \\ h_{(|B \cup C \cup D|(|B \cup C \cup D|-1))} &= g_{(|C \cup D|(|C \cup D|-1))} \end{aligned}$$

one has

$$h_{(|B \cup C \cup D|(|B \cup C \cup D|-1))} \ Q \ a_{(|A|(|A|-1))}$$

and, therefore, $A \succeq_* B \cup C \cup D$.
Now let $\succeq_d$ be defined by

$$A \ \succeq_d B \Longleftrightarrow a_{(|A|(|A|-1))} \ Q \ b_{(1)}$$

where $a_{(|A|(|A|-1))} \in O^Q(A \times A)$ and $b_{(1)} \in O^Q(B \times B)$. This rule says that set A offers at least as much diversity as B if and only if the two most similar distinct *objects in A are at least as dissimilar as the two most dissimilar distinct objects in B. It is immediate to see that $\succeq_d$ satisfies axioms 1 and is transitive. To see that it verifies axiom 3, assume that $A \succeq_d (B \cup C) = E$, $A \succeq_d (B \cup D) = F$ and $A \succeq_d C \cup D = G$. Then*

$$a_{(|A|(|A|-1))} \ Q \ e_{(1)},$$

$$a_{(|A|(|A|-1))} \ Q \ \ f_{(1)}$$

and

$$a_{(|A|(|A|-1))}\ Q\ \ g_{(1)}$$

from which, it follows that:

$$a_{(|A|(|A|-1))}\ Q\ h_{(1)}$$

where $H = B \cup C \cup D$ *and* $e_{(1)}$, $f_{(1)}$ $g_{(1)}$ *and* $h_{(1)}$ *denote, respectively, the first elements of the sets* $O^Q(E \times E)$, $O^Q(F \times F)$, $O^Q(G \times G)$ *and* $O^Q(H \times H)$. *To see that* $\succeq_d$ *violates axiom 2, let* $B = \{b_1,\ b_2\}$ *and* $A = \{b_1,\ b_2,\ b_3\}$ *and assume that* Q *is such that* $(b_1,\ b_2)\ Q_A\ (b_1,\ b_3)\ Q\ (b_2,\ b_3)$. *Clearly,* $b_{(1)} = (b_1,\ b_2)$, $b_{(6)} = (b_2, b_3)$ *and, since* $(b_2, b_3)\ Q\ (b_1,\ b_2)$ *does not hold,* $A \succeq_d B$ *does not hold either.*
Finally, let $\succeq_{add}$ *be defined by:*

$$A \succeq_{add} B \Longleftrightarrow \sum_{i=1}^{|A|^2} v(a_{(i)}) \geq \sum_{i=1}^{|B|^2} v(b_{(i)})$$

for some function $v \in X \times X \to \mathbb{R}_+$ *such that, for all* (w, z), $(x, y) \in X \times X$, $v(w, z) \geq v(x, y) \Leftrightarrow (w, z)\ Q\ (x, y)$. *The existence of such a (distance) real valued function representing* Q *is guaranteed by Debreu (1954) theorem. The binary relation* $\succeq_{add}$ *is reflexive, transitive and complete and satisfies axioms 1 and 2. Yet, it may violate axiom 3 if, for instance,* $X = \{w, x, y, z\}$ *and* v *is such that* $v(w, z) = 7$, $v(w, y) = 5$, $v(w, x) = 3 = v(x, y)$. *In such a case, defining* $A = \{w, z\}$, $B = \{w\}$, $C = \{y\}$ *and* $D = \{x\}$, *one has* $A \succeq_{add} B \cup C \Leftrightarrow 7 \geq 5$, $A \succeq_{add} B \cup D \Leftrightarrow 7 \geq 3$ *and* $A \succeq_{add} C \cup D \Leftrightarrow 7 \geq 3$. *Yet* $A \prec_{add} B \cup C \cup D$ *as* $v(w, z) = 7 < v(w, y) +\ v(w, x) + v(x, y) = 11$. ■

4 Diversity and freedom of choice

The diversity of options available for choice to a decision maker can arguably be seen as an essential element of his or her *freedom of choice*. Yet most rankings of opportunity sets examined in the freedom of choice literature mentioned in introduction have not exhibited a great sensitivity to diversity. In this section, we briefly explore the extent to which the methodology used in this paper can serve to bridge the gap between concerns for diversity and concerns for freedom.

There are roughly two approaches to the issue of defining freedom of choice in the literature.

In the *first* approach, freedom of choice is conceived as an *intrinsic* criterion for comparing opportunity sets, roughly related to the "size" of the opportunity sets, and whose importance is, using the words of Sen (1988) (p. 290), "beyond that of providing the means of choosing the particular alternative that happens to be chosen". Hence, in this approach, the freedom of choice offered by a particular opportunity set is conceived as being completely independent from the preferences that the decision maker will use for choosing from that set. The widely discussed *cardinality ranking* of sets (characterized differently by Jones and Sugden (1982), Pattanaik and Xu (1990) and Suppes (1987)) as well as their additive generalization (see e.g. Gravel, Laslier and Trannoy (1998) or Klemisch-Ahlert (1993)) belong clearly to this approach, as do the definition of freedom as entropy in Suppes (1996) or the examination, made by VanHees (1997), of the distinction between negative and positive freedom.

In the *second* approach, freedom of choice is defined with respect to a set of possible preferences that the decision maker could have when making its choice. In this approach, freedom of choice is important only in so far as it enables the decision maker to make better choice from the view point of some of the possible preference that he may use when making the choice. For this reason, when evaluating the freedom of choice offered by some opportunity set, this approach attaches a particular attention to the set of options which would be considered best in this set from the view point of some of the possible preference of the decision maker. We refer the reader to Arrow (1995), Barberà et al. (2004), Foster (1993), Jones and Sugden (1982), Nehring and Puppe (1999), Pattanaik and Xu (1998), Puppe (1998), Romero-Medina (2001) and Sugden (1998) for further justification of this *multi-preferences approach* to freedom of choice.

The methodology presented in this paper is directly relevant for the first approach if one accepts the view that the diversity of options contained in a particular opportunity set is a natural measure of the freedom of choice offered by that opportunity set. And there is some rationale for this view. After all someone who has only the choice between two slightly different cars for commuting from home to work can arguably be considered to have less freedom of choice - in terms of means of transportation - than an individual who can go to work either by one car or by a suburban train. Hence it is quite possible to interpret the *maxi-max* criterion and its lexicographic extension as freedom of choice rankings rather than diversity ones. Of course the acceptability of the rankings, both from a diversity or a freedom of choice perspective, rides upon the acceptability of the underlying dissimilarity quaternary relation that is taken as given.

But diversity can also contribute to defining freedom of choice in the context of the multi-preference approach. To see how, adopt Pattanaik and Xu (1998) framework and let $\mathfrak{R} = \{R_1, ..., R_n\}$ be the set of all possible preference orderings over X that a "reasonable person" may have. For all $i = 1, ..., n$, let P_i and I_i denote, respectively, the asymmetric factor and the symmetric factor of R_i. In this setting, the binary relation $\succeq$ on $\mathfrak{P}(X)$ is explicitly interpreted in terms of freedom of choice rather than of diversity.

For all $A \in \mathfrak{P}(X)$, let $Max_{\mathfrak{R}}A = \{a \in A : \exists\ R_i \in \mathfrak{R}$ for which $a\ R_i\ a'$ $\forall\ a' \in A\}$ be the set of all options in A that are maximal for some of the possible preferences in $\mathfrak{R}$.

Taking as given the set $\mathfrak{R}$, Pattanaik and Xu (1998) characterize the ranking of all sets in $\mathfrak{P}(X)$ defined by the comparison of the *cardinality* of their sets of elements which are *maximal* from the view point of at least one preference in $\mathfrak{R}$. Formally, this ranking $\succeq_{card}^{\mathfrak{R}}$ is defined by $A \succeq_{card}^{\mathfrak{R}} B \Longleftrightarrow |Max_{\mathfrak{R}} A| \geq |Max_{\mathfrak{R}} B|$. In this paper, taking as given *both* the set of possible preferences *and* the primitive notion of dissimilarity between options Q, we provide a characterization of the ranking $\succeq_{\max}^{\mathfrak{R}}$ defined as follows.

Definition 2 $A \succeq_{\max}^{\mathfrak{R}} B \Longleftrightarrow Max_{\mathfrak{R}} A \succeq_{\max} Max_{\mathfrak{R}} B$

Hence the ranking $\succeq_{\max}^{\mathfrak{R}}$ considers that opportunity set A offers at least as much freedom of choice as opportunity set B if and only if the set of elements in A that any reasonable person would choose is at least as diverse, in the sense of the ordering $\succeq_{\max}$ of definition 1, than the set of options in B that any reasonable person would choose. This ranking provides therefore an alternative to the ranking $\succeq_{card}^{\mathfrak{R}}$ of Pattanaik and Xu (1998) which, contrary to the later, attaches intrinsic importance to the diversity of the options that a reasonable person could choose.

The characterization that we provide of $\succeq_{\max}^{\mathfrak{R}}$ uses the following axioms.

Axiom 4 *$\forall$ A, $B \in \mathfrak{P}(X)$, $\forall$ $x \in X$, if $x \notin Max_{\mathfrak{R}} A \cup \{x\}$, then $[A \succeq B \Longleftrightarrow A \cup \{x\} \succeq B]$ and $[B \succeq A \Longleftrightarrow B \succeq A \cup \{x\}]$.*

Axiom 5 *$\forall$ w, x, y and $z \in X$, if $\{w, z\} = Max_{\mathfrak{R}} \{w, z\}$ and $\{x, y\} = Max_{\mathfrak{R}} \{x, y\}$, then $(w,\ s)\ Q\ (x,\ y) \Longleftrightarrow \{w, z\} \succeq \{x, y\}$.*

Axiom 6 *$\forall$ A, $B \in \mathfrak{P}(X)$, if $B \subseteq Max_{\mathfrak{R}} A$, then $A \succeq B$.*

Axiom 4 has been introduced in Pattanaik and Xu (1998). It requires that if x is an option that no reasonable preference in $\mathfrak{R}$ would consider choosing against the options available in a set A, then the ranking of A with respect to B should not be affected by the addition of x to A.

Axiom 5 is a weakening of axiom 1. Like axiom 1, axiom 5 requires the ranking of sets that are made of two elements, each of which being a best choice over the other by some of the possible preferences in $\mathfrak{R}$, to coincide with the dissimilarity ranking of the pair made of these two elements as per the quaternary relation Q. On the other hand, and contrary to axiom 1, axiom 5 does *not* require the coincidence of dissimilarity comparisons and sets comparisons for pairs in which one element - say the fact of being beheaded at dawn - is considered worse than the other by all preferences in $\mathfrak{R}$. It should be noticed that, as for axiom 1, the formal statement of axiom 5 does not require the options to be distinct. Hence, since $Max_{\mathfrak{R}}\{x\} = \{x\}$ for all $x \in X$, axiom 5 implies also Pattanaik and Xu (1990)'s axiom of *indifference to no-choice situations.*

Axiom 6 is a weakening of axiom 2 which considers that a set A offers weakly more freedom of choice than any subset of the sets of options which, in A, could be considered best by some of the preferences in $\mathfrak{R}$.

These axioms, along with axiom 3, characterize the ordering $\succeq^{\mathfrak{R}}_{\max}$, as established in the following theorem.

Theorem 2 *Let $\succeq$ be a transitive binary relation on $\mathfrak{P}(X)$. Then $\succeq$ satisfies axioms 3, 4, 5 and 6 for a given dissimilarity quaternary relation $Q \in \mathbb{Q}$ and a given set $\mathfrak{R}$ of possible preference orderings if and only if $\succeq \; = \; \succeq^{\mathfrak{R}}_{\max}$.*

Proof. *We leave to the reader the task of verifying that $\succeq^{\mathfrak{R}}_{\max}$ is transitive and satisfies axioms 3, 4, 5 and 6.*

We seek to establish that, for all transitive binary relations $\succeq$ on $\mathfrak{P}(X)$, $A \sim^{\mathfrak{R}}_{\max} B \Longrightarrow A \sim B$ and $A \succ^{\mathfrak{R}}_{\max} B \Longrightarrow A \succ B$. Given the completeness of the relation $\succeq^{\mathfrak{R}}_{\max}$, this suffices to prove the result.

For every sets A and $B \in \mathfrak{P}(X)$, let us write $A = C \cup Max_{\mathfrak{R}}A$ and $B = D \cup Max_{\mathfrak{R}}B$ where $C = \{c_1, ..., c_l\} = A \backslash \; Max_{\mathfrak{R}}A$ and $D = \{d_1, ..., d_m\} = B \backslash Max_{\mathfrak{R}}B$. We also write $Max_{\mathfrak{R}}A = \{a_1, ..., a_g\}$ and $Max_{\mathfrak{R}}B = \{b_1, ..., b_h\}$. We recall that, as the elements of $\mathfrak{R}$ are orderings, neither $Max_{\mathfrak{R}}A$ nor $Max_{\mathfrak{R}}B$ is empty while either C or D can be empty.

Assume first that $A \sim^{\mathfrak{R}}_{\max} B$ and, therefore, that $Max_{\mathfrak{R}}A \sim_{\max} Max_{\mathfrak{R}}B$ and consider first the case where $|Max_{\mathfrak{R}}A| = |Max_{\mathfrak{R}}B| = 1$. By axiom 5, we must then have $Max_{\mathfrak{R}}A \sim Max_{\mathfrak{R}}B$. By axiom 4, $(Max_{\mathfrak{R}}A) \cup \{c_1\} \sim Max_{\mathfrak{R}}A$. Using axiom 4 repeatedly with as many elements in C as necessary, we obtain $A \sim Max_{\mathfrak{R}}A$. Analogously, using axiom 4 with set B, we obtain $B \sim Max_{\mathfrak{R}}B$ and, by the transitivity of $\succeq$, $A \sim B$.

We cannot have $|Max_{\mathfrak{R}}A| > 1$ and $|Max_{\mathfrak{R}}B| = 1$ or $|Max_{\mathfrak{R}}A| = 1$ and $|Max_{\mathfrak{R}}B| > 1$ when $A \sim^{\mathfrak{R}}_{\max} B$ because of the fact that $(x, y) \; Q_A \; (z, z)$ for every x, y and z with x and y distinct. Consider therefore the case where $|Max_{\mathfrak{R}}A| > 1$ and $|Max_{\mathfrak{R}}B| > 1$. Since $Max_{\mathfrak{R}}A \sim_{\max} Max_{\mathfrak{R}}B$, we have $a_{(1)} \; Q_S \; b_{(1)}$ where $a_{(1)} = (a_1, a_2)$ and $b_{(1)} = (b_1, b_2)$ are the most dissimilar pair in $Max_{\mathfrak{R}}A$ and $Max_{\mathfrak{R}}B$ respectively. By axiom 5, $\{a_1, a_2\} \sim \{b_1, b_1\}$. For every $a_i \in Max_{\mathfrak{R}}A$, we have $\{a_1, a_2\} \succeq \{a_1, a_i\}$ and $\{a_1, a_2\} \succeq \{a_2, a_i\}$. Furthermore, $\{a_1, a_2\} \succeq \{a_1, a_2\}$ so that we can use axiom 3 to obtain $\{a_1, a_2\} \succeq \{a_1, a_2, a_i\}$. We can use the same procedure to add all remaining elements from $Max_{\mathfrak{R}}A$, until we have $\{a_1, a_2\} \succeq Max_{\mathfrak{R}}A$. Now $\{a_1, a_2\} \subseteq Max_{\mathfrak{R}}A$ so that, by axiom 6, $Max_{\mathfrak{R}}A \succeq \{a_1, a_2\}$ and, therefore, $\{a_1, a_2\} \sim Max_{\mathfrak{R}}A$. Applying the same treatment to B gives us $\{b_1, b_2\} \sim Max_{\mathfrak{R}}B$. By transitivity, we have $Max_{\mathfrak{R}}A \sim Max_{\mathfrak{R}}B$. Repeated use of axiom 4 guarantees, as in the first case, that $A \sim Max_{\mathfrak{R}}A$ and $B \sim Max_{\mathfrak{R}}B$, which in turn, by transitivity, gives the result.

Assume now that $A \succ^{\mathfrak{R}}_{\max} B$ and, therefore, that $Max_{\mathfrak{R}}A \succ_{\max} Max_{\mathfrak{R}}B$. Using the same notation as above for $a_{(1)} = (a_1, a_2)$ and $b_{(1)} = (b_1, b_2)$, this means that $a_{(1)} \; Q_A \; b_{(1)}$ which implies, by axiom 5, that $\{a_1, a_2\} \succ$

$\{b_1, b_2\}$. *As before, for every* $a_i \in Max_{\mathfrak{R}}A$, *we have* $\{a_1, a_2\} \succeq \{a_1, a_i\}$, $\{a_1, a_2\} \succeq \{a_2, a_i\}$ *and* $\{a_1, a_2\} \succeq \{a_1, a_2\}$ *so that, by axiom 3, we obtain* $\{a_1, a_2\} \succeq \{a_1, a_2, a_i\}$. *Repeating the argument with all elements of* $Max_{\mathfrak{R}}A$, *we obtain* $\{a_1, a_2\} \succeq Max_{\mathfrak{R}}A$ *and, using axiom 6,* $\{a_1, a_2\} \sim Max_{\mathfrak{R}}A$. *As the same treatment can be applied to* B, *we are led by transitivity to the statement* $Max_{\mathfrak{R}}A \succ Max_{\mathfrak{R}}B$. *Finally, and just in the same way as before, a repeated use of axiom 4 will give us* $A \succ B$, *as needed.* ■

While $\succeq^{\mathfrak{R}}_{\max}$ provides a method for evaluating freedom of choice in the multi-preference approach that incorporates a concern for diversity, it is worth mentioning that, as its cousin $\succeq^{\mathfrak{R}}_{card}$, it does *not* satisfy the full-fledged weak monotonicity with respect to set inclusion as expressed in axiom 2. As this property appears to be a very minimal requirement to impose on a ranking of opportunity sets based on freedom of choice (which conception of freedom could say that making available for choice an option reduces freedom of choice ?), we believe that the violation of this axiom by both $\succeq^{\mathfrak{R}}_{card}$ and $\succeq^{\mathfrak{R}}_{\max}$ limits somehow the usefulness of these rankings as appropriate measures of freedom of choice.

We conclude this section by establishing, in the following proposition, that axioms 3, 4, 5 and 6 are independent.

Proposition 2 *For any* $Q \in \mathbb{Q}$, *and for any given set* $\mathfrak{R}$ *of possible preference orderings on* X, *axioms 3, 4, 5 and 6 are independent.*

Proof. *One can show the existence of transitive binary relations on* $\mathfrak{P}(X)$ *satisfying axiom 4 and alternative combinations of pairs of the axioms 3, 5 and 6 by modifying suitably the three binary relations considered in the proof of proposition 1 in such a way as to require the condition that define them to apply only to the alternatives which are maximal in their sets from the view point of some preferences in* $\mathfrak{R}$. *As for a transitive binary relation that satisfies 3, 5 and 6 but which violates 4, consider the ordering* $\succeq_{\max}$. *It is immediate to see that it satisfies 3, 5 and 6. It may violate 4 if there exists an option* x *and sets* A *and* B *such that* $A \succeq_{\max} B$, $x \notin Max_{\mathfrak{R}}(B \cup \{x\})$, (x, b) Q (b', b'') *for all* $(b', b'') \in P(B \cup \{x\})$ *and some* $b \in B$ *and* (x, b) Q_A (a, a') *for all* a, $a' \in A$. ■

5 Conclusion

The object of this paper was to explore the possibility of deriving rankings of sets of objects on the basis of their diversity by using only ordinal information on the similarities of the objects. This approach is to be contrasted with that adopted by Weitzman (1992),Weitzman (1993), Weitzman (1998) or Bossert et al. (2003), which assume a cardinally measurable primitive notion of similarities. While this investigation has been proved successful, we are aware that the specific ranking characterized in this paper is not perfect.

As mentioned earlier, a weakness of this ranking is that it does not allow trade-offs between the contributions of alternative pairs of objects to diversity. This ranking gives indeed a "veto power" to the two most dissimilar options in the sets and prevents the dissimilarity of other pairs of options to contribute significantly to the rankings of sets. Notice that, although it slightly weakeness this veto power, the lexicographic extension of the maxi-max criterion alluded to in section 2 shares the same flaws. It would be nice to obtain rankings which are capable of trading off the dissimilarity of the two most dissimilar options with that of other options. An interesting class of such smoother rankings are those which, like the ordering $\succeq_{add}$ considered in the proof of proposition 1, can be represented by a function that is *additive* with respect to a numerical representation of the primitive dissimilarity quaternary relation.

Finding an axiomatic characterization of this class of rankings is clearly a worthwhile, if not overly ambitious, objective. The characterization would use the same primitives as those considered herein (namely, a ranking of sets and an underlying ordinal notion of dissimilarity) and would identify the axioms of the ranking of sets that enables a numerical representation of the ranking as a summation of a numerical representation of the underlying dissimilarity quaternary relation.

Another avenue of research that could be worth exploring would be to escape from the approach of defining diversity as "aggregate dissimilarity" adopted herein. Must any plausible conception of the diversity offered by a set of objects be *reducible* to an aggregation of the pairwise dissimilarity that may exist between the objects ? There are various grounds on which one could stand for answering negatively to such a question.

One of these grounds is clearly the multi-attributes approach proposed to the problem by Nehring and Puppe (2002). The implementation of the approach requires a grouping of the objects into attributes and a (cardinally meaningful) function that assigns to each attribute its contribution to the overall diversity. But it does not ride on an *a priori* notion of pairwise dissimilarity between options.

Yet it is also possible to answer negatively to the question above even if one does not take the view that the objects can be grouped in various attributes. Suppose that we are interested in comparing the diversity of car models offered by various retailers and that we adopt the hedonic perspective of viewing a car as a combination of values taken by, say, k characteristics (such as size, degree of comfort, speed, fuel consumption, etc.) numerically measurable. This amounts to thinking of a model of car as to a point in $\mathbb{R}^k$ and to a car retailer as a to a (finite) set of points in $\mathbb{R}^k$. A reasonably natural notion of dissimilarity between cars in this perspective could be given by the ranking of pairs of points (cars) induced by the comparisons of their Euclidian distance. Furthermore, an equally plausible notion of diversity of car retailers in this setting could be given by the ranking of set of points (retailers) induced by the comparison of the dimension of the subspace spanned by these sets of points. Yet it is clear that this dimension can not be deduced from the information of the distance between these points alone.[3] It would be nice therefore to find the axiomatic properties of a ranking of sets that would characterize the fact that this ranking could be thought of as resulting from the aggregation of the pairwise dissimilarity of its elements for some underlying dissimilarity quaternary relation.

Finally, a line of research that needs further investigation is that of exploring further the connections between measurement of diversity and measurement of freedom of choice. While some results have been presented on this issue in the last section, we believe that much more could be done.

An interesting thing to do in that context would be to dig further behind the "black box" of the primitive quaternary relation of dissimilarity used to define diversity. This appears particularly important in the context of the multiple preference approach to freedom of choice.

If diversity is conceived in the context of a decision theoretic model, it may well be relevant to connect the primitive notion of dissimilarity to the possible preference of the decision maker for the options that she will choose in the various opportunity sets. Why for example do a car and a bicycle look intuitively more different - or dissimilar - than two cars with slightly different characteristics ? It is, maybe, because we think that most users of the modes of transportation are likely to experiment less difference in satisfaction in changing from one type of car to the other than in changing from one type of car to the bike. These differences in satisfaction could, it seems to us, be expressed in terms of a family of *utility functions* that a "reasonable" decision maker could use when choosing between modes of transportation, in just the same fashion as the notion of freedom of choice has been expressed in terms of a family of *possible preferences* for the decision maker. The resort to cardinally meaningful utility functions, rather than mere ordinal preference orderings, to explain a notion of dissimilarity represented by a quaternary relation seems unavoidable. For it seems very difficult *a priori* to produce rankings of *pairs of objects* in terms of dissimilarity from a mere knowledge of a set of rankings of the objects themselves. We think that the exploration of this area is also promising for future research.

[3] We are indebted to Jean-François Laslier for the mathematical skeleton of this example. Notice that this example would be valid for any notion of distance. Knowing the pairwise distance between each pair of points in a finite set is not a sufficient information to recover the dimension of the subspace spanned by the set of points.

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