Risk-taking in financial networks

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Abstract

We analyze the Value-at-Risk based risk-taking behaviors (VaR-RM) of financial institutions linked through cross-holdings relationships. We model Value-at-Risk Management as targeted default probabilities in the event of an extreme adverse shock on assets. We relate risk-taking behaviors to a centrality measure that captures the propagation of losses-in-value in the network, we address the effect of network integration on risk-taking behavior, and we examine the impact of VaR-RM on the expected shortfall of the financial system. We also analyze how the cross-shareholding network affects the implementation of a regulation through capital requirements by identifying the institutions in the network with the highest impact on aggregate investments in risky assets. (JEL: C72; D85)

Keywords: Financial network; Risk-taking; Value-at-Risk.

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1. Introduction

Financial systems play a crucial role in fostering long-term economic growth by facilitating capital accumulation and the efficient allocation of resources.[1] As intermediaries, they capture valuable economic opportunities through investment in ventures that, albeit risky, promise substantial returns. However, these systems are also vulnerable to extreme risks—events of high magnitude but low frequency—that can precipitate economic crises.[2] Such crises not only incur direct costs due to institutional defaults but also trigger widespread negative externalities, undermining confidence in the financial system at large. Understanding the determinants of risk-taking behavior within these systems is therefore paramount. A common approach to managing firm-level risk exposure is Value-at-Risk-based Management (VaR-RM), which has been extensively documented in literature.[3] The interplay between investment risk and risk exposure is nuanced, often mitigated by diversification and risk-sharing, which are in turn influenced by the financial network among institutions. This network plays a pivotal role in shaping risk-taking behaviors under VaR-RM, especially in the context of extreme risks. Despite its importance, the influence of financial networks on the risk-taking behavior of financial institutions, particularly in managing acceptable risk levels amid extreme risks, has not been thoroughly explored. This gap in the research underscores the need for a

1. See, for instance, Levine (2005) and, more recently, Petra Valickova and Horvath (2015).

2. These extreme risks are at the heart of actual financial regulation. For example, Solvency II, the directive that harmonizes European Union insurance regulation, calibrates prudential regulation on the notion of bicentenary events.

3. See for instance Dowd (1998) and Saunders (1999). Bodnar et al. (1998) document the use of VaR-RM practise in non-financial corporations in the US. Basak and Shapiro (2001) stress that VaR estimates are crucial not just as decision-making tools but also for controlling risk, aiming to keep market exposure within predefined levels.

deeper examination of how financial linkages affect institutional approaches to risk management under VaR-RM.

In this study, we delve into the dynamics of financial networks through the lens of cross-holding contracts, an increasingly significant feature among financial institutions.[4] Our approach involves developing a straightforward model that captures the essence of these interactions. In this model, financial institutions are primarily financed by equity, a portion of which is owned within the financial sector itself, thereby creating a network of cross-holdings. This arrangement forms the backbone of our analysis, mapping out the intricate web of financial interdependencies. We focus on the scenario where the financial system is exposed to an extreme risk event—characterized by its high magnitude and low frequency—that poses a significant threat to individual institutions by potentially inflicting substantial losses on their risky assets. To simplify our analysis and maintain tractability, we intentionally exclude the possibility of default contagion. Within this framework, financial institutions respond to the prospect of extreme risk by employing Value-at-Risk-based Management (VaR-RM) strategies. These strategies involve setting a level of risk-taking that aligns with an externally imposed cap on the probability of default. By constructing this model, we aim to illuminate how the structure of financial networks influences the risk-taking behaviors of institutions that utilize VaR-RM, all while adhering to specified default probability thresholds. This exploration allows us to assess the implications of network organization for managing risk in the face of potentially catastrophic financial events.

In our analysis, we systematically unravel the dynamics of financial risk-taking within cross-holding networks, structured across several key stages.

4. (Pollak and Guan, 2017) argue that *"Between 2000 and 2015 the number of institutions with ownership in other institutions doubled in the United States. [...] Between 2011 and 2015 the total value of ownership of institutions by institutions increased by 211 percent"*.

Firstly, we demonstrate how cross-holding networks foster interdependencies among financial institutions' risk-taking decisions. Specifically, in the face of low-probability, high-impact risks, these networks serve as a form of mutual insurance. This mechanism not only facilitates strategic complementarities in risk-taking behaviors but also encourages institutions to engage in riskier activities than they would independently, due to the safety net provided by their investments in unaffected institutions. This interaction between catastrophic risk and cross-shareholding creates a scenario where the network's presence incentivizes increased risk-taking. We further explore these strategic complementarities by introducing a centrality measure that captures an institution's position within the network and its influence on risk-taking behavior. This measure considers both the benefits of risk-sharing among well-connected institutions and the potential drawbacks of negative feedback from distressed entities. Our analysis indicates a nuanced relationship between network structure and risk-taking, with centrally located institutions in star networks consistently exhibiting higher levels of risk.[5]

Secondly, our findings reveal that more integrated cross-shareholding networks—characterized by denser cross-holdings—correlate with heightened risk-taking under VaR-RM. This suggests that the positive effects of increased integration and mutual support within the network outweigh potential negative feedback loops, thereby enhancing the resilience of affected institutions.

Thirdly, we assess the expected shortfall for the financial institutions under VaR-RM, identifying a paradoxical role of networks. While networks can

5. Star networks are prototypical architectures belonging to core-periphery networks, which have been shown to be reasonably representative of real financial networks, both in terms of inter-institution lending; see, e.g.,Craig and von Peter (2014) and cross-shareholding Rotundo and D'Arcangelis (2014). Core-periphery networks are networks in which highly interconnected nodes, called the core, coexist with nodes loosely connected (both to the core and among themselves), called the periphery.

mitigate individual risk through diversification and support, they also magnify the institutions' overall vulnerability to shocks, especially when considering the density of the network and the nature of the underlying risk distribution.

Finally, we contemplate prudential regulation strategies aimed at mitigating excessive risk-taking while encouraging healthy levels of investment in risky assets. By simulating a scenario where regulatory bodies adjust liability-side balance sheet regulations, we propose a 'key-player' policy approach. This involves pinpointing specific institutions where targeted capital injections could optimally increase aggregate risk-taking within the bounds of VaR-RM, thereby enhancing the system's stability without compromising on necessary risk engagement.

Through this structured exploration, our study sheds light on the intricate interplay between cross-shareholding networks, risk management strategies, and regulatory interventions, offering valuable insights into optimizing financial stability in the face of systemic risks.

Relationship to the literature. Our study adds to distinct, although in some respect complementary, strands of research. First, our paper contributes to the fast-growing literature on cross-holding networks;[6] in particular, it complements the line of work investigating endogenous, and often excessive, risk-taking by financial institutions. Galeotti and Ghiglino (2021) consider a portfolio choice in a model of equity-holding in networks, assuming away default. They show that institutions may overinvest or underinvest with respect to the social welfare optimum, depending on the position of the institution in the financial network. Jackson and Pernoud (2019) incorporate both risk-taking and default, but limited to examples with binary (and independent or fully correlated) returns on the risky assets. Our paper is the first to study

6. See Brioschi et al. (1989), Fedenia et al. (1994), or, more recently, Elliott et al. (2014).

the impact of cross-holding networks on VaR-RM, in a setup with possible institution defaults.

Our paper also adds to the literature on VaR-RM by linking financial networks to VaR-RM. The value-at-risk was originally introduced by Markowitz (1952) and Roy (1952) in an attempt to optimize profit so as to incorporate the risk of high losses. In its current form, VaR was presented in 1989 by JP Morgan in their risk management tool called the RiskMetrics. Jorion (1998) finds that LTCM [thus VaR RM] has severely underestimated its risk due to its reliance on short-term history and risk concentration. Recent extension of the VAR approach include the CVAR -C for conditional-, also known as mean excess loss or mean shortfall or tail VAR; See Rockafellar and Uryasev (2000), Rockafellar and Uryasev (2002).

This paper also adds to the literature on prudential regulation. Prudential regulation through cash or capital requirements has been shown to be a useful and powerful tool to deal with excessive risk-taking by institutions and to reduce default risk (Hellmann et al., 2000; Decamps et al., 2004).[7] Implemented by financial regulators since the early 1990s (through the 1988 Basel Accord, also known as Basel I), such regulation gained in complexity thereafter to account for specific risks (e.g., market risk, liquidity risk, and operational risk). It dampens solvency risk without the social cost of bailouts, or their effects induced through moral hazard when anticipated (Freixas and Rochet, 2013). However, during the 2007 financial crisis, prudential regulation proved insufficient to limit excessive risk-taking, notably because of the extent of financial linkages – see, for example, the cases of Lehman Brothers and American International Group discussed in Glasserman and Young (2016). There is also a nascent literature on public intervention in financial networks. Elliott et al. (2014) study the effect of reallocations of cross-holdings that

7. Cash requirements correspond to constraints on the asset side, whereas capital requirements affect the liability side.

leave the market value of institutions unchanged and find that they are not effective in avoiding the first failure. Leduc and Thurner (2017) study the effect of transaction-specific taxes when institutions are connected through debt contracts and subject to liquidity shocks and show that this can reduce contagion. Finally, Demange (2018) and Jackson and Pernoud (2019) discuss the optimal ex-post intervention, through bailouts or cash injection. We complement these literatures by analyzing a prudential policy consisting in a capital injection intervention taking into account the interdependent risk-taking behaviors of financial institutions under VaR-RM.

We conclude by discussing the recent literature on contagion, namely, the spread of shocks between linked institutions.[8] Although we do not incorporate contagion, our model has close connections to some papers in that literature. Our structure of risk, with one large negative shock hurting one institution at a time, is similar to Cabrales et al. (2017), who model financial linkages as investments by institutions in each other's projects and analyze the optimal network structure depending on projects' riskiness. Moreover, our comparative statics on integration is discussed in Elliott et al. (2014), who consider additional frictions through default costs in a model of linear cross-holdings.[9] In all of the above papers, the initial risk faced by each institution is exogenous.[10] Our paper rather considers, instead, that shocks are endogenous to investment

8. The effect of financial networks on contagion, when institutions are linked through debt contracts is analyzed in Allen and Gale (2000), Acemoglu et al. (2015), Glasserman and Young (2016), Acemoglu et al. (2015), and Glasserman and Young (2016). For a recent survey on the transmission of liquidity shocks in large networks, see Gai and Kapadia (2019).

9. Although they model links as shareholding, Elliott et al. (2014) view them as "debt contracts around and below organizations' failure thresholds" and assume that default costs spread in the network.

10. In an unpublished paper, Shu (2019) models unsecured inter-institution debt contracts, mostly on regular networks, and obtains complementarities in risk-taking behaviors (that paper does not envisage prudential regulation).

choices by institutions. Moreover, our work stresses both the positive and negative aspects of cross-shareholding networks with respect to risk-taking and the challenge it poses for public policies.

The remainder of this paper is organized as follows. The model is presented in Section 2. In Section 3, we characterize the optimal levels of risk-taking under VaR-RM, we undertake a comparative statics on integration, and we analyze simple network structures. Section 4 examines the situation in which the shock induces a stress to all financial institutions. Section 5 explores the impact of VaR-RM behaviors on the expected shortfall of the financial system. Policy interventions are analyzed in Section 6. All proofs are relegated to Appendix A. Appendix B examines risk-taking decisions in the absence of VaR-RM. Appendix C explores the case where several firms suffer the shock at once, and Appendix D analyzes directed shareholding networks.

2. The model

We consider a network of $n \geq 2$ financial institutions potentially linked through cross-shareholding. These institutions can be, for example, banks, insurance companies or pension funds. We consider a two-period model in which every institution is liquidated after asset return realization. This simple model allows to capture the effects of cross-shareholding. We extend the model in the end of the paper when bringing the model to data.

We introduce the following notation. Matrices are written in block and bold letters, and vectors in lower case and bold letters; the superscript ^{T} stands for the transpose operator. Numbers and entries of matrices are in lower case. We let $\mathbf{I}$ be the identity matrix of order n; $\mathbf{0}$ and $\mathbf{1}$ represent the vectors of zeros and ones of dimension n, respectively.

The financial network. At $t = 0$, each financial institution $i \in \mathcal{I} = \{1, 2, \cdots, n\}$ is financed by debt (or deposit) d_i, by equity held by outside investors e_i, and by equity held by other financial institutions in the network.[11] We let $\mathbf{d} = (d_i)_{i \in \mathcal{I}}$ represent the vector of external debts, and $\mathbf{e} = (e_i)_{i \in \mathcal{I}}$ represents the profile of external equities. We set $\{p_{ij}\}_{j \in \mathcal{I} \setminus \{i\}}$, with p_{ij} representing the amount invested by institution i in institution j, and we let $\mathbf{P} = (p_{ij})_{i,j \in \mathcal{I}^2}$ represent the matrix of investment in equity among financial institutions. Call the binary network $\mathbf{G} = (g_{ij})$ supporting cross-investment in equities; I.e., $g_{ij} \in \{0, 1\}$ and we assume $g_{ij} = g_{ji}$ for all i, j (in section D we relax this bilateral symmetry). We denote by $\delta_i = (\mathbf{G1})_i$ the degree of institution i in this network. We then consider cross-shareholding networks such that $p_{ij} = p \cdot g_{ij}$. The investment of each institution in another institution is fixed, with a ticket of size p.

Each financial institution say i divides its resource between investment in a risk-free asset (with normalized return equal to 1), $x_i \geq 0$; investment in a institution-specific risky asset, $z_i \geq 0$; and investment in the equity of other institutions in the network, $\{p_{ij}\}_{j \in \mathcal{I}}$. The balance sheet of institution i at $t = 0$ (i.e., before realization of risk) can then be represented as in Fig. 1 (left panel). This leads to the following accounting equation at $t = 0$ (taking into account that $\sum_{j \in \mathcal{I}} p_{ij} = \sum_{j \in \mathcal{I}} p_{ji}$):

$$x_i + z_i = e_i + d_i \tag{1}$$

Letting $\mathbf{z} = (z_i)_{i \in \mathcal{I}}$ represents the profile of investments in risky assets, we have $\mathbf{z} \in [\mathbf{0}, \mathbf{e} + \mathbf{d}]$ from the balance sheet equation (1). At $t = 1$, the risks are realized institutions liquidated, and their values (if any) are distributed among their shareholders. We denote by $a_{ij} = \frac{p_{ij}}{\sum_k p_{kj} + e_j}$ the share of the value of institution j's equity held by institution i. Matrix $\mathbf{A} = (a_{ij})_{(i,j) \in \mathcal{I}^2}$ represents the set of shares between financial institutions. We let $\rho \geq 1$ represent the

11. To isolate pure cross-shareholding effects and to save on notation, we disregard debt contracts among financial institutions.

deterministic (gross) return on debt[12] (or deposit) and $\tilde{\mu}_i$ be the stochastic return on the risky asset of institution i. Figure 1 (right panel) presents the balance sheet of institution i for a given realization μ_i at $t = 1$, where the equity value v_i accounts for equity held by both the financial system and external investors.

Assets	Liabilities
x_i	d_i
z_i	e_i
$\sum_j p_{ij}$	$\sum_j p_{ji}$

Assets	Liabilities
x_i	ρd_i
$\mu_i z_i$	
$\sum_j a_{ji} v_j^+$	v_i

FIGURE 1. Balance sheet of financial institution i. Left panel: at $t = 0$. Right panel: at $t = 1$.

Equity values. Let $\mathbf{v} = (v_i)_{i \in \mathcal{I}}$ represent the vector of their equity values. The equity value $v_i = x_i + \mu_i z_i - \rho d_i + \sum_{j \neq i} a_{ij} v_j^+$, where $v_j^+ = v_j$ if $v_j > 0$ and 0 otherwise.[13] In the event that $v_i < 0$, all assets go to debt repayment. Let $\eta_i = e_i - (\rho - 1) d_i$ for convenience. Using the accounting equation (1) at $t = 0$, the accounting equation for every institution i at $t = 1$ becomes

$$v_i = \max\left((\mu_i - 1) z_i + \eta_i + \sum_{j \neq i} a_{ij} v_j, 0\right) \tag{2}$$

ASSUMPTION 1. $\eta_i = e_i - (\rho - 1) d_i > 0$ for all i

Under Assumption 1, vector $\eta = (\eta_i)_{i \in \mathcal{I}}$ is positive, and when a financial institution does not invest in risky assets, it remains solvent (i.e., $v_i > 0$ when $z_i = 0$).[14]

12. Returns on debt are assumed to be homogeneous and independent of default risk. As financial institutions will end up with the same default probability, this last assumption is reasonable.

13. The market value of institution i, that is, the share of the value held by external equity holders, is given by $v_i * e_i / \left(e_i + \sum_{k \in \mathcal{I}} p_{ki}\right)$.

14. As will be clear, under Assumption 1, each institution invests a positive amount in risky assets at equilibrium when returns on risky assets are greater than 1.

The structure of risk. We focus here on extreme and rare events, likely to put one institution into financial distress. We therefore assume that only one institution at a time can be hurt by this large negative shock. Still, this shock can also have a negative impact on other institutions' risky investment (on top of the effect through cross-shareholding), such as through a fire-sale mechanism.

With probability $1 - q_0$, the system is not stressed, and the return on every bank's risky asset is $r > 1$. However, with probability q_0, the financial system is stressed: The return on the risky investment of all banks falls to $\underline{r} < r$, and a large negative shock hits a single bank at random (with uniform probability). The bank hit by the shock suffers a stochastic loss $\tilde{s}$, distributed on the non-negative support $[s_0, +\infty)$, $s_0 > \underline{r} - 1$ (leading to $\mu_i < 1$), with cumulative function H and average value $\overline{s}$.[15] Formally, we assume that for every bank i,

$$\tilde{\mu}_i = \begin{cases} r & \text{with probability } 1 - q_0 \\ \underline{r} & \text{with probability } \frac{n-1}{n} q_0 \\ \underline{r} - \tilde{s} & \text{with probability } \frac{q_0}{n} \end{cases} \quad (3)$$

Figure 2 illustrates the structure of the stochastic return of bank i's risky asset.

Assumption 2. $\mathbb{E}(\tilde{\mu}_i) > 1 \ \forall i$.

Assumption 2 implies that investment in the risky assets is still worthwhile.

Value-at-Risk Management. In this setup, the institution's decision reduces to allocating its resources $e_i + d_i$ between the risk free asset and its specific risky asset. This optimal portfolio management by financial institutions is assumed to follow a Value-at-Risk Management principle. That is, each financial institution

15. This structure of risks echoes that of Cabrales et al. (2017), who model rare and large shocks on gross return through a deterministic return with fixed probability and two alternatives with either a small or a large shock.

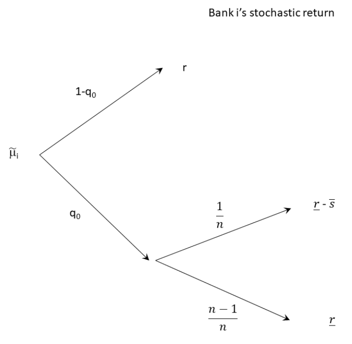

FIGURE 2. The stochastic return $\tilde{\mu}_i$ of institution i

maximizes its expected equity value $\mathbb{E}(v_i)$[16], under the constraint of complying to a maximum acceptable probability of default.[17] Denoting by β the maximum acceptable default probability[18] (e.g., that value can be set by the regulator), each institution i maximizes its expected value $\mathbb{E}(\tilde{v}_i)$ under the constraint $\mathbb{P}(\tilde{v}_i < 0) \leq \beta$. Focusing on environments in which the managerial constraint is

16. Managers' and equity-holders' objectives are assumed to be aligned. We thus ignore agency issues inside the institution.

17. The Value-at-Risk of financial institutions is taken into account by financial regulators; e.g., in Basel III and Solvency II. The value at risk is defined by the Basel Committee on institutioning Supervision as "A measure of the worst expected loss on a portfolio of instruments resulting from market movements over a given time horizon and a pre-defined confidence level" (BCBS, 2019).

18. To isolate pure network effects, the acceptable default probability is here assumed to be homogeneous across institutions. The model can easily be extended to heterogeneous values β_i, to account for individual characteristics such as institution size.

binding for all institutions[19], institutions solve the following program:

$$\max_{z_i \in [0, e_i + d_i]} \quad \mathbb{E}\Bigg(\max \Big((\tilde{\mu}_i - 1) z_i + \eta_i + \sum_{j \neq i} a_{ij} \tilde{v}_j, 0 \Big) \Bigg) \qquad (4)$$

$$\text{s.t.} \qquad \mathbb{P}(\tilde{v}_i < 0) = \beta$$

where $\tilde{v}_i = (\tilde{\mu}_i - 1) z_i + \eta_i + \sum_{j \neq i} a_{ij} \max(\tilde{v}_j, 0)$. In the above expression, the realizations of any $\tilde{v}_j$ is necessarily nonnegative through equation (2). As is clear from the above program, the risk-taking level chosen by each institution depends on the entire shareholding network $\mathbf{A}$.

Remark 1. In the absence of value-at-risk management, institutions are inclined to take more risk. Appendix B illustrates this by providing a sufficient condition, related to the probability of occurrence of the extreme event, such that institutions put all their resource to the risky asset.

3. Risk-taking under Value-at-Risk Management

In this section, we solve the system of optimal risk-taking under VaR-RM, and we undertake a comparative statics analysis with respect to the level of integration of the cross-shareholding network.

For this core section of the paper, and to shed light on pure network effects, we assume $d_i = d, e_i = e$ for all i.Then, the share of the value of institution j's equity held by institution i is given by $a_{ij} = \frac{g_{ij}}{\delta_j + \frac{e}{p}}$.

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19. The upper bound on β for which the constraint is binding depends on the cross-shareholding network. For any institution i, β has to be lower than the probability that institution i has a negative value when receiving the shock and given risk-taking levels are set at their upper bounds; I.e., $\beta < \frac{q_0}{n} \mathbb{P}\Big(s > \alpha_i\Big)$, where $\alpha_i = \frac{1}{m_{ii}} \sum_k m_{ik}((r-1)(e_k + d_k) + \eta_k)$ with $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$; or, letting H be the cumulative distribution of the shock s, $\beta < \frac{q_0}{n}(1 - H(\alpha_i))$. A sufficient condition on the $\max_i \alpha_i$ follows directly.

3.1. Characterization

We describe now how VaR-RM shapes institutions' risk-taking. Since the negative shock hits a single institution and under Assumption 1, the value of a financial institution can be negative only when it suffers the large negative shock on its asset; and in that case, the values of the other institutions are necessarily positive. We define matrix $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$, and matrix $\mathbf{C}$ with null diagonal ($c_{ii} = 0$) and off-diagonal entry $c_{ij} = \frac{m_{ij}}{m_{ii}}$.

In the absence of default (if $v_i \geq 0$ for all i), inverting the system given by equation (2), institution i's value can be expressed as a Bonacich centrality over the shareholding network:

$$v_i = \sum_{j \in \mathcal{I}} m_{ij} \big((\mu_j - 1) z_j + \eta \big)$$

The Bonacich issued from the cross-shareholding matrix, $\mathbf{b} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{1}$, takes a remarkably simple form.[20] Indeed,

PROPOSITION 1. *For all $p, e, \mathbf{G}$,*

$$\mathbf{b} = \mathbf{1} + \frac{p}{e} \mathbf{G} \mathbf{1}$$

Hence, in absence of default, values are aligned with Bonacich centralities.

Proposition 1 shows that centrality is exclusively driven by the number of partnerships. This means that if assets' returns were homogeneous, those institutions with larger amounts invested in the financial system would end up with a larger share (net of the part of the value issuing to external equity holders, that does not depend on the cross-shareholding network).

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20. Assuming $\mathbf{P}^T = \mathbf{P}$ is crucial to get this simple formulation of centralities.

Institution i's VaR constraint is then that the probability to survive the adverse event is $1-\beta$. Formally:

$$\frac{q_0}{n}.\mathbb{P}\left(\underbrace{(r-\tilde{s}-1)z_i+\eta+\sum_{j\neq i}c_{ij}\left[(r-1)z_j+\eta\right]<0}_{\tilde{v}_i \ \mid \ \mu_i=r-s, \ \mu_j=r \ \forall j\neq i}\right)=\beta \tag{5}$$

Now define $t_{1-\frac{n\beta}{q_0}}$ as the $(1-\frac{n\beta}{q_0})$th quantile of the distribution of $\tilde{s}$ and $\ell = r - t_{1-\frac{n\beta}{q_0}}$; ℓ can then be understood as the value at risk at level $(1-\beta)$ of each institution (see footnote 17). A low value of ℓ then reflects tight regulation (low value of β) or large market risk (a distribution of $\tilde{s}$ with heavy right tail). Equation (5) then becomes

$$(\ell-1)z_i+\eta+\sum_{j\neq i}c_{ij}\left[(r-1)z_j+\eta\right]=0 \ \forall i \tag{6}$$

Throughout this section, we consider that the shock hiting one bank does not strongly deteriorate the health of the financial system (in Section 4, we examine a more general setup in which, once a firm is hit by the shock, other institutions also suffer a negative shock on their returns). Formally,

Assumption 3. $\underline{r} > 1$.

Together with Assumption 1, Assumption 3 guarantees that v_i is positive when i is not hit by the large negative shock, even in a stressed environment.

To address interesting cases, we assume that the value at risk of each institution, ℓ, is bounded from above:

Assumption 4. $\ell < 1$.

Under Assumption 4, tight regulation leads to a sufficiently low value of β (the maximum acceptable default probability)[21] and the risk-taking levels under VaR-RM are strategic complements. Indeed, let $\varepsilon = \frac{r-1}{1-\ell}$; we have $\varepsilon > 0$ by Assumption 4. We obtain

$$z_i - \varepsilon \sum_{j \neq i} c_{ij} z_j = \frac{\eta}{1-\ell} \frac{e + p\delta_i}{e\ m_{ii}} \tag{7}$$

This pattern of strategic complementarities stems from the fact that as one institution suffers a negative shock, the other institutions in the network always provide support to institution i through cross-shareholding links. Since the value received by institution i through its shares in the financial system is increasing with their own investment in risky assets, the higher this investment, the higher institution i's investment in its risky asset, for a given default probability. Note that the risk-taking under VaR-RM for an isolated institution is $z_i^* = \frac{\eta}{1-\ell}$, which is positive under Assumptions 1 and 4.

The complementarities of the interactions, together with the upper bounds on values of z_i, guarantee the existence of a solution $\mathbf{z}^*$ to the system of programs (4) $\forall i$. Assumptions 2 to 4 guarantee that the solution is unique and positive (see Belhaj et al., 2014). Some levels of risk-taking can still reach the upper bound $e + d$. Now, considering the system of best-responses functions F, with $F_i(\mathbf{z}) = \varepsilon \sum_{j \neq i} c_{ij} z_j + \frac{\eta}{1-\ell} \frac{e+p\delta_i}{e\ m_{ii}}$ $\forall i$, any interior solution satisfies $\mathbf{z}^* = F(\mathbf{z}^*)$. We can then focus on interior solutions through the following assumption.

Assumption 5. $F((e+d)\mathbf{1}) < (e+d)\mathbf{1}$.

21. In the current regulation, β is set to 1% in the institutioning sector (Basel II) and 0.5% in the insurance industry (Solvency II).

Through complementarity in risk-taking levels, Assumption 5 guaranties that the VaR-RM imposes a binding VaR for all institutions. The risk-taking behaviors are then given by the following theorem:

THEOREM 1. *Under Assumptions 1 to 5, there is a unique solution to program (4). This solution is interior, and risk-taking levels under Value-at-Risk Management are given by*

$$\mathbf{z}^* = \frac{\eta}{r-1}\Big[(1+\varepsilon)(\mathbf{I}-\varepsilon\mathbf{C})^{-1}\mathbf{1}-\mathbf{1}\Big] \tag{8}$$

The interior solution $\mathbf{z}^*$ builds on a centrality measure, $(\mathbf{I}-\varepsilon\mathbf{C})^{-1}\mathbf{1}$, that expresses institutions' risk-taking levels as a function of their position in the (weighted) network of cross-shareholding $\mathbf{A}$ – recall that $c_{ij} = \frac{m_{ij}}{m_{ii}}$ where $\mathbf{M} = (\mathbf{I}-\mathbf{A})^{-1}$.

This centrality echoes the so-called Bonacich centrality, but self-loops play a role in relation with the adverse event. On the one hand, central institutions benefit from other institutions' values in case of shock, which tends to increase their risk-taking level; on the other hand, following a shock, the network also amplifies the loss in value of the shocked institution through feedback effects, which tends to increase the risk-taking of institutions with low feedback effects (this is why self-loops play a role). The centrality measure presented in Theorem 1 captures these complex networked interactions between the shocked institution and the other institutions.

To illustrate the distinction between Bonacich centrality, that drives institutions' values, and the centrality that drives risk-takings under VaR-RM, consider the network depicted in Figure 3. The degree of institution 3, and thus its Bonacich centrality, is larger than that of institution 4, but its risk-taking level is lower.

This tradeoff is even more transparent under tight regulation. We obtain the following corollary:

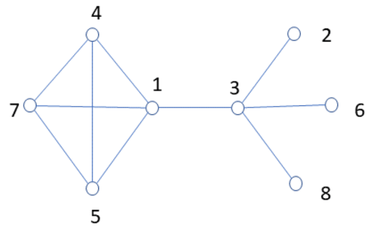

FIGURE 3. Bonacich vs centrality of regulation. For $\rho = 1.4$, $e = 200$, $d = 200$, $p = 20$, $\ell = 0.6$, $R = 1.7$, $\mathbf{z}^* \simeq (887, 445, 761, 765, 765, 445, 765, 445)$. Here z_3 is lower than z_4 whereas the Bonacich of institution 3, which is aligned with its degree, is larger.

COROLLARY 1. *Under Assumptions 1 to 5, when regulation is tight ($\varepsilon = \frac{r-1}{1-\ell} \to 0$), the risk-taking level z_i^* stemming from Value-at-Risk Management is proportional to the ratio $\frac{b_i}{m_{ii}} = \frac{e+p\delta_i}{e\ m_{ii}}$.*

Bonacich centrality aggregates the share of other institutions' values held by one institution ($b_i = \sum_j m_{ij}$ and $\mathbf{M} = \sum_{q=0}^{\infty} \mathbf{A}^q$). Institutions with higher Bonacich centrality receive more from others through the shareholding network and can therefore take more risk (for a given default probability). Now, the network may also make an institution more exposed to its own value, and therefore to its own level of risk-taking, through self-loop (m_{ii}). Institutions with higher self-loop centrality then suffer more from a shock on their risky asset and can therefore take less risk (for a given probability of default). Proposition 1 states that under tight regulation (i.e., a large value-at-risk ℓ in absolute term:

$\varepsilon \to 0$ when $\ell \to -\infty$), the pure risk-sharing effect results from a trade-off between these two effects.

Note that, on regular networks, for which Bonacich centralities are homogeneous, risk-taking levels can be differentiated, when self-loop centralities differ. Furthermore, the ratio $\frac{b_i}{m_{ii}}$ that stems from the cross-shareholding network is not necessarily favorable to more central institutions. The next example depicted in Figure 4 illustrates that the ordinal ranking of this ratio can differ from the ranking of degrees.

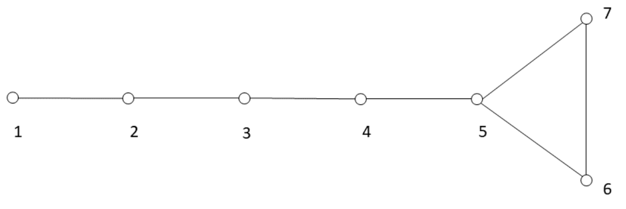

FIGURE 4. In this network, degree and ratio $\frac{b_i}{m_{ii}}$ are not aligned. Here, parameters $e = p = 20$ have been used to generate the cross-shareholding matrix. The profile of ratio is $(3.62, 4.45, 5.02, 5.26, 5.16, 4.51, 4.51)$. Whereas institution 5's degree is larger than that of institution 4, we have $\frac{b_5}{m_{55}} < \frac{b_4}{m_{44}}$.

Among all network structures, core-periphery networks[22] have been shown to be reasonably representative of real financial networks, both in terms of inter-institution lending; see, e.g.,Craig and von Peter (2014) and cross-shareholding

22. Core-periphery networks are networks in which highly interconnected nodes – called the core – coexist with nodes loosely connected (both to the core and among themselves) – called the periphery.

Rotundo and D'Arcangelis (2014). The simplest case is a star network, where the core is reduced to a single institution. We find:

PROPOSITION 2. *Under Assumptions 1 to 5, in an undirected star network with n institutions, the risk-taking of the central institution under VaR-RM is higher than that of peripheral institutions.*

The proof of Proposition 2 rests on the asymmetry of matrix $\mathbf{A}$ and its specification. We first prove that the ratio b_i/m_{ii} for the center is greater than in any peripheral institution.[23] By Corollary 1, this statement proves the result for sufficiently large negative shocks. We then extend the proof to arbitrary values of ε by using the ranking of centralities in an argument by induction.

REMARK 2 (*Multiple shocks*). Allowing for more than one shock makes the network less useful to the institutions suffering the shocks, thereby leading to more restrictions on risk-taking. See Appendix C for more details.

REMARK 3 (*Low levels of investment*). General network structures can be studied under low levels of cross-shareholding, that is, when p is sufficiently close to 0. We obtain $\mathbf{A} = p/e\mathbf{G} + o(p)$ and therefore $\mathbf{M} = \mathbf{I} + p/e\mathbf{G} + o(p)$. Risk-taking levels are then approximated under fixed participation by $z_i^* = \frac{1}{1-\ell}\left(e + (1+\varepsilon)p\delta_i\right) + o(p)$. Then, risk-taking is increasing in the number of financial institutions that invest in a given institution.

23. The ratio of Bonacich centrality over self-loop centrality is not necessarily favorable to central institutions in models of networked interactions à la Ballester et al. (2006). For instance, in a star network, this ratio can be favorable to peripheral agents under sufficiently high values of interaction. Let $\Delta = 0.44$, $\mathbf{G}$ be the adjacency matrix of the 4-player star network, where agent 1 is the central agent, $\mathbf{M} = (\mathbf{I} - \Delta\mathbf{G})^{-1}$, and $\mathbf{b} = \mathbf{M}\mathbf{1}$. Then $b_1/m_{11} = 2.32$, whereas $b_j/m_{jj} = 2.35$ for $j \neq 1$.

3.2. Comparative statics on the cross-shareholding matrix

How does an increase in the cross-shareholding matrix **A** affect risk-taking under VaR-RM? Increasing shareholding has an ambiguous effect a priori: (i) it propagates the negative shock on one institution's asset to the whole network, but (ii) it propagates the (necessarily positive) value of other institutions to the institution hit by the negative shock.

We find:

PROPOSITION 3. *Under Assumptions 1 to 5, an increase of the cross-shareholding network* **A** *induces increased optimal risk-taking under VaR-RM.*

By Proposition 3, the adverse increased feedback effects are always dominated by the positive increased complementarities helping institutions to survive to the bad shock. One implication of this proposition is that every nonempty shareholding network is always helpful with regard to the no shareholding case.

REMARK 4 (*Directed shareholding network*). Directed shareholding networks bring a resource effect in the accounting equation of the institutions' balance sheet: institutions with more investors benefit from higher resource. Taking into account this resource effect adds a term in the risk-taking under VaR-RM. Some of our results are kept unchanged: central institutions in directed stars take more risk under regulation, and simulations over a large number of networks generated by popular random network generation models confirm that more integration fosters risk-taking in general (see Appendix D for more details).

4. A shock stressing the system

In the benchmark model, the shock hitting an institution does not lower other asset retuns (as $\underline{r} > 1$ for other assets by Assumption 3). Here we rather assume that, once a firm is hit by the shock, the system enters into stress in the sense that the other institutions also suffer bad returns (indirect economic mechanisms can explain this, like a firewall effect). That is, formally, we relax Assumption 3. We will see that when the shock spreads to other institutions, the nature of strategic interaction between risk-takings is qualitatively affected: in a stressed environment, risk-takings can become strategic substitutes. However, the network is still always a factor enhancing risk-taking levels with respect to the no-network case.

The magnitude of the stressed environment proves key to understand risk-taking. Recalling that $\ell < 1$, it is immediate that when $\underline{r} > 1$ (or $\varepsilon > 0$), risk-taking levels are strategic complements; When $\underline{r} = 1$ (or $\varepsilon = 0$), risk-taking levels are independent; When $\underline{r} < 1$ (or $\varepsilon < 0$), risk-taking levels are strategic substitutes.

The first question that comes with the presence of strategic substitutes is uniqueness.[24] However, the model does not bring multiplicity, i.e., uniqueness still holds for any $\underline{r} \geq \ell$ (or $\varepsilon \geq -1$); the proof is immediate from Proposition 5 thereafter.[25] Second, the presence of substitutes invalidates the comparative statics result given in Proposition 3, in that more cross-shareholding may not always lead to increased risk-taking levels. However, the statement holds in average:

24. See Bramoullé et al. (2014) for sufficient conditions.

25. Multiplicity requires corners, which do not emerge here.

Proposition 4. *Under Assumptions 1, 2, 4 and 5, any increase of the shareholding matrix that keeps the risk-taking levels interior induces an increase of the average risk-taking level.*

A less demanding issue is to know the extent of which the network enhances risk-taking with respect to the no-network case (as described in Remark **??** when $\underline{r} = r$). Indeed, the network could potentially reduce risk-taking levels at a point that is below the no-network case. Yet the impact of the network on risk-taking is not ambiguous in this respect:

Proposition 5. *Consider Assumptions 1, 2, 4 and 5, and consider any non-empty and undirected cross-shareholding network. When $\underline{r} > \ell$ (or $\varepsilon > -1$), the network favors risk-taking with respect to isolated institutions. When $\underline{r} = \ell$ (or $\varepsilon = -1$), risk-taking levels are identical to those taken in isolation.*

By Proposition 5, when the stress is low so that returns still exceed unity, the impact of the network is similar to the no-stress case; I.e., the high returns guarantee complementarities and risk-taking levels are higher than those taken in isolation. For intermediate returns $\ell < \underline{r} < 1$ however, the financial system faces bad returns as a whole and risk-takings become strategic substitutes. Still the network provides value through equities, and Proposition 5 delivers a positive message: the cross-shareholding network still fosters risk-taking with respect to the no-network case whatever the network structure. This sharp result is tightly linked to the fact that only positive values can circulate in the cross-shareholding network – recall that a defaulting institution does not generate value.

Moreover, the higher the level of stress of the system, the less institutions can deliver value to the shocked institution, and when $\underline{r} = \ell$, the shock is common and the network does not affect risk-taking anymore.[26]

5. Impact of VAR-RM on the expected shortfall of financial institutions

In this section, we examine how VaR-RM impacts the expected shortfall of financial institutions, a measure of the expected debt conditional on defaults. For clarity, we assume that institutions are homogeneous in characteristics excepted their network position.

Under risk-taking vector $\mathbf{z}$, the expected shortfall of institution i is given by

$$\mathbb{E}S_i(\mathbf{z}) = -\mathbb{E}(\tilde{v}_i(\mathbf{z})|v_i(\mathbf{z}) < 0)$$

This is the expected debt due to a default caused by the shock hitting the institution. This expression depends on the structure of the shareholding network and on the nature of the shock hitting the financial system. To evaluate expected shortfall when financial institutions comply to VaR-RM risk-taking $\mathbf{z}^*$, we need to specify two probability distribution functions, exponential and power law. Recalling the equilibrium risk-taking $\mathbf{z}^*$ given by equation (8) and that $\ell = r - t_{1-\frac{n\beta}{q_0}}$, we obtain:

PROPOSITION 6. *Assume firms have homogeneous characteristics e, d, and suppose that Assumptions 1 to 5 hold.*

26. The symmetry of the cross-shareholding network is key. Relaxing the symmetry, the network can be detrimental to some institutions with respect to the no-network case by the presence of a negative resource effect.

Under Pareto distribution $P_a(s) = \frac{as_0^a}{s^{a+1}}$ *over the interval* $[s_0, +\infty)$*, for* $a > 1$*, the expected shortfall of institution* i *under VaR-RM is given by*

$$\mathbb{E}S_i(\mathbf{z}^*) = \left(\frac{t_{1-\frac{n\beta}{q_0}}}{a-1}\right) m_{ii} z_i^*$$

Under exponential distribution $P_\lambda(s) = \lambda e^{-\lambda(s-s_0)}$ *over the interval* $[s_0, +\infty)$*, for* $\lambda > 0$*, the expected shortfall of institution* i *under VaR-RM is given by*

$$\mathbb{E}S_i(\mathbf{z}^*) = \frac{m_{ii} z_i^*}{\lambda}$$

Proposition 6 shows that the shareholding network has a sensible impact on institutions' expected shortfall. On the one hand, the network alleviates the shortfall by transmitting to the shocked firm the positive values of others; on the other hand, the shareholding network acts as a multiplier of the defaulting position of the hit institution by propagating the default to other institutions. The impact of the shareholding network on the expected shortfall sharply depends on the nature of the shock. Still, it is immediate that more cross-shareholding, i.e. augmenting matrix $\mathbf{A}$, has an unambiguous effect (since matrix $\mathbf{M}$ and vector $\mathbf{z}^*$ are increased):

Corollary 2. *Under both Pareto and exponential distribution of the shock, more integration entails increased expected shortfall for all financial institutions.*

Table 5 illustrates the role of networks by comparing the average expected shortfall in the star network, in the circle network, in the core with two central agents, and in the complete network, for some particular parameter values; Parameter values are such that the probability distributions have the same mean.[27] The table shows that both network and nature of the shock have a

	Star	Wheel	two-Core	Complete
Power law	596	600	629	711
Exponential	170	171	179	203

FIGURE 5. Expected Shortfall of the financial system for different networks and different probability distributions of the shock, with $n = 8, \rho = 1.1, d = 1000, e = 500, \ell = 0.6, r = 1.3, p = 10, \beta = 0.1, s_{min} = 1, a = 2, \lambda = 1$.

strong impact on the average expected shortfall. First, the table confirms that the expected shortfall is increasing in network integration, as denser network are more likely to amplify the negative impact of the shock. Second, the impact of network structure can be highly differentiated according to the nature of the shock; overall, the Pareto distribution induces a substantially larger expected shortfall in average than the exponential distribution.

6. Capital injection

In the real world, regulatory authorities may want to boost investment in risky assets while keeping risks at acceptable levels. In such circumstances, the regulator may not want to directly impose constraints on firms' investments in risky assets, preferring instead to regulate the liability side of firms' balance sheets by defining adequate capital requirements. We can rewrite equation (8), which characterizes risk-taking under VaR-RM, as a relationship between one institution's initial risky asset (at $t = 0$, through z_i) and its liability (through e_i):

$$\mathbf{e}(\mathbf{z}) = (1 - \ell)(\mathbf{I} - \mathbf{A})\mathbf{M}_\varepsilon \mathbf{z} + (\rho - 1)\mathbf{d} \tag{9}$$

27. The mean of the Pareto (resp. exponential) distribution is $\frac{a s_{min}}{a-1}$ (resp. $s_{min} + \frac{1}{\lambda}$). Hence, for a given value $a > 1$ and $s_{min} > 0$, all probability distributions have the same means when $\lambda = \frac{a-1}{s_{min}}$.

Then, for any value of $\mathbf{z}$, $\mathbf{e}(\mathbf{z})$ represents the vector of capital requirements that keep default probabilities below a prescribed level (i.e., at value β). Equation (9) specifies the minimum external equity e_i an institution needs so as to be allowed to invest z_i in its risky asset. Here, the capital requirement for a given institution depends on its leverage, its resource effect, the overall risk profile, the cross-shareholding network, the value-at-risk ℓ, and the asset returns of other institutions when it receives a shock (r). Importantly, two firms with the same level of risk and leverage will not be required to hold the same level of capital if they have different network positions.

A regulator can then intervene by injecting capital into a single institution in such a way as to boost aggregate risky investments, subject to a constraint on the probability of default. This raises the question of which institution to target. Concretely, suppose that, while keeping default risk at the prescribed level corresponding to default probability β, the regulator chooses only one institution in which to inject equity, with the objective of maximizing aggregate investment in risky assets. The next proposition defines the institution that should be targeted. Defining matrix $\mathbf{W}$ such that $w_{ii} = m_{ii}$ and $w_{ij} = -\varepsilon m_{ij}$ for all i, j, and vector $\mathbf{w}^S = \mathbf{W}^{-1}\mathbf{1} = (w_i^S)_{i \in \mathcal{I}}$, the impact of adding one unit of external equity to institution i on the total investment in risky assets is given by

$$-\frac{1}{\varepsilon} + \left(\frac{1+\varepsilon}{\varepsilon}\right) m_{ii} w_i^S$$

We thus obtain:

Proposition 7. *The institution to target is the one with the highest index* $m_{ii} w_i^S$.

Proposition 7 is useful to determine the optimal institution to target on the basis of network properties and the relative magnitude of the negative shock (through parameter ε) only. To illustrate, consider again the network depicted in Figure 3, and take the same parameters. Then the optimal target, that maximizes the index given in Proposition 7, is institution 1. Note that this index is not aligned with risk-taking. For instance, institution 3's index is higher than that of institution 4, whereas the ranking of respective risk-taking levels is reversed.

7. Conclusion

Our investigation reveals that the configuration of cross-shareholding networks exerts a significant influence on the risk-taking behaviors of financial institutions governed by Value-at-Risk Management (VaR-RM). Specifically, we discovered that cross-shareholding relationships can mitigate some of the constraints imposed by VaR-RM on risk-taking activities. Furthermore, the particular arrangement of these cross-shareholdings can lead to a diverse range of risk-taking behaviors among individual institutions.

Importantly, these findings are predicated on the assumption of no contagion effects. However, in scenarios where the economy faces substantial exposure to adverse events, financial institutions relying on VaR-RM may not fully account for their systemic risk exposure. This oversight complicates the assessment of VaR-RM's role in influencing the financial system's susceptibility to crises and default contagions.

From an empirical standpoint, a critical takeaway from our study is the pressing need for comprehensive data collection on cross-investments among a broad spectrum of financial institutions—not limited to banks. Such data is crucial for a more accurate evaluation of the cross-holding network structure and its effect on the true risk exposure faced by these entities. Gathering this information will significantly enhance our understanding of systemic risks and inform more effective risk management strategies within the financial sector.

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Appendix A: Proofs

The following lemma – reminiscent of Eisenberg and Noe (2001) – establishes the uniqueness of values satisfying the system of equations (2):

LEMMA A.A.1. *For any financial network* $(\mathbf{d}, \mathbf{e}, \mathbf{P})$*, any investment profile* $\mathbf{z} \in [\mathbf{0}, \mathbf{e}+\mathbf{d}]$*, and any realization of risks* $(\mu_i)_{i\in\mathcal{I}}$*, there is a single set of values* $\mathbf{v}$ *solving system (2) for all* i *(with possible defaults).*

Proof of Lemma A.A.1.

We define $h_i = (\mu_i - 1)z_i + \eta_i$ and $\mathbf{h} = (h_i)_{i\in\mathcal{I}}$. Equation (2) then simply writes:

$$v_i = h_i + \sum_{j\neq i} a_{ij}\max(v_j, 0) \tag{A.1}$$

that is, in the absence of default (if $v_i \geq 0$ for all i):

$$\mathbf{v} = \mathbf{M}\mathbf{h} \tag{A.2}$$

where $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$. The largest eigenvalue of any sharing matrix $\mathbf{A}$ is lower than unity (as the sum of every column is lower than 1). Therefore, $(\mathbf{I} - \mathbf{A})^{-1} = \sum_{q=0}^{\infty} \mathbf{A}^q$.

Consider the system:

$$v_i = \max\Big(0, h_i + \sum_{j\in\mathcal{I}} a_{ij}v_j\Big) \quad \forall i \in \mathcal{I}$$

In v_is, this corresponds to a game of strategic complementarities with lower and upper bounds (for given μ_is), i.e. a supermodular game. Therefore, it possesses a minimum and a maximum equilibrium.

Now, consider an equilibrium with $\mathcal{S}$ non defaulting institutions, i.e. with $\mathbf{v}_{\mathcal{S}} = (v_1, \cdots, v_s) > 0$ and let $\bar{a}_i = 1 - \sum\limits_{k\in\mathcal{S}} a_{ki}$. Then,

$$\sum_{i\in\mathcal{S}} \bar{a}_i v_i = \sum_{i\in\mathcal{S}} \Big(1 - \sum_{k\in\mathcal{S}} a_{ki}\Big) v_i$$

and given that $\sum\limits_{i\in\mathcal{S}} v_i = \sum\limits_{i\in\mathcal{S}} h_i + \sum\limits_{i\in\mathcal{S}}\sum\limits_{k\in\mathcal{S}} a_{ik}v_k$:

$$\sum_{i\in\mathcal{S}} \bar{a}_i v_i = \sum_{i\in\mathcal{S}} h_i$$

Last, suppose that the minimum equilibrium, say $\mathcal{S}$, is distinct from the maximum equilibrium, say $\mathcal{S}'$. Then $\mathbf{v}_{\mathcal{S}} < \mathbf{v}_{\mathcal{S}'}$ (we use here vectorial inequality) and

$$\sum_{i\in\mathcal{S}} h_i = \sum_{i\in\mathcal{S}} \bar{a}_i v_i < \sum_{i\in\mathcal{S}} \bar{a}_i v_i' < \sum_{i\in\mathcal{S}'} \bar{a}_i v_i' = \sum_{i\in\mathcal{S}'} h_i$$

However, by construction, for all institutions $i \in \mathcal{S}' \setminus \mathcal{S}$: $h_i < 0$. Indeed, by (A): $h_i > 0 \Rightarrow v_i > 0$ and all institutions with $h_i > 0$ always belong to the surviving set. Then, $\sum_{i\in\mathcal{S}'} h_i < \sum_{i\in\mathcal{S}} h_i$, which is in contradiction with (A). The equilibrium values are then unique.

□

The proof of Lemma A.A.1 rests on the complementarities between institutions' values, which, under multiplicity, would imply a minimum and a maximum configuration solving the system. Now, the total equity invested in the financial system is identical in both configurations, while there would be greater debt repayment in the maximum configuration, due to a larger number of survivors. This would leave less wealth to distribute in the maximum configuration than in the minimum configuration, despite the higher values in the maximum configuration. Hence, the two configurations coincide, implying uniqueness.[28]

Proof of Proposition 1.

In this proof, the matrix $\mathbf{P}$ can be asymmetric. Denote $\mathbf{P}_i = \sum_k p_{ki}$ for convenience. We can write $\mathbf{A} = \mathbf{PW}$, where $\mathbf{W}$ is a diagonal matrix with

28. Complementarity in values also leads to a simple algorithm that pins down the equilibrium set of surviving institutions. Start with an initial set containing all institutions with positive constant h_i, and compute their values in this initial setting. Then extend the set by systematically testing neighbors as newcomers, and check whether each newcomer has a positive value. If so, include it in the set of survivors. This is an efficient algorithm: A newcomer to the current set of survivors never forces other survivors out of the set, which thus only expands during the process.

diagonal entry $\mathbf{W}_{ii} = \frac{1}{P_i+e}$. We have

$$\mathbf{b} = \mathbf{1} + (\mathbf{PW})\mathbf{1} + (\mathbf{PW})^2\mathbf{1} + \cdots$$

That is,

$$\mathbf{b} = \mathbf{1} + \mathbf{P}\Big(\mathbf{I} + \mathbf{WP} + (\mathbf{WP})^2 + \cdots\Big)\mathbf{W1}$$

Now, $\mathbf{WP} = \tilde{\mathbf{A}} = (\frac{p_{ij}}{P_i+e})$ (note that $\tilde{\mathbf{A}} = \mathbf{A}^T$ if $\mathbf{P}^T = \mathbf{P}$). Then,

$$\mathbf{b} = \mathbf{1} + \mathbf{P}(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{W1}$$

The solution $\mathbf{x}$ of $(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{x} = \mathbf{1}$ satisfies

$$\mathbf{x} = (\mathbf{I} - \tilde{\mathbf{A}})\mathbf{1}$$

i.e., for entry i, $\mathbf{x}_i = 1 - \sum_j \frac{p_{ij}}{P_i+e} = e[\mathbf{W1}]_i$ (exploiting $\mathbf{P}^T = \mathbf{P}$), or

$$\mathbf{x} = e\mathbf{W1}$$

We thus obtain that

$$\mathbf{b} = \mathbf{1} + \frac{1}{e}\mathbf{P}\underbrace{(\mathbf{I} - \tilde{\mathbf{A}})^{-1}e\mathbf{W1}}_{=\mathbf{1}} = \mathbf{1} + \frac{1}{e}\mathbf{P1}$$

□

Proof of Theorem 1.

The matrix form of the system of equations (7) is given by:

$$(\mathbf{I} - \varepsilon\mathbf{C})\mathbf{z} = \frac{\eta}{1-\ell}(\mathbf{I} + \mathbf{C})\mathbf{1}$$

i.e.,

$$(\mathbf{I} - \varepsilon\mathbf{C})\mathbf{z} = \frac{\eta}{1-\ell}\left[-\frac{1}{\varepsilon}(\mathbf{I} - \varepsilon\mathbf{C})\mathbf{1} + \frac{1+\varepsilon}{\varepsilon}\mathbf{1}\right]$$

i.e., noting that $\frac{\eta}{(1-\ell)\varepsilon} = \frac{\eta}{r-1}$,

$$\mathbf{z} = \frac{\eta}{r-1}\left[(1+\varepsilon)(\mathbf{I}-\varepsilon\mathbf{C})^{-1}\mathbf{1}-\mathbf{1}\right]$$

Uniqueness is guaranteed by $\eta > 0$ and $\varepsilon > 0$ (see Belhaj et al., 2014), a direct implication from Assumption 4. Assumption 5 guarantees interiority.

□

Proof of Corollary 1.

In our setting $\varepsilon \to 0$ when $|\ell|$ is large, that is when β is low, which corresponds to situations with tight regulation. The Corollary stems from observing that

$$\lim_{\varepsilon\to 0} \mathbf{z}^* = \frac{\eta}{1-\ell}(\mathbf{I}+\mathbf{C})\mathbf{1} \tag{A.3}$$

and by remarking that entry i of vector $(\mathbf{I}+\mathbf{C})\mathbf{1}$ is equal to $\frac{b_i}{m_{ii}}$. Recalling that $b_i = \frac{e+p\delta_i}{e}$, the result follows.

□

Proof of Proposition 3.

The following lemma shows an increasing relationship between matrix $\mathbf{A}$ and matrix $\mathbf{C}$:

Lemma A.A.2. *If* $\mathbf{A}' \leq \mathbf{A}$, *then* $\mathbf{C}' \leq \mathbf{C}$.

Proof of Lemma A.A.2.

The proof relies on the Sherman-Morrison formula, that states: Suppose $\mathbf{Q}$ is an invertible n-square matrix with real entries and $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n$ are column vectors. Then $\mathbf{Q}+\mathbf{r}\mathbf{s}^T$ is invertible if and only if $1+\mathbf{s}^T\mathbf{Q}^{-1}\mathbf{r} \neq 0$. If $\mathbf{Q}+\mathbf{r}\mathbf{s}^T$ is invertible, its inverse is given by

$$(\mathbf{Q}+\mathbf{r}\mathbf{s}^T)^{-1} = \mathbf{s}^{-1} - \frac{\mathbf{Q}^{-1}\mathbf{r}\mathbf{s}^T\mathbf{Q}^{-1}}{1+\mathbf{s}^T\mathbf{Q}^{-1}\mathbf{r}} \tag{A.4}$$

We apply this formula with $\mathbf{Q} = \mathbf{I} - \mathbf{A}$ and $\mathbf{r}\mathbf{s}^T = -\mathbf{\Omega}$, where $\mathbf{\Omega} = [\omega_{ij}]$ is such that $\omega_{ij} = \omega$ if $(i,j) = (r,s)$, $\delta_{ij} = 0$ otherwise. Then matrix $\mathbf{\Omega}$ has a single non-zero entry, corresponding to a positive impulsion at the entry (r,s). It is easily shown that $\mathbf{\Omega} = -\mathbf{r}\mathbf{s}^T$ for $\mathbf{r} = (0, \cdots, 0, \omega, 0, \cdots, 0)^T$ with ω at entry r, and $\mathbf{s}^T = (0, \cdots, 0, 1, 0, \cdots, 0)^T$ with 1 at entry s.

Applying the formula, noting $(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{M}$ and $\mathbf{s}^T\mathbf{M}\mathbf{r} = -m_{rs}\omega$, we get

$$(\mathbf{I} - \mathbf{A} - \mathbf{\Omega})^{-1} = \mathbf{M} + \frac{\mathbf{M}\mathbf{\Omega}\mathbf{M}}{1 - m_{rs}\omega}$$

Now the entry (i,j) of matrix $\mathbf{M}\mathbf{\Omega}\mathbf{M}$ is given by $[\mathbf{M}\mathbf{\Omega}\mathbf{M}]_{ij} = m_{ir}m_{sj}\omega$. Then,

$$[(\mathbf{I} - \mathbf{A} - \mathbf{\Omega})^{-1}]_{ij} = m_{ij} + \frac{m_{ir}m_{sj}\omega}{1 - m_{sr}\omega}$$

We want to prove that the ratio $\frac{m_{ij}}{m_{ii}}$ increases for all i, j when $\mathbf{A}$ becomes $\mathbf{A}' = \mathbf{A} + \mathbf{\Omega}$. Note that $\frac{m_{ij}}{m_{ii}} \leq \frac{m_{ij}+a}{m_{ii}+b}$ if and only if $\frac{m_{ij}}{m_{ii}} \leq \frac{a}{b}$. Then it is sufficient to prove that $\frac{m_{ij}}{m_{ii}} \leq \frac{m_{ir}m_{sj}\omega}{m_{ir}m_{si}\omega}$, i.e.

$$\frac{m_{ij}}{m_{ii}} \leq \frac{m_{sj}}{m_{si}} \tag{A.5}$$

Now the path product property of any inverse M-matrix $\mathbf{Y}$ (see for instance Johnson and Smith, 2007, p. 329) writes

$$y_{ij}y_{jk} \leq y_{ik}y_{jj} \tag{A.6}$$

Equation (A.5) can be written:

$$m_{si}m_{ij} \leq m_{ii}m_{sj}$$

that is, permuting labels i and j:

$$m_{sj}m_{ji} \leq m_{jj}m_{si}$$

and, permuting labels i and q:

$$m_{ij}m_{js} \leq m_{jj}m_{is}$$

which corresponds to the path product property with i, j, s as shown by equation (A.6). $\mathbf{M}$ being an inverse M-matrix, we therefore have that $\mathbf{A}' > \mathbf{A}$ leads to $\mathbf{C}' > \mathbf{C}$, where $c_{ij} = m_{ij}/m_{ii}$ and $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$.

□

By Lemma A.A.2, increasing the integration of the network of cross-shares induces an increase in the entries of matrix $\mathbf{C}$. The proof of Lemma A.A.2 relies on the path-product property of inverse M-matrices.[29] In particular, for any $p' \geq p$, we obtain $\mathbf{A}' \geq \mathbf{A}$ on a fixed network $\mathbf{G}$. That is, increasing the amount of investment from any existing investment increases the cross-share matrix.

We can now examine the impact of an increase in the cross-shareholding matrix $\mathbf{A}$ risk-taking under VaR-RM. At the interior equilibrium, the matrix $(\mathbf{I} - \varepsilon\mathbf{C})^{-1}$ is well-defined and nonnegative, so that $(\mathbf{I} - \varepsilon\mathbf{C})^{-1} = \sum_{k\geq 0} \varepsilon^k \mathbf{C}^k$. This implies that increased matrix $\mathbf{C}$ induces increased matrix $(\mathbf{I} - \varepsilon\mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})$. Therefore, the risk-taking $\mathbf{z}^*$ under VaR-RM increases when cross-shareholding increases.

□

Proof of Proposition 2.

Step 1. Let us first show that the ratio b_i/m_{ii} is higher for the center of the star under the fixed participation case.

Consider a star network with n agents. We denote by 1 the center of the star and by 2 the representative periphery. We denote $a = a_{12}$ and $b = a_{21}$.

29. An M-matrix is a n-by-n matrix with non-positive off-diagonal entries and has an entry-wise non-negative inverse. In our case, $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$ is then an inverse M-matrix.

Under fixed participation: $a = 1/(1 + e/p)$ and $b = 1/(n - 1 + e/p)$. Then,

$$\mathbf{I} - \mathbf{A} = \begin{pmatrix} 1 & -a & -a & \cdots & -a \\ -b & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -b & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1} = \left(\frac{1}{1-(n-1)ab}\right) \mathbf{Q}$ with

$$\mathbf{Q} = \begin{pmatrix} 1 & a & a & \cdots & a \\ b & 1-(n-2)ab & ab & \cdots & ab \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b & ab & ab & \cdots & 1-(n-2)ab \end{pmatrix}$$

This gives $b_1^O/m_{11} = 1 + (n - 1)a$ and $b_2^O/m_{22} = (1 + b)/(1 - (n - 2)ab)$; so that the ratio b_i^O/m_{ii} is higher for the center when

$$1 + (n - 2)a + (n - 2)(n - 1)a^2 < (n - 1)\frac{a}{b} \tag{A.7}$$

Considering the values of a and b for the fixed participation case, this corresponds to

$$(n-1)\frac{n-1+\frac{e}{p}}{1+\frac{e}{p}} > \frac{(n-1)(n-2)}{(1+\frac{e}{p})^2} + \frac{n-2}{1+\frac{e}{p}} + 1$$

Multiplying both sides by $(1 + e/p)$ and simplifying, we get

$$\left(1+\frac{e}{p}\right)\left(n-1+\frac{e}{p}\right) > n-1$$

which holds true as $e/p > 0$, meaning that the ratio b_i/m_{ii} is higher for the center of a star in the fixed participation case.

Step 2. We now prove by induction that $z^*_{RS,1} > z^*_{RS,2}$, i.e. that the risk-taking level is higher for the center of the star than for any of the periphery.

To do so, we simply need to show that $\forall q\ (\mathbf{C}^q\mathbf{1})_1 > (\mathbf{C}^q\mathbf{1})_2$. Now, by step 1, we know that $(\mathbf{C1})_1 > (\mathbf{C1})_2$. For convenience, let us $\psi_1 = (\mathbf{C1})_1$, $\psi_2 = (\mathbf{C1})_2$, and more generally, $\psi_1^{(q)} = (\mathbf{C}^q\mathbf{1})_1$, $\psi_2^{(q)} = (\mathbf{C}^q\mathbf{1})_2$ for all $q \geq 1$.

Let property $\mathcal{P}(q) : \varphi_c^{(q)} > \varphi_p^{(q)}$. Assume $\mathcal{P}(1), \cdots, \mathcal{P}(q-1)$. We will prove $\mathcal{P}(q)$. First note that

$$\psi_1^{(q)} = \psi_1 \psi_1^{(q-1)}$$

and

$$\psi_2^{(q)} = c_{21}\psi_1^{(q-1)} + (\psi_2 - c_{pc})\psi_2^{(q-1)}$$

The inequality $\psi_1^{(q)} > \psi_2^{(q)}$ then means

$$(\psi_1 - \psi_2)\,\psi_2^{(q-1)} > c_{21}\left(\psi_1^{(q-1)} - \psi_2^{(q-1)}\right) \tag{A.8}$$

Now, by $\mathcal{P}(q-1)$, we have

$$\psi_1\psi_2^{(q-2)} > c_{21}\psi_1^{(q-2)} + (\psi_2 - c_{21})\psi_2^{(q-2)}$$

and inequality (A.8) also writes

$$(\psi_1 - \psi_2)\psi_2^{(q-1)} > c_{21}\left((\psi_1 - \psi_2)\varphi_2^{(q-2)} - c_{21}\left(\psi_1^{(q-2)} - \psi_2^{(q-2)}\right)\right)$$

that is

$$(\psi_1 - \psi_2)\left(c_{21}\left(\psi_1^{(q-2)} - \psi_2^{(q-2)}\right) + (\psi_2 - c_{21})\psi_2^{(q-2)}\right) > -c_{21}^2\left(\psi_1^{(q-2)} - \psi_2^{(q-2)}\right)$$

which holds whenever $\psi_2 - c_{21} > 0$. Now $\psi_2 > c_{21}$ corresponds to

$$\frac{\sum_{j\neq 2} m_{2j}}{m_{22}} > \frac{m_{21}}{m_{22}}$$

which always holds as $m_{ij} \geq 0\ \forall i,j$. Therefore $\mathcal{P}(q)$ holds, whenever $\mathcal{P}(q-1)$ holds. As $\mathcal{P}(1)$ holds by Step 1, we have that regulated risk-taking is always higher for the center of the star than for the periphery.

□

Proof of Proposition 4.

Suppose that $\varepsilon < 0$, and then define $e = -\varepsilon > 0$ for convenience. Define also $\alpha = \frac{\eta}{1-\ell} > 0$. The optimal risk-taking then solves

$$(\mathbf{I} + e\mathbf{C})\mathbf{z} = \alpha(\mathbf{I} + \mathbf{C})\mathbf{1}$$

i.e.,

$$(\mathbf{I} + e\mathbf{C})\mathbf{z} = \frac{\alpha}{e}(\mathbf{I} + e\mathbf{C})\mathbf{1} - \alpha\frac{1-e}{e}\mathbf{1}$$

i.e.,

$$\mathbf{z} = \frac{\alpha}{e}\left[1 - (1-e)(\mathbf{I} + e\mathbf{C})^{-1}\mathbf{1}\right]$$

That is, risk-taking is a decreasing function of the solution of a linear system of substitute interaction. Now, it is well-known that, in a classical STS system, increasing interaction decreases the average output. This implies that risk-taking increases, in average, under increased matrix $\mathbf{C}$. And, by Lemma A.A.2, an increase in the shareholding matrix entails an increase in matrix $\mathbf{C}$.

□

Proof of Proposition 5.

Consider that institution i is hit by the shock. By equation (2), the value of a surviving institution i exerting risk-taking level z_i, and given that others' VaR-RM risk-taking level, is given by

$$v_i(z_i) = f_i(z_i) + g(z_i)$$

where $f_i(z_i) = (\ell - 1)z_i + \eta$, and where $g(z_i) = \sum_{j \neq i} a_{ij} v_j^+(z_j^*(z_i)) \geq 0$.

When there is no network, VaR-RM entails $f_i(z_i^0) = 0$; While, when there is a network, VaR-RM entails $f_i(z_i^*) = -g(z_i^*) \leq 0$. Since $\ell < 1$, function f_i is decreasing, which implies that $z_i^0 \leq z_i^*$.

□

Proof of Proposition 6.

Under risk-taking vector $\mathbf{z}$, the expected shortfall of institution i is given by

$$\mathbb{E}S_i(\mathbf{z}) = -\mathbb{E}(\tilde{v}_i(\mathbf{z})|v_i(\mathbf{z}) < 0)$$

Denoting

$$\Gamma_i(\mathbf{z}) = (r-1)\sum_{k\in\mathcal{I}} m_{ik}z_k + b_i\eta$$

we get

$$\tilde{v}_i(\mathbf{z}) = \Gamma_i(\mathbf{z}) - m_{ii}z_i\tilde{s}$$

When a shock hits institution i, we have $v_i = 0$ for a realization of the shock $s_i^0(\mathbf{z})$ such that

$$s_i^0(\mathbf{z}) = \frac{\Gamma_i(\mathbf{z})}{m_{ii}z_i}$$

The expected value of an institution i, conditional on defaulting, is thus given by

$$\mathbb{E}(\tilde{v}_i|v_i < 0) = \Gamma_i(\mathbf{z}) - m_{ii}z_i\mathbb{E}\left(\tilde{s}|s > \frac{\Gamma_i(\mathbf{z})}{m_{ii}z_i}\right)$$

Hence, the expected shortfall of institution i is written as

$$\mathbb{E}S_i(\mathbf{z}) = -\Gamma_i(\mathbf{z}) + m_{ii}z_i\mathbb{E}\left(\tilde{s}|s > \frac{\Gamma_i(\mathbf{z})}{m_{ii}z_i}\right)$$

We explore now the impact of risk-taking behavior $\mathbf{z}^*$, issued from VaR-RM, on the expected shortfall, for both Pareto and exponential probability distributions.

- Assume that the shock $\tilde{s}$ has a Pareto density distribution over $[s_0, +\infty)$: $P_a(s) = \frac{as_0^a}{s^{a+1}}$ for $a > 1$. Then it is well-known that

$$\mathbb{E}(\tilde{s}|s > s_i^0(\mathbf{z})) = \frac{as_i^0(\mathbf{z})}{a-1}$$

This implies that

$$\mathbb{E}(\tilde{v}_i | v_i < 0) = \Gamma_i(\mathbf{z}) - m_{ii} z_i \cdot \frac{a s_i^0(\mathbf{z})}{a-1}$$

which, given $m_{ii} z_i s_i^0 = \Gamma_i(\mathbf{z})$, is simplified as

$$\mathbb{E}(\tilde{v}_i | v_i < 0) = -\frac{\Gamma_i(\mathbf{z})}{a-1}$$

We therefore get, for the Pareto distribution of the shock of parameter a,

$$\mathbb{E}S_i(\mathbf{z}^*) = \frac{1}{a-1}\Gamma_i(\mathbf{z}^*)$$

Recalling that $\ell = r - t_{1-\frac{n\beta}{q_0}}$ and that the FOC defining VaR-RM risk-taking gives $\Gamma_i(\mathbf{z}^*) = t_{1-\frac{n\beta}{q_0}} \cdot m_{ii} z_i^*$, we deduce

$$\mathbb{E}S_i(\mathbf{z}^*) = \left(\frac{t_{1-\frac{n\beta}{q_0}}}{a-1}\right) m_{ii} z_i^*$$

- Assume that the shock $\tilde{s}$ has an exponential density distribution: $P_\lambda(s) = \lambda e^{-\lambda(s-s_0)}$ for $\lambda > 0$. Then

$$\mathbb{E}(\tilde{s} | s > s_i^0(\mathbf{z})) = s_i^0(\mathbf{z}) + \frac{1}{\lambda}$$

This implies that

$$\mathbb{E}(\tilde{v}_i | v_i < 0) = \Gamma_i(\mathbf{z}) - m_{ii} z_i \cdot \left(s_i^0(\mathbf{z}) + \frac{1}{\lambda}\right)$$

which, given $m_{ii} z_i s_i^0(\mathbf{z}) = \Gamma_i(\mathbf{z})$, is simplified as

$$\mathbb{E}(\tilde{v}_i | v_i < 0) = -\frac{m_{ii} z_i}{\lambda}$$

We thus obtain, for the exponential distribution of parameter λ,

$$\mathbb{E}S_i(\mathbf{z}^*) = \frac{m_{ii} z_i^*}{\lambda}$$

□

Proof of Proposition 7.

Defining $\upsilon_i = \frac{1}{1-\ell}\left(m_{ii}\eta_i + \sum_{j\neq i} m_{ij}\eta_j\right)$, the initial $\mathbf{z}^*$ solves

$$m_{ii}z_i^* - \varepsilon\sum_{j\neq i} m_{ij}z_j^* = \upsilon_i$$

Or, in matrix notation,

$$\mathbf{W}\mathbf{z}^* = \boldsymbol{\upsilon}$$

where $\mathbf{W}$ is a n-dimensional square matrix such that $w_{ii} = m_{ii}$ and $w_{ij} = -\varepsilon m_{ij}$; and $\boldsymbol{\upsilon} = (\upsilon_i)_{i\in\mathcal{I}}$.

Suppose now that one $1-\ell$ unit of cash in the external equity of institution 1 (for ease of exposition, all the following addresses institution i). Letting $\mathbf{m}_1 = (m_{11}, m_{21}, \cdots, m_{n1})^T$ be the first column of matrix $\mathbf{M}$, the optimal risk-taking $\mathbf{z}'^*$ then writes

$$\mathbf{W}\mathbf{z}'^* = \boldsymbol{\upsilon} + \mathbf{m}_1$$

and the change in total investment in the risky asset is

$$\mathbf{1}^T(\mathbf{z}' - \mathbf{z}) = \mathbf{1}^T\mathbf{W}^{-1}\mathbf{m}_1$$

Observing that

$$\mathbf{m}_1 = -\frac{1}{\varepsilon}\begin{pmatrix} m_{11} \\ -\varepsilon m_{21} \\ \cdots \\ -\varepsilon m_{n1} \end{pmatrix} + \frac{1+\varepsilon}{\varepsilon}\begin{pmatrix} m_{11} \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

we obtain

$$\mathbf{W}^{-1}\mathbf{m}_1 = -\frac{1}{\varepsilon}\begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix} + \frac{1+\varepsilon}{\varepsilon}\mathbf{W}^{-1}\begin{pmatrix} m_{11} \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

Thus, defining $\mathbf{1}^T\mathbf{W}^{-1} = (w_1^S, w_2^S, \cdots, w_n^S)$, so that w_i^S is the sum of entries of column i in matrix $\mathbf{W}^{-1}$, we obtain that

$$\mathbf{1}^T(\mathbf{z}' - \mathbf{z}) = -\frac{1}{\varepsilon} + \left(\frac{1+\varepsilon}{\varepsilon}\right) m_{11} w_1^S$$

The highest effect on total investments in risky assets $(\mathbf{1}^T(\mathbf{z}' - \mathbf{z}))$ is achieved by targeting for capital injection the institution with the highest index $m_{ii}w_i^S$.

□

Appendix B: Optimal risk-taking in the absence of VaR-RM

In this appendix, we explore the behavior of financial institutions in the absence of Value-at-Risk Management. Each institution is risk neutral and maximizes its expected equity value $\mathbb{E}(v_i)$.[30] In this setup, obviously firms take more risk than under VaR-RM. However, the presence of the extreme event may lead them not to put all resource in the risky asset. We give an upper bound on the probability of the bad event under which institutions still put all their resource in the risky asset.

In this setup, the institution's decision consists in allocating its resources between the risk free asset and its specific risky asset. Using equation (2), this comes to:

$$\max_{z_i \in [0, e_i + d_i]} \mathbb{E}\left(\max\left((\tilde{\mu}_i - 1)z_i + \eta_i + \sum_{j \neq i} a_{ij}\tilde{v}_j, 0 \right) \right) \tag{B.1}$$

In the above expression, the realizations of any $\tilde{v}_j$ is necessarily nonnegative through equation (2).

Even if the average return on the risky asset is larger than the one of the risk-free asset, through network effects, risk-neutral institutions may not want

30. Managers' and equity-holders' objectives are assumed to be aligned. We thus ignore agency issues inside the institution.

to put all their resource to the risky asset. This phenomenon arises when the probability that the shock hits the financial system, q_0, is high. Indeed, on top of the classical direct effect (the first term of (B.1), positive under Assumption 2), the level of risk-taking by one institution also impacts its value through self-loops in the risk-sharing network. This last effect can dominate when the probability of shock q_0 is large. As we want to focus here – consistently with current regulation – on extreme event that occurs with low probability, we assume that this q_0 is low enough (Assumption B.1), so that the first classical effect dominates.

Assumption B.1. $q_0 \leq \frac{1}{1-\frac{1}{n}\left(\frac{r-\overline{s}-1}{r-1}\right)}$.

Indeed:

Proposition B.B.1. *Under Assumptions 1, 2 and B.1, the expected value of a financial institution is increasing with its risk-taking level. Then unregulated institutions optimally allocate all their resources to the risky asset:* $z_i^{*u} = u_i\ \forall i$.

By Proposition B.B.1, institutions invest their whole resource in the risky asset when the probability that the negative shock hits the financial system is sufficiently low.

Proof of Proposition B.B.1.

Lemma B.B.1. *For all* μ, $\mathbf{z}, \mathbf{z}' = (z_i', z_{-i})$ *such that* $z_i \leq z_i'$,

$$v_i(\mathbf{z}') - v_i(\mathbf{z}) \geq (\mu_i - 1)m_{ii}(z_i' - z_i)$$

Proof of Lemma B.B.1.

Call $\mathcal{I}$ the set of surviving institutions under $(\mu, \mathbf{z})$, $\mathcal{I}'$ the set of surviving institutions under $(\mu, \mathbf{z}')$, and $\mathbf{M}$ and $\mathbf{M}'$ the respective invert matrices of the systems. We have

$$\begin{cases} v_i = \sum_{j\in\mathcal{I}} m_{ij}((\mu_j-1)z_j+\eta_j) \\ v_i' = \sum_{j\in\mathcal{I}'} m_{ij}'((\mu_j-1)z_j'+\eta_j) \end{cases}$$

Given the structure of risk of the model, there are three cases to consider. Either the change in institution i's risk-taking does not affect the set of surviving institutions (Case (i)), or it implies one more surviving institution (Case (ii)), or it implies one less surviving institution (Case (iii)).

Case (i) $\mathcal{I}=\mathcal{I}'$. Then $\mathbf{M}'=\mathbf{M}$, and:

$$v_i'-v_i=(\mu_i-1)m_{ii}(z_i'-z_i)$$

Case (ii): $\mu_i>1$ and $\mathcal{I}'=\mathcal{I}\cup\{k\}$. Hence, $\mathbf{M}\leq\mathbf{M}'(\mathcal{I}')$. Consider the level $z_i^c\in(z_i,z_i')$, at which institution k becomes knife-edge, i.e. such that its value is equal to zero under both systems $\mathcal{I}$ and $\mathcal{I}'$. Such a value exists by continuity. Denote by v_i^c the value of institution i at (z_i^c,z_{-i}). Then,

$$v_i'-v_i=v_i'-v_i^c+v_i^c-v_i$$

Consider $v_i^c-v_i$. We are here in Case (i), and then

$$v_i^c-v_i=(\mu_i-1)m_{ii}(z_i'-z_i)$$

Now consider $v_i'-v_i^c$. We are here in Case (i) again, but with matrix $\mathbf{M}'$; we deduce

$$v_i'-v_i^c=(\mu_i-1)m_{ii}'(z_i'-z_i)$$

Therefore,

$$v_i'-v_i=(\mu_i-1)\Big(m_{ii}'(z_i'-z_i^c)+m_{ii}(z_i^c-z_i)\Big)$$

And since $\mu_i > 1$ and $m_{ii} < m'_{ii}$, we obtain

$$v'_i - v_i \geq (\mu_i - 1)m_{ii}(z'_i - z_i)$$

Case (iii): $\mu_i < 1$ and $\mathcal{I} = \mathcal{I}' \cup \{k\}$. Hence, $\mathbf{M} \geq \mathbf{M}'(\mathcal{I})$; note that $\mu_i < 1$ cannot induce that higher risk-taking from i hurts institution k's health. Consider the level $z_i^c \in (z_i, z'_i)$, at which institution k becomes knife-edge, i.e. such that its value equal to zero under both systems $\mathcal{I}$ and $\mathcal{I}'$ (like Case (ii), such a value exists by continuity). Replicating the same argument as Case (ii), we find

$$v'_i - v_i = (\mu_i - 1)\Big(m'_{ii}(z'_i - z_i^c) + m_{ii}(z_i^c - z_i)\Big)$$

And since $\mu_i < 1$ and $m_{ii} > m'_{ii}$, we obtain

$$v'_i - v_i \geq (\mu_i - 1)m_{ii}(z'_i - z_i)$$

□

By Lemma B.B.1, following an increase in z_i (from z_i to z'_i), the gap in the expected value of institution i is bounded from below; i.e., denoting $\Delta z_i = z'_i - z_i$ and $\underline{m}_{ii} = \min_{j \neq i} m_{ii}^{-j}$:

$$\mathbb{E}(v'_i) - \mathbb{E}(v_i) \geq (1 - q_0)(r - 1)m_{ii}\Delta z_i + \frac{q_0}{n}(r - \overline{s} - 1)m_{ii}\Delta z_i + \frac{q_0(n-1)}{n}(r - 1)\underline{m}_{ii}\Delta z_i \quad \text{(B.2)}$$

In the RHS, the first term corresponds to the no-shock case, the second term corresponds to institution i being hit by the shock, and the third term corresponds to another institutions being hit. Importantly, this is adapted from Case (ii) in lemma B.B.1 and leads to bound the value from below with the term $\underline{m}_{ii}$, which is such that $\underline{m}_{ii} < m_{ii}$ (under return greater than unity, the self-loop of institution i allowing to give a lower bound is that of the smallest network).

Since $r > 1$, from inequality (B.2), a sufficient condition for $\mathbb{E}(v_i') - \mathbb{E}(v_i) \geq 0$ (after dropping the third negative term and the negative quantities associated with the unit return in both first and second term) is given by

$$(1 - q_0)(r - 1) + \frac{q_0}{n}(r - \overline{s} - 1) \geq 0$$

That is,

$$q_0 \leq \frac{1}{1 - \frac{1}{n}\left(\frac{r - \overline{s} - 1}{r - 1}\right)}$$

□

Appendix C: Multiple shocks

This appendix presents a possible modeling of risk management under multiple shocks hitting the financial system. The overall model generates complementarities in risk-taking levels, but multiple shocks bring equilibrium multiplicity. Even under equilibrium multiplicity, the comparative statics presented in the single-shock case, like Lemma A.A.2, still generically hold at least locally.

In this extension, the catastrophic event affects $q + 1$ institutions at the same time, with $q \in \{0, 1, \cdots, n - 1\}$ ($q = 0$ in the benchmark model with a single shock)[31], but the shock hits the financial system at random with uniform probability across institutions. A prudent risk management imposes an upper bound on the default probability of each bank conditionally on being shocked and any other q banks shocked and defaulted.[32] This objective leads to the

31. For simplicity, this value is assumed to be common knowledge among institutions.

32. Alternatively, firms may want to bound the *unconditional* probability of default, rather than the probability of default conditional to the worst state of nature. In this case, institutions should take into account the default probabilities of other shocked institutions. This alternative scenario can hardly be explored analytically.

following managerial constraints:

$$\mathbb{P}(\tilde{v}_i < 0 | v_{k_1} = 0, \cdots, v_{k_q} = 0) \leq \beta \ \forall i, \forall \{k_1, \cdots, k_q\} \subset \mathcal{I} \setminus \{i\} \tag{C.1}$$

where all institutions in $\{k_1, \cdots, k_q\}$ are shocked and defaulted. Hence, institution i should survive with probability β to a negative shock hitting it with certainty plus any q other simultaneous shocks hitting other institutions. We define the set of critical institutions to institution i as the set of institutions such that the above equation is binding.

Conforming to the worst-case-scenario basis of VaR-RM, critical institutions to any institution i are those whose dropout hurts institution i's expected value the most.[33] The set of critical institutions of any institution i, as well as institution i's best-response risk-taking, are determined jointly. We define the finite set $\mathcal{S}_i$ of all subsets of q distinct institutions out of the set $\mathcal{I} \setminus \{i\}$; $\mathcal{S}_i = \emptyset$ in the single-shock case $q = 0$.

To evaluate how the dropout of a given group of shocked institutions $\mathcal{S} \in \mathcal{S}_i$ affects the value of institution i, we need to take into account that the dropout restricts the interactions system generating institution i's best-response risk-taking. To take into account that a defaulting institution does not transmit any value to others (particularly a negative value), we introduce the modified cross-holding matrix $\mathbf{A}^{\mathcal{S}}$, in which each share invested in a defaulting institution in the set $\mathcal{S}$ is put to zero; that is, for every institution $k \in \mathcal{I}$, for all $j \in \mathcal{S}$, $a^{\mathcal{S}}_{kj} = 0$. We denote by $\mathbf{C}^{\mathcal{S}}$ the analogous matrix to matrix $\mathbf{C}$ associated with cross-holding matrix $\mathbf{A}^{\mathcal{S}}$. When the shocked institutions are in the set $\mathcal{S}$, equation (C.1) becomes

$$q_1.\mathbb{P}\left((r - \tilde{s} - 1)z_i + \eta_i + \sum_{j \in \mathcal{N} \setminus \mathcal{S}} c^{\mathcal{S}}_{ij} \left[(r-1)z_j + \eta_j\right] < 0\right) \leq \beta \tag{C.2}$$

33. In what follows, we will abuse the notation by assuming a single maximizor; under multiple maximizors, choosing any set among them is indifferent.

where $q_1 = q_0\binom{n}{q-1}$ (recall that q_0 represents the probability that the shocks hit the financial system).

Recalling that $t_{1-\frac{n\beta}{q_1}}$ is the $(1-\frac{n\beta}{q_1})$th quantile of the distribution of $\tilde{s}$ and $\ell = r - t_{1-\frac{n\beta}{q_1}}$, and taking into account that equation (C.2) is binding at the optimum, we can determine the risk-taking of institution i, $z_i^*(\mathbf{z}_{-i})$, which makes condition (C.1) binding:

$$z_i^*(\mathbf{z}_{-i}) = \frac{1}{1-\ell}\left(\eta_i + \min_{\mathcal{S}\in\mathcal{S}_i} \sum_{j\in\mathcal{I}\setminus\mathcal{S}} c_{ij}^{\mathcal{S}}\Big((r-1)z_j + \eta_j\Big)\right) \tag{C.3}$$

By equation (C.3), optimal risk-taking decisions are still strategic complements (as in the case of a single shock). Like the case of a single shock, the institutions that survive in the network always provide support to institution i through cross-shareholding links. However, with multiple shocks, each institution has its own relevant network of complementarities, induced from the whole cross-holding network by dropping its set of critical institutions.

Even if strategic complementarities resist the introduction of shock multiplicity, system (C.3) is highly non-linear, and both cycles and equilibrium multiplicity may emerge, as illustrated by the six-institution example shown in Fig. C.1. Consider the fixed-participation case and the following parameters: $q = 1$, $\rho = 1.01$, $d = 1000$, $e = 100$, $l = 0.85$, $r = 1.02$, and $p = 5$. Consider a sequential best-response algorithm (SBRA) with discrete periods, where a single institution reacts at a time in any pre-definite order, starting from any initial risk-taking vector. A Nash equilibrium is a fixed point of such a SBRA. Then, numerical computations show that both $\mathbf{z}_1^* = (697.45, 694.19, 631.33, 663.97, 659.83, 662.67)$ and $\mathbf{z}_2^* = (697.51, 664.49, 664.36, 631.33, 662.67, 659.83)$ are fixed points of the SBRA. If equilibrium multiplicity asks for the question of equilibrium selection, there is no simple answer, because, as suggested in the above examples, equilibria might not be ranked.

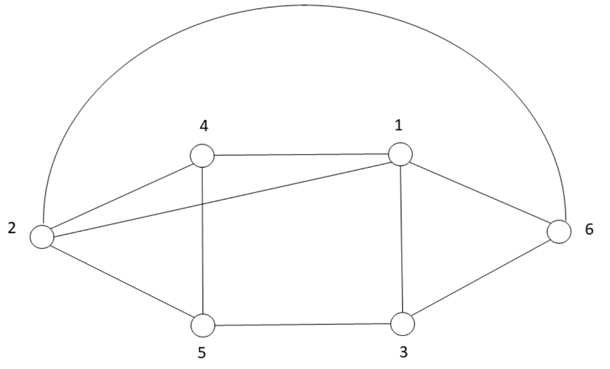

FIGURE C.1. Two shocks in the economy ($q = 1$). This six-institution network can generate multiple equilibria.

Lastly, even under equilibrium multiplicity, the comparative statics presented in the single-shock case, like Lemma **??**, still hold at least locally; that is, for any change of parameter sets that keeps unchanged the set of critical institutions for each institution, the system of interactions describing institutions' values, and thus risk-raking levels, is of the same qualitative nature as for the single-shock benchmark (i.e., complementarities), so our proofs extend straightforwardly.

Appendix D: Directed shareholding network

In this section, we allow for cross-investments to be bilaterally asymmetric; i.e., $p_{ij} \neq p_{ji}$ is now possible. We examine the impact on risk-taking and on the comparative statics on integration. Under asymmetric cross-shareholding, contagion of a single default is possible, thus we assume that cross-investment relationships are sufficiently low as compared to leverage ratio so as to avoid contagion effects.

With asymmetric bilateral relationships, we obtain the following accounting equation at $t = 0$:

$$x_i + z_i + \sum_{j \in \mathcal{I}} p_{ij} = d_i + e_i + \sum_{j \in \mathcal{I}} p_{ji} \tag{D.1}$$

meaning that the accounting equation at $t = 1$ becomes

$$v_i = \max\Big((\mu_i - 1)z_i + \eta_i + \underbrace{\sum_{j \in \mathcal{I}} (p_{ji} - p_{ij})}_{\text{Resource effect}} + \sum_{j \neq i} a_{ij} v_j, 0\Big) \tag{D.2}$$

The additional term is a resource effect, by which institutions with more investors from the financial system benefit from more resource to allocate between risk-free and risky asset.

We impose a generalized version of Assumption 1 as follows.

Assumption D.1. $\eta_i + \sum_{j \in \mathcal{I}} (p_{ji} - p_{ij}) > 0$ for all i

Under Assumption D.1, when an institution does not invest in risky assets, it remains solvent (i.e., $v_i > 0$ when $z_i = 0$). When the network of cross-shareholding is balanced ($\sum p_{ji} = \sum p_{ij}$), Assumption D.1 reduces to the condition $e_i > (\rho - 1)d_i$ for all i (i.e., Assumption 1), meaning that banks' equity suffices to finance the interest paid on debt. More generally, this assumption also depends on $\sum p_{ji} - \sum p_{ij}$, hereafter called the resource effect, which must be of sufficiently low magnitude.

Risk-taking under VaR-RM.

The expression of interior regulated risk-taking is the same as that of Theorem 1, except for the resource effect (i.e., the term $(\mathbf{P}^T - \mathbf{P})\mathbf{1}$ hereafter):

$$\mathbf{z}^* = \frac{1}{1 - \ell} (\mathbf{I} - \varepsilon \mathbf{C})^{-1} (\mathbf{I} + \mathbf{C})(\mathbf{e} - (\rho - 1)\mathbf{d} + (\mathbf{P}^T - \mathbf{P})\mathbf{1}) \tag{D.3}$$

We now analyze the characteristics of the centrality measure $\mathbf{z}^*$ as a function of the network topology. Equity holdings impact risk-taking twice: (i) through the shareholding matrix via $\mathbf{C}$, and (ii) through the accounting balance via $(\mathbf{P}^T - \mathbf{P})\mathbf{1}$. This second effect arises from differences in the resources that can be allocated toward the risky asset when investments in equities by institutions, $e_i + d_i + \sum_{j\in\mathcal{I}} p_{ji} - \sum_{j\in\mathcal{I}} p_{ij}$, are not balanced. The entry i of vector $(\mathbf{P}^T - \mathbf{P})\mathbf{1}$ reflects the difference between the investment of other institutions in institution i's equity and the investment of institution i in other institutions' equities. It is useful to decompose risk-taking levels into $\mathbf{z}^* = \mathbf{z}^*_{\mathbf{RS}} + \mathbf{z}^*_{\mathbf{RE}}$ (where "RS" stands for the risk-sharing effect and "RE" stands for the resource effect):

$$\begin{cases} \mathbf{z}^*_{\mathbf{RS}} = \frac{1}{(1-\ell)}(\mathbf{I} - \varepsilon\mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})(\mathbf{e} - (\rho - 1)\mathbf{d}) \\ \mathbf{z}^*_{\mathbf{RE}} = \frac{1}{(1-\ell)}(\mathbf{I} - \varepsilon\mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})(\mathbf{P}^T - \mathbf{P})\mathbf{1} \end{cases}$$

Bow-tie networks. To complement the previous discussion on specific network structures to directed networks, bow-tie cross-holding networks have been identified in the empirical literature on industrial and finance economics (see Galeotti and Ghiglino (2021) and references therein; for a typical example, see the seven-institution network in Galeotti and Ghiglino (2021) in figure 3 therein).[34] They are particularly interesting to illustrate the powerful role of resource effects in shaping risks. Consider the following network shown in Fig. D.1: The in-section institution (institution 1) benefits from risk-sharing from the core institutions (institutions 2 and 3), but suffers from a negative resource effect (in that the sum of received investments is lower than the sum of investment in other institutions); core institutions benefit from each other only and have a null resource effect; the out-section institution (institution 4) benefits from no risk-sharing effect, but has a positive resource effect. Consider $\rho = 1.01$, $d = 100$, $e = 10$, $r = 1.02$, and $p = 5$. Then, for $l = 0.9$,

34. These networks have three classes of institutions: in-section, core, and out-section. In-section institutions invest in core institutions, core institutions invest among themselves and in out-section institutions, and out-section institutions do not invest in other institutions.

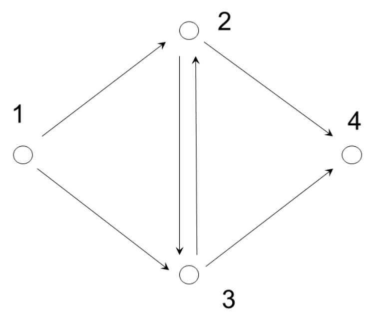

FIGURE D.1. A bow-tie network with four institutions.

$\mathbf{z}^* \simeq (113, 193, 193, 190)$; and for $l = 0.8$, $\mathbf{z}^* \simeq (48, 91, 91, 95)$. In this example, the out-section institution takes more risk than the in-section institution in both parameter sets. Furthermore, for a sufficiently high shock magnitude, the out-section institution also takes more risk than the core institutions. This example illustrates that the resource effect can dominate the risk-sharing effect.

Statics on integration. When the cross-shareholding network is undirected, there is no resource effect and Proposition 3 implies that integration increases optimal risk-taking. However, this result does not extend to the directed network case, that is when shareholding links are not reciprocated, nor when the amount invested in other institutions varies across institutions. We present an example where increased integration can decrease the contribution of resource effects to total optimal risk-taking ($\mathbf{1}^T\mathbf{z}^*_{RE}$) in directed networks. Consider indeed the following cross-shareholding network:

$$\mathbf{P} = \begin{pmatrix} 0 & p/3 & p/3 & p/3 \\ 0 & 0 & 0 & 0 \\ p & 0 & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}$$

Then, the resource effect is given by $p\boldsymbol{\gamma}$ where $\boldsymbol{\gamma} = (\mathbf{G}^T - \mathbf{G})\mathbf{1}$. To simplify, consider $\varepsilon = 0$, so that $\mathbf{z}^* = (\mathbf{I} + \mathbf{C})\left(\frac{e-(\rho-1)d}{1-\ell}\mathbf{1} + \frac{p}{1-\ell}\gamma\right)$. The effect of p on the total contribution of resource effect to optimal risk-taking is then captured by: $p/(1-\ell)\mathbf{1}^T\mathbf{C}\boldsymbol{\gamma}$, and can be decreased when the level of integration of the financial network is increased. Recall here that matrix $\mathbf{C}$ depends on parameter p. Denoting $\mathbf{C}_p$ the value of this matrix under parameter p, we have: $\mathbf{1}^T\mathbf{C}_1\boldsymbol{\gamma} \sim -0.0328$ and $2\mathbf{1}^T\mathbf{C}_2\boldsymbol{\gamma} \sim -0.0789$. Therefore, the contribution of resource effects to total optimal risk-taking is here negative and decreasing with p. This comes from the negative correlation between the vector of out-degrees $\delta^O = (3,1,1,0)^T$ and the vector of resource effects $\boldsymbol{\gamma} = (-1,0,0,1)^T$.