

Vote trading and subset sums

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Abstract

We analyze the complexity of vote trading problems with equal-sized voting districts. For two allied vote-swapping parties, the problem is polynomially solvable. For three parties, the problem is NP-complete.

Keywords: Vote swapping; vote trading; manipulation; subset sum problem.

1 Introduction

During the 2000 presidential elections in the USA, an internet mechanism organized the swapping of votes for candidate Albert Gore with votes for candidate Ralph Nader across state borders. Gore was running for the democrats and a serious contender for winning the election. Nader was the Green Party nominee, and only an outsider (who in the end received less than 3% of the total votes). The central idea of *vote trading* was that Gore should become stronger in states where this would help him, while Nader should become stronger in states where this would not hurt Gore.

Hartvigsen [5] presents a mathematical model for such vote trading problems, and analyzes a variety of algorithmic and combinatorial concepts in this area. In particular, Hartvigsen establishes the NP-hardness of optimal vote trading in the case where two allied parties are swapping votes and where different voting districts may have different sizes. Bervoets & Merlin [1, 2] perform an axiomatic analysis of democratic swap-proof and gerrymander-proof voting rules.

In this short technical note, we discuss vote trading in the cases where all the voting districts are of identical size. We show that then the best vote trading can be found in polynomial time, if there are only two allied parties that are swapping votes. For three allied parties, however, the problem becomes NP-complete. Our results draw a sharp separation line between easy and hard cases. Furthermore, they yield yet another example for Lawler's mystical power of twoness; see Lenstra [6].

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The note is organized as follows. Section 2 discusses a variant of the classical subset sum problem, and identifies a polynomially solvable special case of this variant. Section 3 establishes a connection between vote swapping with two allied parties (and equal-sized voting districts) and the subset sum variant from Section 2; this connection yields the polynomial time result. Section 4 establishes NP-hardness of vote swapping with three allied parties (and equal-sized voting districts).

2 A subset sum variant

Subset sum problems are centered around n items with positive integer sizes u_1, \dots, u_n , and ask certain questions about the values attained by $u(I) := \sum_{i \in I} u_i$ as I ranges over the item subsets $I \subseteq \{1, \dots, n\}$. As a rule of thumb, typical subset sum problems are computationally intractable. For example, the problem of deciding whether $u(I)$ attains *all* integer values between two given bounds V^- and V^+ is Π_2^p -complete; see Eggermont & Woeginger [3]. As another example, the problem of deciding whether $u(I)$ attains some concrete given integer goal value V is NP-complete; see Garey & Johnson [4]. This latter example with goal value V constitutes the classical SUBSET-SUM problem, which plays a fundamental and prominent role in combinatorial optimization.

In general, we should not expect to find simple certificates for NO-instances of SUBSET-SUM that are easy to verify (as this would imply $\text{NP} = \text{coNP}$). But for certain well-behaved special cases the NO-instances are easy to recognize. For example, if all the item sizes u_1, \dots, u_n are even while the goal value V is odd, then the answer certainly must be NO. For another example, if the sum of the largest three values among u_1, \dots, u_n is strictly smaller than V while the sum of the smallest four values among u_1, \dots, u_n is strictly larger than V , then the answer also must be NO. In the rest of this section, we will consider the following subset sum variant and we will identify a polynomially solvable special case that is centered around this latter observation.

Problem: SUBSET-SUM INTERVAL

Instance: Items with positive integer sizes u_1, \dots, u_n ; two integers $V^- \leq V^+$.

Question: Does there exist $I \subseteq \{1, \dots, n\}$ with $V^- \leq u(I) \leq V^+$?

Note that for $V^- = V^+$, problem SUBSET-SUM INTERVAL boils down to problem SUBSET-SUM; consequently SUBSET-SUM INTERVAL is NP-complete.

Lemma 2.1 *The special case of SUBSET-SUM INTERVAL with*

$$V^+ - V^- \geq \max_{i=1}^n u_i - \min_{i=1}^n u_i \tag{1}$$

is polynomially solvable.

Proof. First renumber the items so that $u_1 \leq u_2 \leq \dots \leq u_n$ holds. For $1 \leq p \leq n$ define $S_p^{\min} = \sum_{i=1}^p u_i$ and $S_p^{\max} = \sum_{i=n-p+1}^n u_i$ as the sum of the p smallest respectively the

p largest items; furthermore let $S_0^{\max} = -\infty$ and $S_{n+1}^{\min} = +\infty$. Consider the following polynomial time algorithm:

- Determine the largest index r ($0 \leq r \leq n$) for which $S_r^{\max} < V^-$.
- If $V^+ < S_{r+1}^{\min}$ then output NO, and otherwise output YES.

First assume that the algorithm outputs NO, so that $S_r^{\max} < V^- \leq V^+ < S_{r+1}^{\min}$ holds. Note that any set $I \subseteq \{1, \dots, n\}$ with cardinality $|I| \leq r$ satisfies $u(I) \leq S_r^{\max} < V^-$, and that any set I with cardinality $|I| > r$ satisfies $u(I) \geq S_{r+1}^{\min} > V^+$. Hence there is no I with $V^- \leq u(I) \leq V^+$, and the output of the algorithm is correct.

Next assume that the algorithm outputs YES. This implies $r + 1 \leq n$, and also $V^- \leq S_{r+1}^{\max}$ and $S_{r+1}^{\min} \leq V^+$. If $S_{r+1}^{\min} \geq V^-$ then the set $I = \{1, \dots, r + 1\}$ constitutes a feasible solution, and if $S_{r+1}^{\max} \leq V^+$ then the set $I = \{n - r + 1, \dots, n\}$ constitutes a feasible solution. It remains to consider the cases with

$$S_{r+1}^{\min} < V^- \leq V^+ < S_{r+1}^{\max}.$$

We start with the set $I = \{1, \dots, r + 1\}$ that contains the $r + 1$ smallest items, and then step by step replace some item by a larger one. Every step raises $u(I)$ by at most $\max_i u_i - \min_i u_i$, so that by (1) the value $u(I)$ eventually must fall between the bounds V^- and V^+ . Hence also in this case, the output of the algorithm is correct. \square

3 The vote trading problem

We consider the following special case of vote trading with three political parties A , B , C and with m equal-sized voting districts. The number of voters in the i th district that respectively vote for A , B , C is denoted by a_i , b_i , c_i . As all voting districts have equal size s , we have $a_i + b_i + c_i = s$ for $1 \leq i \leq m$. Every district is won by the party that receives the relative majority of votes. For the sake of simplicity we assume that ties are always broken to the disadvantage of party A ; therefore party A wins the i th district if and only if $a_i > \max\{b_i, c_i\}$ holds. The question is whether parties B and C can repartition their votes such that they reach the relative majority in at least k of the districts. Here is a formal description of this question.

Problem: VOTE TRADING

Instance: Non-negative integers a_1, \dots, a_m , b_1, \dots, b_m , and c_1, \dots, c_m with $a_i + b_i + c_i = s$ for $1 \leq i \leq m$; an integer k .

Question: Do there exist non-negative integers b'_1, \dots, b'_m and c'_1, \dots, c'_m , with $\sum_{i=1}^m b'_i = \sum_{i=1}^m b_i$, and $\sum_{i=1}^m c'_i = \sum_{i=1}^m c_i$, and $b'_i + c'_i = b_i + c_i$ for $1 \leq i \leq m$, such that the following holds: there exists an index set $I \subseteq \{1, \dots, m\}$ with $|I| = k$, such that $a_i \leq \max\{b'_i, c'_i\}$ for all $i \in I$?

For later reference, we note that $a_i + b_i + c_i = s$ for $1 \leq i \leq m$ implies

$$\sum_{i=1}^m a_i + \sum_{i=1}^m b_i + \sum_{i=1}^m c_i = ms. \quad (2)$$

Furthermore, we will assume without loss of generality that the numbering of the districts satisfies

$$a_1 \leq a_2 \leq \dots \leq a_{m-1} \leq a_m. \quad (3)$$

Under (3) it is straightforward to see that parties B and C can win k districts if and only if they can win the first k districts. Finally, we will assume that

$$a_k \leq s/2. \quad (4)$$

Note that whenever (4) is violated with $a_k > s/2$, there is no way for parties B and C to win the districts k, \dots, m ; hence in these cases the answer to VOTE TRADING is trivially negative.

Lemma 3.1 *An instance of VOTE TRADING has answer YES, if and only if there exists a set $I \subseteq \{1, \dots, k\}$ such that*

$$-ms + \sum_{i=1}^k a_i + \sum_{i=1}^m a_i + \sum_{i=1}^m b_i \leq \sum_{i \in I} a_i \leq \sum_{i=1}^m b_i. \quad (5)$$

Proof. For the only-if part, assume that the VOTE TRADING instance has answer YES. Consider the corresponding integers b'_1, \dots, b'_m and c'_1, \dots, c'_m for which parties B and C win the first k districts. Define I as the set of all $i \in \{1, \dots, k\}$ with $b'_i \geq c'_i$, and define $J = \{1, \dots, k\} \setminus I$. Then $a_i \leq b'_i$ for $i \in I$ yields $\sum_{i \in I} a_i \leq \sum_{i \in I} b'_i$, which implies the right hand inequality in (5). Similarly, $a_i \leq c'_i$ for $i \in J$ yields $\sum_{i \in J} a_i \leq \sum_{i \in J} c'_i$ and consequently

$$\sum_{i \in J} a_i \leq \sum_{i=1}^m c_i. \quad (6)$$

By using (2) and $\sum_{i=1}^k a_i = \sum_{i \in I} a_i + \sum_{i \in J} a_i$, inequality (6) can be rewritten into the left hand inequality in (5).

For the if part, consider a set $I \subseteq \{1, \dots, k\}$ that satisfies (5). For $i \in I$, we initialize $b'_i := a_i$ and for $i \in \{1, \dots, k\} \setminus I$ we initialize $c'_i := a_i$. The right hand inequality in (5) yields that after this initialization the sum of all b'_i is at most $\sum_{i=1}^m b_i$, while the left hand inequality in (5) yields that the sum of all c'_i is at most $\sum_{i=1}^m c_i$. We then increase the values b'_i and c'_i appropriately so that they satisfy the constraints $\sum_{i=1}^m b'_i = \sum_{i=1}^m b_i$, $\sum_{i=1}^m c'_i = \sum_{i=1}^m c_i$, and $b'_i + c'_i = b_i + c_i$ for $1 \leq i \leq m$. \square

Theorem 3.2 *The VOTE TRADING problem with two allied political parties and equal-sized voting districts is polynomially solvable.*

Proof. By Lemma 3.1, VOTE TRADING is a special case of SUBSET-SUM INTERVAL with k items of size a_1, \dots, a_k . The bounds are $V^- = -ms + \sum_{i=1}^k a_i + \sum_{i=1}^m a_i + \sum_{i=1}^m b_i$ and $V^+ = \sum_{i=1}^m b_i$, the smallest item size is a_1 and the largest item size is a_k . From (3) and (4) we derive the upper bounds $a_i \leq s/2$ for $1 \leq i \leq k$ and $a_i \leq s$ for $k+1 \leq i \leq m$. This yields

$$ms = 2k \frac{s}{2} + (m-k)s \geq (a_1 + 2 \sum_{i=2}^{k-1} a_i + 3a_k) + \sum_{i=k+1}^m a_i. \quad (7)$$

Since (7) is equivalent to the condition $V^+ - V^- \geq \max_i a_i - \min_i a_i$ in (1), the polynomial time result now follows from Lemma 2.1. \square

4 A hardness result

Finally, let us discuss the generalization of the above VOTE TRADING problem to four political parties A, B, C, D in an odd number $2m-1$ of equal-sized voting districts. In the i th district ($1 \leq i \leq 2m-1$) there are a_i votes for party A , and parties B, C, D have respectively T_B, T_C, T_D votes at their disposal that they may repartition. Since every district has exactly s voters, these numbers are assumed to satisfy

$$\sum_{i=1}^{2m-1} a_i + T_B + T_C + T_D = (2m-1)s. \quad (8)$$

Parties B, C, D are plotting up against party A , and their goal is to reach the relative majority in at least m of the $2m-1$ districts. Formally, the problem is to decide whether there exist non-negative integers b_i, c_i, d_i with $\sum_{i=1}^{2m-1} b_i = T_B$, $\sum_{i=1}^{2m-1} c_i = T_C$, $\sum_{i=1}^{2m-1} d_i = T_D$, and with $a_i + b_i + c_i + d_i = s$ in all districts, and $a_i \leq \max\{b_i, c_i, d_i\}$ in at least m of the districts?

We establish NP-hardness of this four party variant, by means of a reduction from the NP-complete PARTITION problem; see Garey & Johnson [4].

Problem: PARTITION

Instance: Positive integers u_1, \dots, u_{2n} that satisfy $\sum_{i=1}^{2n} u_i = 2U$, and $u_i \leq U$ for $1 \leq i \leq 2n$.

Question: Does there exist $I \subseteq \{1, \dots, 2n\}$ with $|I| = n$ and $u(I) = U$?

Theorem 4.1 *The vote trading problem with three allied political parties and equal-sized voting districts is NP-complete.*

Proof. From an instance of PARTITION, we construct $4n-1$ equal-sized voting districts with $s = (8n + 2)U$ voters. In the i th district, we define $a_i = 4nU + u_i$ for $1 \leq i \leq 2n$ and $a_i = s$ for $2n + 1 \leq i \leq 4n - 1$. Furthermore, we set $T_B = T_C = (4n^2 + 1)U$ and $T_D = (4n - 4)U$. It is easily verified that these numbers satisfy (8). We claim that the considered PARTITION instance has answer YES, if and only if the constructed instance of vote trading allows parties B, C, D to win the first $2n$ districts.

Assume that parties B, C, D can indeed win the first $2n$ districts. As $a_i \geq 4nU > T_D$ for $1 \leq i \leq 2n$, party D is not able to win any of these districts. As for winning $n + 1$ districts against party A one needs more than $(n + 1)4nU$ votes, parties B and C each must win precisely n districts. Since $T_B + T_C = \sum_{i=1}^{2n} a_i$, the i th district can only be won by B, C, D if (i) $a_i = b_i$ and $c_i = 0$ or if (ii) $a_i = c_i$ and $b_i = 0$. This implies that $I = \{i : a_i = b_i\}$ satisfies $|I| = n$ and $u(I) = U$.

Next assume that the PARTITION instance has a solution I with $|I| = n$ and $u(I) = U$. For $i \in I$ we set $b_i := a_i$; for $i \in \{1, \dots, 2n\} \setminus I$ we set $c_i := a_i$; and for $1 \leq i \leq 2n$ we set $d_i := s - 2a_i$. All remaining values a_i, b_i, c_i are set to 0. With these settings, parties B, C, D will win the first $2n$ districts. \square

5 Final remarks

Our results precisely pinpoint the computational complexity of the considered vote trading problem: it is easy for $p = 2$ allied parties, and hard for $p \geq 3$ allied parties. If the number p of allied parties is fixed (and not part of the input), the problem can be solved in pseudo-polynomial time by a routine dynamic programming approach. If the number p is part of the input, the problem is strongly NP-hard. This can be proved by a reduction from the strongly NP-hard THREE PARTITION problem [4]; as the arguments are very similar to those in Section 4, we leave all details to the interested reader.

Acknowledgements. Part of this work was started during the Schloss Dagstuhl seminar 12101 on “Computation and Incentives in Social Choice”. We are grateful to the organizers of this meeting and to the Dagstuhl staff for providing a stimulating atmosphere.

Gerhard Woeginger acknowledges support by DIAMANT (a mathematics cluster of the Netherlands Organization for Scientific Research NWO), and by the Alexander von Humboldt Foundation, Bonn, Germany.

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