Sharing Opportunities under Externalities

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Abstract
A subset of economic agents in a society is aware of the existence of an economic opportunity, and compete for exploiting the opportunity. We study incentives to communicate about the existence of this economic opportunity when the exploitation of the opportunity by the winner generates externalities to other agents. We characterize the equilibria of the communication game, identify conditions under which more externalities generates more communication, and analyze welfare. (JEL: C72; D83; D85)

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1. Introduction

The discovery and exploitation of economic opportunities are essential to economic activity, requiring both the ability to identify them and the right technology for effective utilization. When these opportunities are not publicly disclosed or when the technology remains privately owned, economic agents often rely on their network of peers for access. These peers might be motivated to share such opportunities if they anticipate significant positive externalities, despite the competitive nature of these opportunities potentially limiting the incentive to share information.[1] Yet, there are situations where the incentive to share information is strong. For instance, in Research and Development (R&D) activity, companies aware of a potential innovation may foster competitive dynamics in the innovation race to disadvantage other competitors. In the context of job searches, individuals frequently share information about job openings with their social networks, driven by altruism[2] or career concerns. Additionally, in cases where information dissemination serves the public good, it can lead to enhanced public services. Therefore, understanding the diverse motivations for sharing information about economic opportunities is critical, reflecting its significant influence on innovation and economic efficiency.

This paper delves into the diffusion of information regarding economic opportunities across social and economic networks, prompted by the illustrative examples provided earlier. We propose a novel framework to understand the underlying mechanisms incentivizing the communication of such opportunities. Central to our analysis are three key elements: the competitive nature of exploiting rival opportunities, the division within society between those aware and those unaware of these opportunities, and the externalities arising from their exploitation. Our model suggests that agents may find it advantageous to inform others about an opportunity when they stand to gain from the externalities ensuing from its utilization by the informed party. The drive to inform is fundamentally rooted in these externalities, which exhibit diverse characteristics across various economic settings. The aim of this study is to explore how the structure of externalities influences the incentives to share information about economic opportunities.

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1. For convenience, we will speak about information transmission throughout the paper; however, the paper also covers technology transfer.

2. There is an experimental literature showing evidence of prosocial behaviors by winners of contests; see for instance Engelman and Strobel (2008), or Binzel and Fehr (2013).

We model this question through a simple normal-form communication game. The set of externalities obtained conditional on who exploits the opportunity is formally represented by a *matrix of externalities*. This is simply a matrix whose i, j entry is equal to the externality received by agent j if agent i wins the contest. The existence of the economic opportunity, as well as the matrix of externalities, are common knowledge among a set of initially informed agents. These agents then simultaneously choose to inform a set of uninformed agents. All informed agents compete for the opportunity, the winner is selected from an exogenous probability distribution, and the exploitation of the opportunity by the winner generates externalities to others according to the relevant row in the externality matrix.

We present a series of applications to this general model. We examine differentiated oligopolies[3], a public good game environment[4], and a network of altruism.[5] In each application, we introduce a contest for an opportunity (an innovation in R&D oligopolies, a job offer for the two other applications) and we build the matrix of externalities from the primitives of each model.

We study the Nash equilibria of the communication game and analyze implications for welfare. Our first result pertains to equilibrium existence. While the model does not exhibit strategic complementarity in communication strategies, we show that an agent's best-response can only be increased (in group inclusion sense) when others communicate more. The reason is that, if not informing an agent is a best-response strategy, the fact that this agent gets informed by another agent can only lower her communication threshold (which is given by the average externality obtained over the set of informed agents in the society), which thus fosters incentives. We derive from this fundamental monotonicity property that the communication game admits a minimum equilibrium and a maximum equilibrium in terms of the set of informed agents, a standard property of games with strategic complementarities. Hence, equilibria are partially ordered. Moreover, the minimum equilibrium Pareto-dominates all other equilibria over the initially informed agents; whereas information

3. See Goyal and Joshi (2003) and Goyal and Moraga-González (2006).

4. See Allouch (2015), Bramoullé and Kranton (2007) and Bramoullé et al. (2014).

5. See Bourlès et al. (2017).

receivers might be better off in other equilibria.[6] This sharp result holds for any externality matrix and any set of initially informed agents.

To give a more comprehensive characterization of equilibria, and understand deeper who informs who, we examine more specific externality matrices. Firstly, we present the class of *common-preference externality matrices*, for which there is always a unique equilibrium in communication. In this class, all agents have the same ordinal ranking in preference over the winners of the contest. We show that, for any externality matrices in this class, there is a unique equilibrium in communication whatever the set of initially informed agents. Then we examine the case of *binary input externality matrices*. Here externalities are constrained by an undirected network, in that only direct neighbors[7] provide a non-null externality; and furthermore, the intensity of the externality depends on a characteristics of the receiver of the externality. We find that equilibrium communication strategies are bang-bang: players either don't communicate, or communicate with all uninformed neighbors. We then analyze who communicates. We identify an individual index, decreasing in the received externality and increasing in the degree, that proves key: players who communicate are those with lower index.[8]

Turning to comparative statics, we examine whether larger externalities is always beneficial to communication. Actually it may not because larger externalities also higher communication thresholds. However, we identify conditions on the inflation of externalities which can only foster communication for any set of players. This necessarily happens when externalities are subject to an increasing and concave transformation. In such situations, increasing externalities enhances communication unambiguously from any equilibrium.

Finally, we study efficiency, by focusing on an ex ante utilitarian criterion. Determining the efficient communication profile resorts to finding the optimal strategy of a representative agent among initially informed agents, whose objective would be to maximize the aggregate externality at the society level. The analysis shows

6. Symmetrically, the maximum equilibrium is Pareto-dominated by all other equilibria over the subgroup of initially informed agents.

7. Two agents are said to be neighbors whenever there is a link between them.

8. Appendix C studies the class of *multi-level externality matrices*, in which externalities between neighbors belong to a set of discrete levels. The analysis stresses the emergence of partial communication, and shows that, to understand incentives, degree centrality is not sufficient; the communication decisions of agents positioned on paths of length larger than one must be taken into account.

that equilibria can exhibit either over-communication or under-communication with respect to the efficient communication, depending on whether the players generate a high amount of welfare or not.

Related literature. This paper is contributes to distinct strands of literature. Firstly, our paper adds to the literature addressing strategic communication on networks. In that literature, the need for communication comes from seeking to influence others' actions under differentiated individual preferences and, in some contexts, coordination issues. Recent extensions to networks include Hagenbach and Koessler (2010), Galeotti et al. (2013), Calvó-Armengol et al. (2015). The two former focus on costless, non-verifiable information (cheap talk model as in Crawford and Sobel (1982)), whereas the latter models the endogenous acquisition of a communication technology under costly and verifiable information. The main focus of that literature is on organizational economics (for decentralized decisions making within organizations, see Dessein and Santos (2006), Alonso et al. (2008), or Rantakari (2008)); or on political economy (See Dewan and Myatt (2008) for a study related to political parties). Focusing rather on social networks, Bloch et al. (2018) examine the strategic spread of rumors in a model in which agents can decide whether to pass on the received information, and find that, when agents, say partisans, diffuse false information, other agents can block messages coming from parts of the network with many partisans.[9] Our main contribution to that literature is to propose a new rationale for strategic communication, by identifying incentives to communicate about the existence of a rival opportunity in presence of externalities.

There is also a literature on strategic experimentation and social learning (Keller et al. (2005)). Heidhues et al. (2015) introduce privacy of payoffs, and agents can communicate via cheap-talk messages. Marlats and Ménager (2021) introduce strategic costly observation of actions and outcomes. In contrast to that literature, we suppose that the value of the opportunity is known with certainty.

Our paper also adds to the literature on information acquisition through peers. Galeotti and Goyal (2004) model information acquisition about a public good through social networks, to explain the empirical observation that individuals acquire information from a small subset of their social contacts. Herskovic and Ramos

9. Merlino et al. (2023) introduce incentives to verify information status when false information spread in networks.

(2021) model information acquisition from peers in a beauty contest setting, in which agents form connections to acquire information. In both models, there is no strategic communication consideration, because connecting to another agent allows to observe her signal. We contribute to that literature by incorporating strategic communication, focusing rather on contexts in which accessing information requires the consent of the information provider.

Our paper also contributes to literatures in which the information on an economic opportunity is private. In that respect, our paper adds to the literature on innovation in industries. Goyal and Joshi (2003) and Goyal and Moraga-González (2006) model the formation of R&D partnerships among rival firms. In their setting, partnerships lead to innovation-processes of the partners. We complement that literature by considering situations in which firms may find profitable to include other firms into a race to innovation without merging R&D effort. In the same spirit, Our paper also adds to the literature on job search through social contacts. Founding their study on the well-known fact that a huge proportion of job offers are transmitted by social contacts, Calvo-Armengol and Jackson (2004), Calvo-Armengol and Jackson (2007) explore unemployment dynamics when social contacts transmit job offers. While, in these models, information transmission is non-strategic, we add to that literature in providing rationale for strategic information transmission.

The paper is organized as follows. The communication game and economic foundations are exposed in Section 2, the characterization of the equilibria of he communication game, as well as the comparative statics and welfare analysis, are presented in Section 3. Section 4 concludes. All proofs are relegated in Appendix A, Appendix B presents our applications in more detail, and Appendix C studies equilibria on the class of multi-level externality matrices.

2. The communication game

2.1. The model

Agents compete for an *opportunity*. An agent is initially either aware of the existence of the opportunity, or not. Hence, the set of agents - $\mathcal{N} = \{1, 2, \cdots, n\}$ - is partitioned as follows: $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$ where $\mathcal{I}$, of cardinal I, is the set of agents informed before the communication stage (called *players*), and $\mathcal{J}$ is the set of agents who are not informed before communication (called *regular* agents). Thereafter we will speak about information transmission, but another interpretation is that the informed agents are those who have the adequate technology to exploit the opportunity.

Informed agents compete for the opportunity (they don't incur a cost to compete), and the single winner is selected from a stochastic rule. Each competitor wins with a uniform probability in the paper; the model is straightforwardly extended to heterogeneous probabilities of winning (see Remark 3 thereafter).

The exploitation of the opportunity generates externalities, which are represented by an n-square matrix $\mathbf{E} = (e_{ij})_{i,j \in \mathcal{N}}$, where entry $e_{ij} \in \mathbb{R}$ is the utility of agent j when agent i wins the contest. We refer to this matrix as the *externality matrix*. At this level of abstraction, externalities can be either positive or negative. In some specific contexts, it may be legitimate to focus on non-negative entries, or even row-stochastic matrices $\mathbf{E}$. In particular, diagonal entries need not be positive to rationalize entry in the contest and communication.

Given a externality matrix $\mathbf{E}$ as well as a subset of initially informed agents $\mathcal{I}$, we define a normal-form game $(\mathcal{I}; (\mathcal{S}_i)_{i \in \mathcal{I}}; (\pi_i)_{i \in \mathcal{I}})$ as follows: agent i chooses a set $\mathbf{S}_i \in \mathcal{S}_i := \mathcal{P}(\mathcal{J})$ of regular agents to inform. Let $\mathbf{S} := (\mathbf{S}_i)_{i \in \mathcal{I}}$ be an action profile. For simplicity, we also denote by $\mathbf{S}$ the set $\underset{i \in \mathcal{I}}{\cup} \mathbf{S}_i$, i.e. the set of agents who have been informed of the opportunity through communication. Adding communication costs or rewards does not significantly alter the analysis (see Remark 5 thereafter). We let $\mathcal{M}(\mathbf{S}) := \mathcal{I} \cup \mathbf{S}$ and $m(\mathbf{S}) := |\mathcal{M}(\mathbf{S})|$. Let $\mathbf{S}_{-i} := (\mathbf{S}_j)_{j \neq i}$ be the profile of actions of all players, except for i.[10] Then, letting u_i represent player i's utility in absence of the contest, player i's payoff is given by

$$\pi_i(\mathbf{S}_i, \mathbf{S}_{-i}) = u_i + \frac{1}{m(\mathbf{S})} \sum_{k \in \mathcal{M}(\mathbf{S})} e_{ki}$$

The quantity $\frac{1}{m(\mathbf{S})} \sum_{k \in \mathcal{M}(\mathbf{S})} e_{ki}$ is the expected externality that player i obtains over all informed agents, including herself, after the communication phase. As it will become clear thereafter, initial utilities $(u_i)_{i \in \mathcal{N}}$ do not affect communication.

10. From the point of view of player i, all that matters in this game is the set of agents to which other players transmitted their knowledge. We characterize equilibria in terms of their set of informed agents. However, there can be many equilibrium strategies generating a given set of informed agents (through appropriate permutations on the label of the informer of a given informed agent). We disregard those permutations in the paper.

REMARK 1. This is not a game with strategic complements. Take the following example. Assume $n = 4$, $\mathcal{I} = \{1, 2\}$, and

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Then,

$$\pi_1(\{3\}, \emptyset) - \pi_1(\emptyset, \emptyset) = \frac{3}{3} - \frac{1}{2} = \frac{1}{2} > \pi_1(\{3\}, \{4\}) - \pi_1(\emptyset, \{4\}) = \frac{4}{4} - \frac{2}{3} = \frac{1}{3}$$

That is, for player 1, it is less valuable to inform agent 3 when agent 4 is informed by player 2 than when agent 4 is not informed.

2.2. *Economic foundations*

The above abstract model is compatible with economic applications with the following features. Agents are differentiated by a trait/type; depending on the economic model, the trait can be a production cost, the quality of a produced good, or an initial wealth endowment. There is a unique equilibrium payoff, that is increasing in own trait. Prior to playing the underlying game, agents can compete for an opportunity captured by an improvement of their own trait. A set of agents is initially aware of this opportunity, and can diffuse the information to other agents. We present few economic applications (for details see Appendix B).

2.2.1. Innovations. Imagine a Research and Development (R&D) department in a firms which finds an innovation, that could potentially reduce production cost or improve product quality. The firm should however invest in R&D with the goal of finding a valuable innovation. The firm could find it profitable to inform other firms about a potential innovation.[11] The benefits from communication can for instance be

11. R&D partnerships are now a widespread activity in industries, especially those with rapid technological change, such as the IT or the pharmaceutical industry; see Hagedoorn (2002), Powell et al. (2005), or Hagedoorn (2006). Such partnerships can take various forms, including crowdsourcing platforms and open innovation: a company facing a specific technical challenge might find it advantageous to share this challenge on an open innovation platform, inviting external innovators to propose solutions. Not only can this lead to creative and effective solutions, but the company can also establish relationships with external talent and potentially discover new opportunities for collaboration.

generated by rivalry (a firm informs another firm to hurt a rival), or from knowledge spillovers. Note that the model can be extended to the case in which the probability to find an innovation is increasing in the number of participants to the R&D race (see Remark 4).

Cost-reducing innovation in horizontally differentiated oligopolies. Consider a differentiated Cournot oligopoly with n firms facing a linear inverse demand. Given a vector of output $\mathbf{q} = (q_i, q_{-i})$, firm i's inverse demand is $p_i = \alpha - q_i - (\mathbf{Bq})_i$, where $\mathbf{B}$ is a n-square matrix with null diagonal terms and $b_{ij} \in [0,1]$ representing the substitutability factor between i and j. Firm i produces at zero fixed cost and constant marginal cost $c_i > 0$; write $\mathbf{c} = (c_i)_{i\in\mathcal{N}}$. Parameter α is assumed to be high enough to generate a positive quantity to all firms at equilibrium. When the initial cost profile is $\mathbf{c}^0$, we get $u_i = \left(q_i^*(\mathbf{c}^0)\right)^2$.

Assume that there exists a contest for an innovation leading the winner to decrease its marginal cost by a fixed amount $\gamma > 0$. Agents $i \in \mathcal{I} \subseteq \mathcal{N}$ are initially aware of the existence of the R&D race. They can beforehand inform a set of regular agents of the existence of the contest.

OBSERVATION 1. *The game described above is a communication game, with externality matrix* $\mathbf{E}$ *given by* $e_{ji} = \left(q_i^*(\mathbf{c}^0) + \gamma m_{ij}\right)^2 - \left(q_i^*(\mathbf{c}^0)\right)^2$, *where* $\mathbf{M} := (2\mathbf{I}_n + \mathbf{B})^{-1}$.

EXAMPLE 1. Consider the substitutability matrix

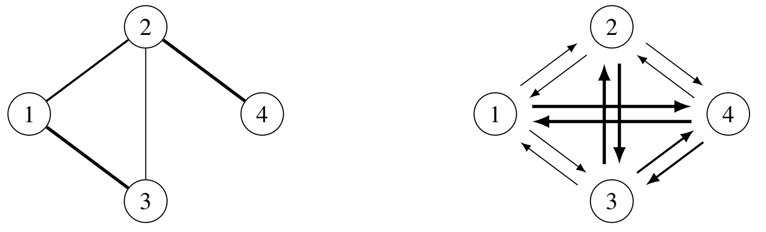
$$\mathbf{B} = \begin{bmatrix} 0 & 3/4 & 1 & 0 \\ 3/4 & 0 & 1/8 & 1 \\ 1 & 1/8 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{1}$$

Then $\mathbf{q}^*(\mathbf{1}) = (0.064, 0.031, 0.131, 0.169)$, and we can compute the externality matrix, with $\gamma = 0.1$:

$$\mathbf{E} = \begin{pmatrix} 0.049 & -0.012 & -0.027 & -0.017 \\ -0.019 & 0.038 & 0.011 & -0.033 \\ -0.019 & 0.005 & 0.055 & -0.006 \\ 0.1 & -0.013 & -0.005 & 0.064 \end{pmatrix}.$$

■

FIGURE 1. **Cournot.** Left panel: substitutability network; Right panel: externality matrix (boldness of arrows represent externalities)



Quality-improving innovation in vertically differentiated oligopolies. Consider a Cournot oligopoly with vertical differentiation à la Sutton (1997). There are n firms and one consumer. Firm i produces a product of quality $a_i > 0$, at marginal cost c. The industry is also horizontally differentiated: parameter $\sigma \in [0, 1)$ captures product differentiation. The consumer's utility is given by:

$$\sum_{i \in N} \left(\alpha q_i - \frac{q_i^2}{a_i^2} \right) - 2\sigma \sum_i \sum_{j<i} \frac{q_i}{a_i} \frac{q_j}{a_j}$$

Parameters are such that all firms produce a positive quantity. Let $\mathbf{B}(\mathbf{a})$ be the matrix of entries $b_{ij} = \frac{1}{a_i a_j}$, let $\mathbf{D}_{\mathbf{B}(\mathbf{a})}$ be the diagonal matrix of diagonal entries $d_{ii} = \frac{1}{a_i^2}$, and $\mathbf{M}(\mathbf{a}) = ((2 - \sigma)\mathbf{D}_{\mathbf{B}(\mathbf{a})} + \sigma \mathbf{B}(\mathbf{a}))^{-1}$. Assume that there exists a contest to increase quality by γ. Then, normalizing costs such that $\alpha - c_i = 1$ for all i, we get

OBSERVATION 2. *The game described above is a communication game, with externality matrix given by*[12] $e_{ji} = \frac{1}{2a_i^2}\Big(\big(\mathbf{M}(\mathbf{a} + \gamma \mathbf{h}^j)\mathbf{1}\big)_i \Big)^2 - \frac{1}{2a_i^2}\Big((\mathbf{M}(\mathbf{a})\mathbf{1})_i \Big)^2.$

2.2.2. Job market opportunities. When people are aware of a job offer, they can transmit this information to their social contacts.[13] Information transmition can benefit the informer through various externalities, including altruism, or for career consideration.[14] We present here externalities originated from a public good dimension or from altruist behavior.

12. $\mathbf{h}_k^j := 0$ if $k \neq j$ and $\mathbf{h}_j^j = 1$.

13. It is well-known that a huge proportion of job offers are transmitted by social contacts. See for instance Calvo-Armengol and Jackson (2004), Calvo-Armengol and Jackson (2007).

14. Informing other economic agents about a job offer can expand a job seeker's professional network. Sharing job offers with others can also demonstrate goodwill and a willingness to help fellow professionals

Local public good. We explore a variant of the local public good game (see Allouch (2015), Bramoullé and Kranton (2007) and Bramoullé et al. (2014)), in which agents allocate a budget between the provision of a private good and a local public good. We will relate the level of externalities received by any agent to the wealth distribution and network structure. This will allow us to understand how the externalities vary when the wealth of a given agent increases after getting a new job.

Agents are endowed with an initial wealth $\mathbf{w}^0 = (w_i^0)_i$. Let g_i be agent i's contribution to a local public good. The binding individual budget constraint imposes private good consumption to be equal to $w_i^0 - g_i$ for all i (there is no waste). Agents are organized in a network, and benefit from neighbors' contributions to the public good. For instance, the public good can be a social activity benefiting neighbors, etc. Formally, the amount of public good enjoyed by agent i is $g_i + (\mathbf{B}\mathbf{g})_i$, where, again $\mathbf{B}$ has null diagonal entries. Individual utilities are shaped by private and public good consumption. For simplicity, assume separability:

$$v_i(\mathbf{g} \mid \mathbf{w}^0) = \frac{1}{2}\varphi_i(w_i^0 - g_i) + \frac{1}{2}\varphi_i\left(g_i + (\mathbf{B}\mathbf{g})_i\right),$$

with φ_i strictly increasing and concave.

Suppose that, starting with wealth profile $\mathbf{w}^0 = (w_i^0)_i$, only agents in $\mathcal{I}$ are aware of the existence of a job offer; if one unemployed neighbor gets hired, she receives a wealth increase of $\gamma > 0$. Then, informing an unemployed neighbor can be profitable when the informed agent, being hired, provides a sufficiently large amount of public good.[15] Letting $\mathbf{M} := (2\mathbf{I}_n + \mathbf{B})^{-1}$ and $\overline{\mathbf{w}}_i^0 := w_i^0 - (\mathbf{M}\mathbf{w}^0)_i$, we have

OBSERVATION 3. *The game described above is a communication game, with externality matrix* $\mathbf{E}$ *given by* $e_{ji} = \varphi_i\left(\overline{\mathbf{w}}_i^0 - \gamma m_{ij}\right) - \varphi_i\left(\overline{\mathbf{w}}_i^0\right)$ *for* $i \neq j$ *and* $e_{ii} = \varphi_i\left(\overline{\mathbf{w}}_i^0 - \gamma m_{ii} + \gamma\right) - \varphi_i\left(\overline{\mathbf{w}}_i^0\right)$.

Altruist network. We consider an altruist network à la Bourlès et al. (2017). There is a society $\mathcal{N} = \{1, 2, \cdots, n\}$, where agents are differentiated by the initial wealth

in their job search, which can foster positive relationships and goodwill within professional networks, which may be reciprocated in the future. Last, actively sharing job opportunities and providing assistance to others can enhance a job seeker's reputation and credibility within their industry or professional community.

15. An employed agent could also find it profitable to communicate, even if she does not participate to the competition for the job. The model can be easily adapted to cover such situation.

profile $\mathbf{y}^0$, as well as the altruist network $(\alpha_{ij})_{i,j}$, with $\alpha_{ij} \in [0,1[$ for $i \neq j$, and (by convention) $\alpha_{ii} = 1$. Agents can transfer wealth to agents they care about. Agents' actions consist in choosing how much wealth they transfer to other agents. Given a transfer profile $\mathbf{t} = (t_{ij})_{i\neq j}$ and a vector of private utilities $(w_i)_{i\in\mathcal{N}}$ increasing and concave in wealth, social utilities are given by:

$$v_i(\mathbf{t} \mid \mathbf{y}^0) = w_i\left(y_i^0 + \sum_{k\neq i} t_{ki} - \sum_{k\neq i} t_{ik}\right) + \sum_{j\neq i} \alpha_{ij} w_j \left(y_j^0 + \sum_{k\neq j} t_{kj} - \sum_{k\neq j} t_{jk}\right)$$

By Theorem 1 in Bourlès et al. (2017), the consumption profile is unique at equilibrium, so that $v_i^*(\mathbf{y}^0)$ is well-defined.

Now suppose that, starting with wealth profile $\mathbf{y}^0 = (y_i^0)_i$, only agents in $\mathcal{I} \subseteq \mathcal{N}$ are aware of the existence of a job offer, and the hired agent of which receives an extra-wealth of $\gamma > 0$. Then an altruist agent can be profitable to inform an unemployed neighbor, whose improved status generates a positive externality to the informer. We need to specify our model, in order to be able to say more about conditional utilities. Suppose that all agents initially have the same wealth, y^0, and that private utilities are of the CARA form: $w_i(y_i) = -\frac{1}{A}e^{-Ay_i}$, for all i. Define $\hat{\alpha}_{jl} = \prod_{s=1}^t \alpha_{i_s,i_{s+1}}$, where $i_1, i_2, ..., i_{t+1}$ is a *least-cost path* from j to l. Also let $\hat{\alpha}_j := \left(\prod_{l=1}^n \hat{\alpha}_{jl}\right)^{1/n}$. When γ is large enough, money flows from the winner to the rest of agents through the altruist network, and after-transfer incomes are well-defined (see Proposition 4 in Bourlès et al. (2021)). We then find:

OBSERVATION 4. *When γ is sufficiently large, the game described above is a communication game, with externality matrix $\mathbf{E}$ given by*

$$e_{ji} = -C\hat{\alpha}_j \sum_k \frac{\alpha_{ik}}{\hat{\alpha}_{jk}}, \quad \textit{where} \quad C := \frac{e^{-A(y^0+\gamma/n)}}{A} - \frac{e^{-Ay^0}}{A}.$$

If we further assume that $\alpha_{ij} = \alpha\, g_{ij}$, with $g_{ij} \in \{0,1\}$, for all $i, j \neq i$, we get an unweighted altruist network. Defining d_{ij} as the length of the shortest path from agent i to agent j in the network, we then have $\hat{\alpha}_j = \alpha^{\frac{1}{n}\sum_{l=1}^n d_{jl}}$, and

$$e_{ji} = -C\alpha^{\frac{1}{n}\sum_{l=1}^n d_{jl}} \left(\frac{1}{\alpha^{d_{ji}}} + \alpha \sum_{k\in\mathcal{N}} g_{ik} \frac{1}{\alpha^{d_{jk}}}\right)$$

3. Results

In this section, we study the existence of equilibria of the communication game, discuss uniqueness, then we address comparative statics, and finally we study welfare.

3.1. The Equilibria

We start by illustrating how strategic communication can emerge and lead to multiple equilibria, by revisiting Example 1 in terms of SNE.

Example 1 continued: non-uniqueness. Consider the Cournot Example 1 described in Section 2.2.1, and suppose that $\mathcal{I} = \{1, 2\}$. One can easily check that both player

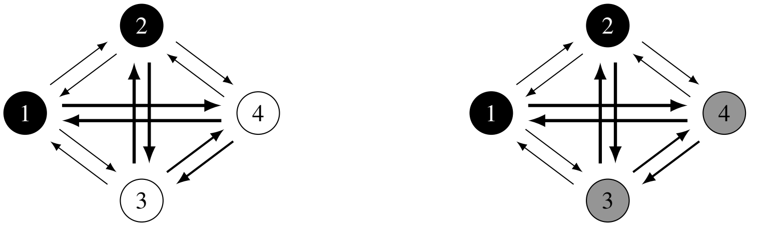


FIGURE 2. **Cournot.** Left panel: empty SNE; Right panel: SNE where $S_1^* = \{4\}, S_2^* = \{3\}$. Black nodes represent players. Grey nodes represent regular agents informed by players. White nodes represent uninformed regular agents. The solid lines represent the externalities generated, thicker links meaning larger externality levels.

informing nobody is a SNE. There is another SNE, where regular agent 4 is informed by player 1, while agent 3 is informed by 2, ■

A best-response $\mathbf{S}_i$ to a profile of actions of other players $\mathbf{S}_{-i}$ is such that $\pi_i(\mathbf{S}_i, \mathbf{S}_{-i}) \geq \pi_i(\mathbf{S}'_i, \mathbf{S}_{-i})$ for all $\mathbf{S}'_i$. Given that the expected payoff is the expected externality among informed agents, it is profitable for agent i to inform a regular agent j whenever u_{ji} exceeds the average externality obtained among already informed agents: given $i \in \mathcal{I}$ and $\mathbf{S}_{-i} \in \mathcal{S}_{-i}$, $\mathbf{S}_i$ is a best-response against $\mathbf{S}_{-i}$ iff[16]

$$e_{ji} \geq \frac{1}{m(\mathbf{S})} \sum_{l \in \mathcal{M}(\mathbf{S})} e_{li} \geq e_{ki}, \;\; \forall j \in \mathbf{S}_i \setminus \mathbf{S}_{-i}, \;\; \forall k \in \mathcal{J} \setminus \mathbf{S}.$$

If agent j has not been informed by any player, including player i, the externality obtained by agent i when agent j wins cannot be strictly larger than agent i's payoff.

16. For convenience $\mathbf{S}_{-i}$ also denotes the set $\underset{j \in \mathcal{I}, j \neq i}{\cup} \mathbf{S}_j$.

Moreover if j is not informed by other players, but is informed by player i, then it must be the case that the externality e_{ji} is larger than his payoff. The current payoff is a threshold above which externalities entail profitable communication. This threshold is endogenous to the agents' communication strategies.

The set of best-responses is never empty. However it is typically not a singleton, because if one player informs an agent then any other player is indifferent between informing this agent or not. As a consequence, the set $Br_i(\mathbf{S}_{-i})$ is stable by intersection: if $\mathbf{S}_i$ and $\mathbf{S}'_i$ both belong to $Br_i(\mathbf{S}_{-i})$ then the same holds for $\mathbf{S}_i \cap \mathbf{S}'_i$. Moreover, there is indifference for player i between informing agent j or not when u_{ji} is equal to agent i's payoff. These observations motivate the following refinement:

DEFINITION 1 (*Strict best-response*). We say that $\mathbf{S}_i \in Br_i(\mathbf{S}_{-i})$ is a *strict best-response* against $\mathbf{S}_{-i}$ if, for any $\mathbf{T}_i \subseteq \mathbf{S}_i$ such that $\mathbf{T}_i \neq \mathbf{S}_i$, we have

$$\pi_i(\mathbf{T}_i, \mathbf{S}_{-i}) < \pi_i(\mathbf{S}_i, \mathbf{S}_{-i})$$

In short, a best-response is strict if none of the current communications of an agent to a set of neighbors can be cut without penalizing the agent's payoff. Note that, if the empty set is a best-response, it is strict by definition. Moreover, since the best-response set is stable by intersection, the strict best-response is the intersection of all best-responses and is therefore unique. We denote this set $\text{SBR}_i(\mathbf{S}_{-i})$. We have:

RESULT 1. *Let $\mathcal{J} \setminus \mathbf{S}_{-i} = \{j_1, j_2, ...j_L\}$ be such that $e_{j_1,i} \geq ... \geq e_{j_L,i}$. Then $SBR_i(\mathbf{S}_{-i}) = \{j_1, ...j_l\}$ iff*[17]

$$e_{j_l i} > Mean\{e_{ji} : j \in \{j_1, ..., j_l\} \cup \mathbf{S}_{-i} \cup \mathcal{I}\} \geq e_{j_{l+1},i}$$

By Result 1, player i's strict best-response is easily identified: player i ranks the externalities obtained from all uninformed agents in the society. Then, she examines the profitability of informing the agent with the highest externality in that pool, say agent 1. If informing this agent is not strictly profitable, the empty set is the strict best-response. Otherwise, agent i should inform agent 1. Then, agent i examines the possibility of informing the agent with the second largest externality in the pool, say agent 2. If this is not profitable, the strict best-response consists in informing agent

17. If R is a set of real numbers, $Mean\, R$ denotes the average value of this set throughout the paper.

1. Otherwise, agent i should inform agent 2. Etc. The process involves no more than $n-1$ steps. To sum up, at every stage of this process, agent i's payoff is strictly increasing; when the process stops, all externalities obtained from informing agents, and only these externalities, exceed player i's payoff at the strict best-response. By Result 1, the strict best-response map

$$\text{SBR} : \mathcal{J}^n \to \mathcal{J}^n,\ \text{SBR}(\mathbf{S}) = (\text{SBR}_1(\mathbf{S}_{-1}), ..., \text{SBR}_n(\mathbf{S}_{-n}))\,.$$

is well defined and one-to-one. A strict Nash equilibrium (or SNE) is a fixed point of the strict best-response map.

Result 1 shows that incentives to communicate depend on both the externality that the informer can get from the information receiver and the communication threshold which is the average externality got from informed agents. In that respect, we provide a simple characterization of any equilibrium in communication:

PROPOSITION 1. *Consider any externality matrix* $\mathbf{E}$ *and any set of players* $\mathcal{I}$. *Every equilibrium* $\mathbf{S}^*$ *is such that:*

$$j \in \mathbf{S}_i^* \Leftrightarrow e_{ji} > \frac{1}{m(\mathbf{S}^*)} \sum_{k \in \mathcal{M}(\mathbf{S}^*)} e_{ki} = \pi_i(\mathbf{S}^*) - u_i$$

That is, at equilibrium, information receivers, and only them, generate a larger externality to the informer than the informer's equilibrium payoff.[18] This characterization indicates that incentives to inform are shaped by two factors: (i) the receiver of the information should provide a sufficiently high externality level to the informer, and (ii) the informer's payoff is sufficiently low.

As said earlier, incentives to communicate are higher when the externality obtained from communicating is larger and when the average externality from informed agents is lower. Hence, it is possible to have an equilibrium in which an agent generating a high externality level is not informed, while an agent generating a

18. Note that the condition $e_{ji} > \pi_i(\mathbf{S}^*) - u_i$ is equivalent to the condition $e_{ji} > \pi_i(\mathbf{S}^* \setminus \{j\}) - u_i$; meaning that the incentives condition, which says that benefit from communication exceeds the before-communication payoff, can also be expressed in terms of the after-communication payoff.

lower externality is. To illustrate, consider the externality matrix

$$\mathbf{E} = \begin{bmatrix} e_0 & 0 & 0 & 0 & 0 \\ e_1 & e_0 & 0 & 0 & 0 \\ e_1 & 0 & e_0 & 0 & 0 \\ 0 & e_2 & 0 & e_0 & 0 \\ e_1 & 0 & 0 & 0 & e_0 \end{bmatrix}$$

with $\frac{e_0}{3} < e_2 < e_1 < \frac{e_0}{2}$ and assume $\mathcal{I} = \{1, 2, 3\}$. Here, the communication strategy profile $\mathbf{S} = \{\emptyset, \{4\}\}$, in which player 2 informs agent 4 while player 1 does not inform agent 5, is an equilibrium. This is because player 1's communication threshold $\left(\frac{e_0+2e_1}{4}\right)$ is larger than player 2's threshold $\left(\frac{e_0}{3}\right)$. In this case, incentives to communicate are not aligned with the ranking of externalities ($e_1 > e_2$).

We turn to the existence of an equilibrium in communication. To show existence, a key property of the strict best-response is that, for any player i, SBR_i is increasing in the following sense[19]:

LEMMA 1 (Monotonicity). *For any player i and any $\mathbf{S}_{-i}, \mathbf{S}'_{-i}$ such that $\mathbf{S}_{-i} \subseteq \mathbf{S}'_{-i}$, we have $SBR_i(\mathbf{S}_{-i}) \subseteq SBR_i(\mathbf{S}'_{-i}) \cup \mathbf{S}'_{-i}$.*

Example 1 illustrates this monotonicity property: if player 1 finds it best to inform regular agent 3 when player 2 does not inform regular agent 5, she still prefers to inform regular agent 3, when regular agent 5 is informed by player 2. The reason why Lemma 1 holds is that, at the strict best-response, the arrival of a new informed agent does not increase the current payoff of the player. Indeed, the very fact that the new informed agent was not informed by player i means that her externality is lower than the average externality that player i experiences from other informed agents; and thus informing that agent can only lower player i's payoff. One important consequence of Lemma 1 is the existence of a minimum and a maximum SNE.

19. Note that simultaneous best-responses $\text{SBR} := (\text{SBR}_1, ..., \text{SBR}_i)$ may not be increasing: we might have $\mathbf{S}_i \subseteq \mathbf{S}'_i\ \forall i$, but $\text{SBR}(\mathbf{S}) \not\subseteq \text{SBR}(\mathbf{S}')$.

THEOREM 1. *There exist two strict Nash equilibria* $\underline{\mathbf{S}}^*, \overline{\mathbf{S}}^*$ *with the respective properties: for any SNE* $\mathbf{S}^*$*, we have* $\underline{\mathbf{S}}^* \subseteq \mathbf{S}^* \subseteq \overline{\mathbf{S}}^*$*. We call* $\underline{\mathbf{S}}^*$ *the minimum SNE and we call* $\overline{\mathbf{S}}^*$ *the maximum SNE.*[20]

The proof is not trivial given that communication strategies are discrete and that the monotonicity property only holds over strict best-responses (see Remark 1). We introduce a sequential best-response map, and show that, starting from the empty strategy set, the iteration of the map converges to a minimum SNE, $\underline{\mathbf{S}}$ (for the maximum SNE, we use a imilar argument, with different initial conditions). This result echoes supermodular games, through the monotonicity property of strict best-responses, although the game is not supermodular, because the payoffs are not supermodular on the partially ordered spaces of actions.

Having shown the existence of a minimum SNE has a major welfare implication[21]:

PROPOSITION 2. *The minimum SNE strictly Pareto-dominates all other SNEs (over the set of players).*

Proposition 2 follows from a simple observation: by construction of best-responses, for any equilibrium with a set of informed agents larger than $\underline{\mathbf{S}}^*$, the expected externality from those informed agents in the larger SNE who are not in set $\underline{\mathbf{S}}^*$ is lower than the expected externality got from agents in set $\underline{\mathbf{S}}^*$. Note that Pareto-dominance applies here to players only, and that regular agents can be better off in larger equilibria.

Example 1 continued: Minimum equilibrium and Pareto-dominance. Let us again go back to Cournot Example 1, with $\mathcal{I} = \{1, 2\}$. The minimum equilibrium (introduced in Theorem 1) is the empty profile $\mathbf{S}^* = \emptyset$, with payoffs vector equal to $(0.079, 0.044)$. In accordance with Proposition 2, it Pareto-dominates the other SNE, whose payoffs vector is $(0.069, 0.035)$.[22]

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20. Formally, $\underline{\mathbf{S}}^*, \overline{\mathbf{S}}^*$ are not unique in terms of action profile. They are unique in terms of set of informed agents.

21. Actually we prove the more general statement that any SNE Pareto-dominates any SNE with a larger set of informed agents.

22. A communication profile $\mathbf{S}'$ Pareto dominates $\mathbf{S}$ if $\pi_i(\mathbf{S}') \geq \pi_i(\mathbf{S})$, for all $i \in \mathcal{I}$.

REMARK 2 (*Sequential game*). If we consider the sequential version of the communication game, the sub-game perfect equilibrium is the minimum SNE. See Appendix A for formal definitions and statements.

REMARK 3 (*Heterogeneous probabilities to win the contest*). Lemma 1 extends to heterogeneous probabilities to win the contest. Assume that agents have an individual technology to compete, captured by the vector of characteristics $\theta = (\theta_i)_{i \in \mathcal{N}}$. Then, assume that the probability for agent i to win the contest is given by the ratio $\frac{\theta_i}{\sum_{k \in \mathcal{M}} \theta_k}$. In that case, player i wishes to inform agent j under communication profile $(\mathbf{S})$ whenever

$$e_{ji} > \frac{\sum_{k \in \mathcal{M}(\mathbf{S})} \theta_k e_{ki}}{\sum_{k \in \mathcal{M}(\mathbf{S})} \theta_k}$$

Therefore, best-responses have the same structure as in the benchmark case. In particular, to see that the monotonicity property holds, it is easily seen that player i's communication threshold can only be decreased when agent j is informed by a third party.[23]

REMARK 4 (*Probability to win the contest increasing with the number of informed agents*). In some circumstances, like a race to innovate, the probability to get a winner can be an increasing function of the number of informed agents. For instance, this can arise in teams or in research activity, where collective searching can boost the emergence of new ideas, or when the pressure of the competition increases incentives. It can be seen that, when the probability to get a winner is an increasing and convex function of the number of informed agents, the monotonicity property holds.

REMARK 5 (*Communication-related transaction*). In general, informing can entail a cost, representing any transaction cost depending on the context. We represent such friction through matrix $\mathbf{F} = (f_{ij})$, with $f_{ij} \in \mathbb{R}$; for instance, $f_{ij} < 0$ can represent a cost to player i induced from informing agent j or for transmitting the

23. In the model, the probability to win the contest is exogenous. If we rather introduce individual costly efforts to win the contest, this brings a new motive shaping incentives to communicate. This new motive is related to the interaction among the efforts of the participants of the contest: agents may want to communicate in the purpose of influencing - typically decreasing - the efforts of competitors. This explains the possible failure of the monotonicity property.

technology necessary to compete in the contest. Alternatively, $f_{ij} > 0$ could represent a reward, like warm-glow effect (i.e. the satisfaction of communicating), or a fixed price, etc, that is auxiliary to the information transmission from player i to agent j. It is straightforward to see that the monotonicity property holds for any matrix $\mathbf{F}$, irrespective of its sign. Indeed, because f_{ij} does not affect player i's communication threshold[24], if player i finds it profitable to inform an agent, further communication on the network can only lower the communication threshold.

3.2. Who informs who? Some polar cases

In this subsection, we investigate further who informs who among players. Our aim is to identify informers as a function of the primitives of the model. If the characterization obtained in Proposition 1 holds on general externality matrices, in order to go beyond and to have a better understanding of who informs who, it is useful to present more specific externality matrices. We first consider the case where agents are ranked by externality level in Section 3.2.1. Then , in Section 3.2.2, we turn to externality matrices in which externalities an agent receive can only take one non-zero value. In Appendix C, we consider another interesting class of externality matrices.

3.2.1. Common preference. We consider the set of externality matrices such that all agents have the same ordinal ranking on externalities: We say that $\mathbf{E}$ is a *common-preference externality matrix* if $u_{ij} \geq u_{i+1,j}$ for all $i < n, j \neq i$ (up to permutation of agents' labelling).

THEOREM 2. *For any common-preference externality matrix, and any set of players, there is a unique SNE.*

By Theorem 2, as soon as winners generate a common ordinal ranking in externalities, and thus a common ranking of preferences in the society, there is a unique equilibrium in communication. The equilibrium can easily be determined. Remembering that in our convention the preferences are given by the index of agents, an equilibrium is characterized by a threshold t^* such that only regular agents of index

24. Every equilibrium $\mathbf{S}^*$ is such that: $j \in \mathbf{S}_i^* \Leftrightarrow e_{ji} + f_{ij} > \frac{1}{m(\mathbf{S}^*)} \sum_{k \in \mathcal{M}(\mathbf{S}^*)} e_{ki}$.

lower than or equal to t^* are informed. This index satisfies

$$\min_{i\in\mathcal{I}} e_{t^*,i} - (\pi_i^* - u_i) > 0 \text{ and } \max_{i\in\mathcal{I}} e_{t^*+1,i} - (\pi_i^* - u_i) < 0$$

3.2.2. Receiving same externalities from neighbors. Consider the class of externality matrices such that $e_{ij} = e_j\, g_{ij}$ for all $i \neq j$, with $g_{ij} \in \{0,1\}$, and $g_{ii} = 1$. Let $\mathcal{N}_i := \{j : g_{ji} = 1\}$ be the set of neighbours of i, and $d_i := |\mathcal{N}_i|$. Such externality matrices are then shaped by three inputs: the vector $(e_{ii})_{i\leq n}$, the vector $(e_i)_{i\leq n}$, as well the network $\mathbf{G}$. We call them *binary input* externality matrices. Since a player can receive only a single externality level from neighbors, it is immediate that a profitable communication to one uninformed neighbor induces a profitable communication to all uninformed neighbors:

PROPOSITION 3. *Consider a binary input externality matrix. Any SNE $\mathbf{S}^*$ is such that $\mathbf{S}_i^* \in \{\emptyset, (\mathcal{N}_i \cap \mathcal{J}) \setminus \mathbf{S}_{-i}^*\}$.*

That is, partial communication is not individually optimal. However, this is not incompatible with partial communication at the society level, when only a subset of players communicate. In the extreme case where the network is complete, there is a unique equilibrium, in which a player i communicates to all uninformed neighbors if $e_i > e_{ii}$, and a player i does not communicate at all if $e_i \leq e_{ii}$. Now for general network structure, it can be that only a subset of players communicate, and there can be equilibrium multiplicity.

In order to explore equilibria further, we need to introduce few notation. Define, for any player $i \in \mathcal{I}$, the individual index

$$\eta_i := \frac{e_{ii}}{e_i} + d_i$$

This simple index, which increases with degree and decreases with received externality, embodies the two-dimensional aspect of incentives. Note that a larger degree and/or a lower received externality are both detrimental to communication and tend to increase the index. Assume without loss of generality that $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_I$ and, given $i_0 \in \{0, 1, ..., I\}$ define

$$M(\eta_{i_0}) := \left(\bigcup_{i\leq i_0} \mathcal{N}_i\right) \cup \mathcal{I}, \quad \text{and} \quad \xi(\eta_{i_0}) := \min_{i>i_0} \left\{ \frac{e_{ii}}{e_i} + |\mathcal{N}_i \cap M(i_0)| \right\},$$

with the convention that $\eta_0 = 0$ and $\xi(n) = +\infty$. Note that $M(0) = \mathcal{I}$ and $\xi(0) = \min_{i \in \mathcal{I}} \left\{ \frac{e_{ii}}{e_i} + |\mathcal{N}_i \cap \mathcal{I}| \right\}$. The next proposition provides a detailed description of equilibria in this class of externality matrices:

PROPOSITION 4. *Consider an binary input externality matrix. To any SNE $\mathbf{S}^*$, we can associate a threshold $\eta^* \in \{\eta_1, ..., \eta_{\mathcal{I}}\}$ such that*

$$\mathbf{S}_i^* = \mathcal{N}_i \setminus (\mathcal{I} \cup \mathbf{S}_{-i}^*),\ \forall \eta_i \leq \eta^*,\ \textit{while}\ \ \mathbf{S}_i^* = \emptyset\ \forall \eta_i > \eta^*.$$

Furthermore a profile such that only agents i with index $\eta_i \leq \eta^*$ communicate is a SNE as soon as $\eta^* < |M(\eta^*)| \leq \xi(\eta^*)$.[25] *All in all, equilibria are nested and communication is driven by low-index players.*

Proposition 4 formally expresses that any SNE is associated to an index η^* with the property that (i) players of index lower than η^* communicate to all uninformed neighbors and (ii) players of index strictly larger than η^* do not communicate to anyone.

Proposition 4 is useful to understand equilibria analysis. Note first that all equilibria are nested. Then the main message is that communication is driven by low-index players. This index embodies the two factors, received externality and communication threshold, shaping incentives to communicate. In particular, the lower the degree and/or the higher the received externality, the lower the index. More precisely, we can check whether an index η^* corresponds to an equilibrium by applying the following procedure. First, compute the value $|M(\eta^*)|$ which is the number of agents informed once all players of index not larger than η^* inform their neighbors, and then check that $\eta^* < |M(\eta^*)|$; Second, among all players of index η strictly larger than η^*, identify the one with smallest index ξ given that all players of index η^* or less communicate, and check whether that value, $\xi(\eta^*)$, satisfies $|M(\eta^*)| \leq \xi(\eta^*)$.

An interesting subclass is the set of externality matrices in which all neighbors generate the same externality level. Consider the subclass such that $e_{ii} = e_0$ for all i, and $e_{ij} = eg_{ij}$ for all $i, j \neq i$. this class generates a communication game of parameters represented by the triplet $(e_0, e, \mathbf{G})$. In this context, incentives are

25. The no-communication strategy profile $\mathbf{S} = \emptyset$ is a SNE as soon as $I \leq \xi(0)$. Also a full communication profile $\mathbf{S} = \mathcal{J}$ is a SNE if $\eta_I < n$.

only driven by communication thresholds[26]: those players with lower communication threshold are more incited to communicate. Threshold being then directly related to degrees, we can state:

COROLLARY 1. *For any externality matrix represented by the triplet* $(e_0, e, \mathbf{G})$*, any equilibrium is characterized by a threshold* d^* *such that*

$$\mathbf{S}_i^* = \mathcal{N}_i \setminus (\mathcal{I} \cup \mathbf{S}_{-i}^*)^{\text{`}},\ \forall i:\ d_i \leq d^*,\ \textit{while}\ \ \mathbf{S}_i^* = \emptyset\ \forall i:\ d_i > d^*.$$

If $\mathbf{G}$ is a *Nested-Split Graph* (i.e. $\forall i \neq j,\ d_j \geq d_i \Rightarrow \mathcal{N}_i \subset \mathcal{N}_j \cup \{j\}$) [27], an externality matrix $\mathbf{E}$ of type $(e_0, e, \mathbf{G})$ is a common-preference externality matrix. Thus, by Theorem 2, there is a unique SNE.

Consider the Nested-Split Graph depicted in Figure 3. We have $d_1 = 5 > d_2 = d_3 = 4$. If $1 \leq \frac{e_0}{e} < 2$ then any profile where players 2 and 3 inform agents 4 and 5, while player 1 does not inform anyone is a SNE.

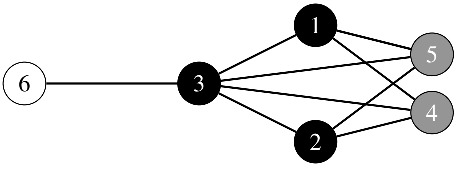

FIGURE 3. $n = 6$; The network is a Nested-Split Graph. Black nodes represent players. Grey nodes represent regular agents informed by players. White nodes represent uninformed regular agents.

Last, one application of binary input externality matrices corresponds to a society of local altruism without transfers; here the externality e_i received by player i when one of her neighbor wins the contest depends on her degree d_i; we then obtain that, under homogeneous wealth, only players with lower degrees transmit information (see Appendix B for more details).

3.3. Comparative statics

We now investigate comparative statics over the externality matrices. A preliminary observation is that, for a given set of players, increasing externalities among players can only reduce communication at the minimum and the maximum equilibria because

26. The ratio $\frac{e_{ii}}{e_i} = \frac{e_0}{e}$ being identical for all players, the index η_i is only differentiated through degrees.

27. See Mahadev and Peled (1995) for a comprehensive introduction.

communication thresholds can only be increased. For that same reason, it is clear that increasing the diagonal entries of a externality matrix can only reduce communication. Second, it might seem at first glance that communication at the minimum equilibrium can only increase with the externalities that players can obtain from regular agents. This is not true because increasing externalities also increases the threshold levels triggering communication.

However we can identify some situations in which externality increases cannot be detrimental to communication. Consider a single increase, say e_{ji}, in the externality matrix. Even though agent j becomes more interesting for i, a possible adverse effect is that, if agent j was not informed before the increase and is informed after, player i's best-response communication can actually be reduced, because her communication threshold has increased. Now suppose that this does not happen, i.e. player i's best-response is not reduced after the inflation. A direct implication of the monotonicity property is that no other player would be better off reducing communication after the increase. Let $\Delta^{(i)}$ denote a matrix such that,$\forall k$, $\Delta^{(i)}_{ki} \geq 0$, and $\Delta^{(i)}_{kj} = 0,\ \forall j \neq i$; We get:

LEMMA 2. *If,* $\forall i = 1, \cdots, n$, $\forall \mathbf{S}_{-i}$, $SBR_i(\mathbf{S}_{-i} \mid \mathbf{E}) \subseteq SBR_i(\mathbf{S}_{-i} \mid \mathbf{E} + \Delta^{(i)})$, *then*

$$\underline{\mathbf{S}}(\mathbf{E}) \subseteq \underline{\mathbf{S}}(\mathbf{E} + \Delta^{(i)}).$$

The proof is omitted and directly related to Lemma 1. Said differently, a decrease in communication following an increase in column i can only be driven by a decrease of communication of player i herself. Which condition on inflation $\Delta^{(i)}$ should be imposed to ensure that player i's optimal communication will increase? Consider the set of agents informed by player i under externality matrix $\mathbf{E}$. It should be that the inflation $\Delta^{(i)}$ puts the communication threshold below the minimum externality generated by agents in this set after the inflation. This condition depends on the communication pattern at the considered equilibrium.

A sufficient condition, for any set of players, consists in concave transformations[28]:

28. Note that the following neutrality result is direct: given two externality matrices $\mathbf{E}$ and $\mathbf{E}'$ such that $\mathbf{E}' = a\mathbf{E} + b\mathbf{J}$ (with $\mathbf{J}$ the matrix of ones), with $a \neq 0$, then $\underline{\mathbf{S}}(\mathbf{E}') = \underline{\mathbf{S}}(\mathbf{E})$.

PROPOSITION 5. *Given two externality matrices $\mathbf{E}$ and $\mathbf{E}'$ such that $e'_{ij} = \varphi_i(e_{ij})$ for all $i \neq j$; and $e'_{ii} \leq \varphi_i(e_{ii})$, for some concave and strictly increasing φ_i. Then*

$$SBR_i(\mathbf{S}_{-i} \mid \mathbf{E}) \subseteq SBR_i(\mathbf{S}_{-i} \mid \mathbf{E}'),\ \forall \mathbf{S}_{-i},\ \textit{and}\ \ \underline{\mathbf{S}}^*(\mathbf{E}) \subseteq \underline{\mathbf{S}}^*(\mathbf{E}').$$

The intuition is as follows: First, by concavity of function φ_i, player i's communication threshold, which is given by averages of externalities, can only be lowered when passing from $\mathbf{E}$ to $\mathbf{E}'$. Second, on top of the concavity transformation, another positive effect occurs when diagonal entries are lowered, as said earlier. This effect is favorable to communication because diagonal entries are necessarily counted to compute the average externality received over all informed agents winning the contest; reducing this entry can thus only decrease the communication threshold.

3.4. Welfare

The question of the diffusion of economic opportunities is of major interest from the perspective of the overall economic performance. An adequate diffusion would contribute to improving a good matching between opportunities and economic agents.[29] In that regard, it is important to address efficiency.

Consider an ex ante utilitarian welfare criterion. More precisely, a welfare function $W(\mathbf{S}|\mathbf{U}, \mathcal{I})$ depends on both the matrix of externalities and the set of players. Then, given a profile of informed agents $\mathcal{M}(\mathbf{S})$, the aggregate payoff is given by (abusing the notation for convenience when there is no confusion)

$$W(\mathbf{S}) = \frac{1}{m(\mathbf{S})} \sum_{j \in \mathcal{M}(\mathbf{S})} \sum_{i \in \mathcal{N}} e_{ji}$$

There is always an efficient communication profile, i.e. one maximizing the welfare function. To each externality matrix $\mathbf{E}$, define a social externality to each agent i: $se_i = \sum_{j \in \mathcal{N}} e_{ij}$. The efficient communication profile can then be characterized:

PROPOSITION 6. *An efficient communication profile $\hat{\mathbf{S}}$ consists in informing agents with social utility larger than the average social externality of informed agents.*

29. Examples abound in the economic literature about the negative impact of mismatch on economic activity. For instance, see Lise and Postel-Vinay (2020) or Fredriksson et al. (2018) in the context of job market; Cortes and Lincove (2016) about student admission in college, or Carlana et al. (2022) about educational choices of children of immigrants; Akcigit et al. (2016) about the patent market.

Formally,

$$\hat{\mathbf{S}} = \{i : se_i \geq \frac{1}{m(\hat{\mathbf{S}})} \sum_{j \in \mathcal{M}(\hat{\mathbf{S}})} se_j\}$$

Determining the efficient communication resorts to considering a representative player of the set of players, having an objective to maximize the social externality, and let that representative agent play her best-response (multiple efficient profile can coexist). The shape of the efficient communication depends on the allocation of players. E.g., if players are those generating the largest social externalities, the efficient communication consists in not informing others; if in opposite players are those generating the lowest social externalities, the efficient communication consists in informing all others; if the players are in-between, the efficient communication can be partial.

The efficient communication configuration may not coincide with any equilibrium. We then discuss the efficiency level of communication equilibria. In general either under-communication or over-communication are possible. For instance, it could be socially desirable to inform an agent generating high social externality, but if that agent delivers a low externality level to players, there can be under-communication with respect to the efficient communication. Similarly, an agent can generate high externality for a player but low social externality, which leads to over-communication. The overall discussion depends on the allocation of players.

To illustrate, consider the following common-preference externality matrix for $n = 4$:

$$\mathbf{E} = \begin{bmatrix} e_0 & e_1 & e_1 & e_1 \\ e_2 & e_0 & e_2 & e_2 \\ e_3 & e_3 & e_0 & e_3 \\ e_4 & e_4 & e_4 & e_0 \end{bmatrix}$$

Then, there is a unique equilibrium by Proposition 2. If $\mathcal{I} = \{1, 2\}$, the efficient allocation consists in informing no agent; but player 1 wants to inform agent 3 if e_0 is sufficiently low; If $\mathcal{I} = \{3, 4\}$, the efficient communication requires to inform all agents, but player 3 does not want to inform agent 2 if e_0 is sufficiently large.

The same conclusions can hold for externality matrices in which preferences are not aligned. Consider for instance

$$\mathbf{E} = \begin{bmatrix} e_0 & e & 0 & e \\ e & e_0 & e & e \\ 0 & e & e_0 & e \\ e & e & e & e_0 \end{bmatrix}$$

and $\mathcal{I} = \{1, 2\}$. Then, for $e > 0$, the efficient communication requires to inform agent 4. For $e_0 < e$, the equilibrium is the full-communication configuration meaning over-communication, while for $e_0 > e$, the equilibrium is no-communication, meaning under-communication.

However, when network $\mathbf{G}$ is a Nested-Split Graph, there is always over-communication (or coincidence) with respect to the efficient communication. Indeed, players with lower degree have larger incentives to communicate. Given that there is a common ordering in preferences in NSGs, stability is driven by the incentives to communicate of the player of lowest degree whatever the set pf players. Hence, the player of lowest degree has stronger incentives than the average over the set of players.

4. Concluding remarks

In this paper, we investigated the private incentives to share information about the existence of competitive opportunities, within a context where an agent's exploitation of such an opportunity generates externalities for others. We found that in this environment, private incentives for communication are amplified when others also engage in sharing information, leading to the emergence of a minimal communication equilibrium that Pareto-dominates all other equilibria among the subgroup of initially informed agents. Our analysis further reveals that larger externalities positively influence communication under conditions of concave increases, and highlights the potential for either over-communication or under-communication relative to the level of communication that would be considered efficient.

Despite the highly stylized nature of our model, delving deeper into the public policy implications of this communication framework in real-world scenarios remains a compelling avenue for future research. There are at least two avenues for policy intervention that merit consideration. Firstly, policy measures could be designed to influence individual communication incentives by adjusting the externalities involved.

For example, in the context of public goods, policymakers could fund enhancements to public service quality; similarly, in the job market, interventions could target wage adjustments to influence job-related information sharing. Secondly, exploring policies that strategically expand the circle of agents aware of opportunities, ensuring that this expansion aligns with social welfare goals, presents an intriguing area of study.

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Appendix A: Proofs

Proof of Result 1. Uniqueness directly follows from the fact that $\text{SBR}_i(\mathbf{S}_{-i})$ is the intersection of all elements of $Br_i(\mathbf{S}_{-i})$.

We now prove the second statement. Note that there cannot be $1 \leq l < l' \leq L$ such that $l' \in \text{SBR}_i(\mathbf{S}_{-i})$ while $l \notin \text{SBR}_i(\mathbf{S}_{-i})$, because deviating to informing l instead of l' would yield an equal or higher payoff, contradicting the fact that $\text{SBR}_i(\mathbf{S}_{-i})$ is the strict best-response. Thus there exists $l^* \geq 0$ such that $\text{SBR}_i(\mathbf{S}_{-i}) = \{j_1, ...j_{l^*}\}$.[30]

Let now

$$f(l) := Mean\left\{(e_{ki})_{k \in \mathcal{I} \cup \mathbf{S}_{-i}}, e_{j_1 i}, \cdots, e_{j_l i}\right\}$$

for $l = 0, ..., L$. Note that

$$f(l) \geq f(l+1) \Leftrightarrow f(l) \geq e_{j_{l+1} i} \Rightarrow f(l+1) \geq e_{j_{l+2} i} \Leftrightarrow f(l+1) \geq f(l+2).$$

As a consequence the map $f(\cdot)$ is quasi-concave in the sense that

$$f(l) \geq f(l+1) \Rightarrow f(l+1) \geq f(l+2).$$

30. Note that fact that the ordering is not uniquely defined does not contradict that the strict best-response is unique.

Hence l^* is the only integer in $\{0, ..., L\}$ such that $f(l^* - 1) < f(l^*)$, and $f(l^*) \geq f(l^* + 1)$[31]. This proves the second statement. □

Proof of Lemma 1. Write $\mathcal{J} \setminus \mathbf{S}'_{-i} = \{j_1, \geq, j_L\}$, where $e_{j_1 i} \geq e_{j_2 i} \geq \cdots \geq e_{j_L i}$. Then $\text{SBR}_i(\mathbf{S}_{-i}) \setminus \mathbf{S}'_{-i} = \{j_1, ..., j_l\}$ for some $l \leq L$. By definition of j_l belonging to the strict best-response to $\mathbf{S}_{-i}$ we must have that $e_{j_l i}$ is strictly greater than $Mean(A)$, where

$$\mathbf{A} := \{e_{ji} : \ j \in \mathbf{S}_{-i} \cup \mathcal{I}\} \cup \{e_{j_1 i}, \cdots, e_{j_l i}\} \cup \{e_{ji} : \ j \in \mathbf{S}'_{-i}, e_{ji} > e_{j_l i}\}$$

We want to prove that j_l belongs to $\text{SBR}_i(\mathbf{S}'_{-i})$. Let $\mathbf{A}' := \{e_{ji} : \ j \in \mathbf{S}'_{-i} \cup \mathcal{I}\} \cup \{e_{j_1 i}, \cdots, e_{j_l i}\}$. Then

$$\mathbf{A}' = \mathbf{A} \cup \{e_{ji} : j \in \mathbf{S}'_{-i}, \ e_{ji} \leq e_{j_l i}\}$$

Hence, since $e_{j_l, i} > Mean(\mathbf{A})$, we necessarily also have that $e_{j_l, i} > Mean(\mathbf{A}')$, because every element in $\mathbf{A}' \setminus \mathbf{A}$ is smaller or equal than $e_{j_l i}$. Thus $j_l \in \text{SBR}_i(\mathbf{S}'_{-i})$. □

A profile $\mathbf{S}$ is *under-informed* if $\mathbf{S}_i \subseteq \text{SBR}_i(\mathbf{S}_{-i})$ for any $i \in \mathcal{I}$. We call $\mathcal{S}_u$ the set of under-informed profiles. Furthermore, for any $i \in \mathcal{I}$, let $\mathbf{B}_i$ be given by

$$\mathbf{B}_i : (\mathbf{S}_i, \mathbf{S}_{-i}) \mapsto (\text{SBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}), \quad \text{and } B_i(\mathbf{S}_i, \mathbf{S}_{-i}) = \text{SBR}_i(\mathbf{S}_{-i}) \cup \mathbf{S}_{-i}.$$

DEFINITION A.1. The *sequential best-response map* is constructed as follows. Let $\mathbf{S} = (\mathbf{S}_i)_{i \in \mathcal{I}}$ be an action profile. Then $\mathbf{B} : \mathcal{S} \to \mathcal{S}$ is defined as[32]

$$\mathbf{B}(\mathbf{S}) := \mathbf{B}_I \circ \mathbf{B}_{I-1} \circ ... \circ \mathbf{B}_1(\mathbf{S})$$

We write $B(\mathbf{S}) = \cup_i (\mathbf{B}(\mathbf{S}))_i$.

31. With the convention that $f(-1) < f(0)$ and $f(L+1) \leq f(L)$

32. Note that map $\mathbf{B}$ depends on the order of players. However, as we will see the important objects do not depend on the order chosen. Note also that $\mathbf{B}_i(\mathbf{S})$ and $(\mathbf{B}(\mathbf{S}))_i$ are different objects; the map $\mathbf{B}$ is not monotonic in the classical sense, as there are simple examples where $\mathbf{S}_i \subseteq \mathbf{S}'_i$ for all i does not imply that $(\mathbf{B}(\mathbf{S}))_i \subseteq (\mathbf{B}(\mathbf{S}'))_i$.

Proof of Proposition 1. The proof is immediate, remarking that the condition $e_{ji} > \pi_i(\mathbf{S}^*)$ is equivalent to the condition $e_{ji} > \pi_i(\mathbf{S}^* \setminus \{j\})$; meaning that the incentives condition, which says that benefit from communication exceeds the before-communication payoff, can also be expressed in terms of the after-communication payoff. □

Before proving Theorem 1, we first prove some useful lemmas.

LEMMA A.1. *If* $\mathbf{S}$ *and* $\mathbf{S}'$ *are such that* $\mathbf{S} \subseteq \mathbf{S}'$ *and* $\mathbf{S}'$ *is under-informed then, for any player* i*, we have* $B_i(\mathbf{S}) \subseteq B_i(\mathbf{S}')$*. More importantly,* $\mathbf{B}(\mathbf{S}')$ *is under-informed and* $B(\mathbf{S}) \subseteq B(\mathbf{S}')$*.*

Proof. By assumption, $\mathbf{S}'$ is such that $\mathbf{S}'_i \subseteq \text{SBR}_i(\mathbf{S}'_{-i})$. Hence $\mathbf{S}_{-i} \subseteq \mathbf{S}' \subseteq B_i(\mathbf{S}') = \text{SBR}_i(\mathbf{S}'_{-i}) \cup \mathbf{S}'_{-i}$. Consequently we only need to prove that $\text{SBR}_i(\mathbf{S}_{-i}) \subseteq \text{SBR}_i(\mathbf{S}'_{-i}) \cup \mathbf{S}'_{-i}$. Without loss of generality, we can write $\mathcal{J} \setminus \mathbf{S}_{-i} = \{j_1, \cdots, j_P\} \cup (\mathbf{S}' \setminus \mathbf{S}_{-i})$ where $\{j_1, \cdots, j_P\} = \mathcal{J} \setminus \mathbf{S}'$ and $e_{j_1 i} \geq \cdots \geq e_{j_P i}$.
The set $\text{SBR}_i(\mathbf{S}_{-i})$ can then be written $B \cup \{j_1, \cdots, j_p\}$ (where $B \subseteq \mathbf{S}' \setminus \mathbf{S}_{-i}$), while $\text{SBR}_i(\mathbf{S}'_{-i}) = \mathbf{S}'_i \cup \{j_1, \cdots, j_{p'}\}$. We need to prove that $j_p \in \text{SBR}_i(\mathbf{S}'_{-i})$. Since $j_p \in \text{SBR}_i(\mathbf{S}_{-i})$, we have

$$e_{j_p i} > Mean\{e_{ji} : \ j \in \mathcal{I} \cup \mathbf{S}_{-i} \cup B \cup \{j_1, \cdots, j_{p-1}\}\}$$

Thus we have

$$e_{j_p i} > Mean\left(e_{ji} : \ j \in \mathcal{I} \cup \mathbf{S}_{-i} \cup (\mathbf{S}' \setminus \mathbf{S}_{-i}) \cup \{j_1, \cdots, j_{p-1}\}\right),$$

because B consists of the elements of the elements of $\mathbf{S}' \setminus \mathbf{S}_{-i}$ who give the largest share to i. This proves that $j_p \in \text{SBR}_i(\mathbf{S}'_{-i})$, and therefore that $B_i(\mathbf{S}) \subseteq B_i(\mathbf{S}')$.

Let us now prove that $B(\mathbf{S}) \subseteq B(\mathbf{S}')$. By a recursive argument, it is enough to show that $\mathbf{B}_i(\mathbf{S}')$ is under-informed, to be able to repeatedly apply the first point of the lemma. Let $j \neq i$. We must prove that $(\mathbf{B}_i(\mathbf{S}'))_j \subseteq \text{SBR}_j((\mathbf{B}_i(\mathbf{S}'))_{-j})$. Since $(\mathbf{B}_i(\mathbf{S}'))_j = \mathbf{S}'_j$, it amounts to proving that $\mathbf{S}'_j \subseteq \text{SBR}_j((B_i(\mathbf{S}'))_{-j})$. Note that $\mathbf{S}'_j \cap (\mathbf{B}_i(\mathbf{S}'))_{-j} = \emptyset$. Hence

$$\mathbf{S}'_j \subseteq \text{SBR}_j(\mathbf{S}'_{-j}) \setminus (\mathbf{B}_i(\mathbf{S}'))_{-j} \subseteq \text{SBR}_j((\mathbf{B}_i(\mathbf{S}'))_{-j}),$$

because $\mathbf{S}'_{-j} \subseteq (\mathbf{B}_i(\mathbf{S}'))_{-j}$, and applying Lemma 1. □

LEMMA A.2. *Let* $\mathbf{S} \in \mathcal{S}_u$. *Then* $\mathbf{S}_i \subseteq (\mathbf{B}(\mathbf{S}))_i$ *for any* $i \in \mathcal{I}$.

Proof. We have

$$(\mathbf{B}(\mathbf{S}))_i = \mathrm{SBR}_i((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{S}_{i+1}, ..., \mathbf{S}_I), \text{ for } i = 1, ..., I.$$

We show the proposition by induction on i. By definition of $\mathbf{S} \in \mathcal{S}_u$ we have $\mathbf{S}_1 \subseteq \mathrm{SBR}_1(\mathbf{S}_{-1}) = (\mathbf{B}(\mathbf{S}))_1$. Assume that $\mathbf{S}_j \subseteq (\mathbf{B}(\mathbf{S}))_j$ for $j = 1, ..., i-1$. Then

$$\mathbf{S}_{-i} \subseteq ((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{S}_{i+1}, ..., \mathbf{S}_I)$$

and $\mathbf{S}_i \cap (\mathbf{B}(\mathbf{S}))_1 \cup ... \cup (\mathbf{B}(\mathbf{S}))_{i-1} \cup \mathbf{S}_{i+1} \cup ... \cup \mathbf{S}_I)$ by construction. Hence

$$\begin{aligned}
\mathbf{S}_i &\subset Br_i(\mathbf{S}_{-i}) \setminus ((\mathbf{B}(S))_1, ..., \mathbf{B}(S))_{i-1}, \mathbf{S}_{i+1}, ..., \mathbf{S}_I) \\
&\subset Br_i(((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{S}_{i+1}, ..., \mathbf{S}_I)) \\
&= (\mathbf{B}(\mathbf{S}))_i
\end{aligned}$$

by Lemma 1. □

LEMMA A.3. *If* $\mathbf{S}_i \subseteq (\mathbf{B}(\mathbf{S}))_i \; \forall i$ *then* $\mathbf{B}^k(\mathbf{S})$ *is non-decreasing. In particular if* $\mathbf{S}$ *is under-informed then* $\mathbf{B}^k(\mathbf{S})$ *is non-decreasing.*

Proof. Suppose that $\mathbf{S}_i \subseteq (\mathbf{B}(\mathbf{S}))_i$ for any $i \in \mathcal{I}$. We only need to prove that $(\mathbf{B}(\mathbf{S}))_i \circ (\mathbf{B} \circ \mathbf{B}(\mathbf{S}))_i$ and the result follows by induction. We can write the terms of $\mathbf{B}(\mathbf{S})$ recursively:

$$(\mathbf{B}(\mathbf{S}))_i = \mathrm{SBR}_i((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{S}_{i+1}, ..., \mathbf{S}_I), \text{ for } i = 1, ..., I.$$

Also

$$(\mathbf{B}^2(\mathbf{S}))_i = \mathrm{SBR}_i\left((\mathbf{B}^2(\mathbf{S}))_1, ..., (\mathbf{B}^2(\mathbf{S}))_{i-1}, (\mathbf{B}(\mathbf{S}))_{i+1}, ...(\mathbf{B}(\mathbf{S}))_I\right)$$

By assumption we have $\mathbf{S}_{-1} \subseteq (\mathbf{B}(\mathbf{S}))_{-1}$. Moreover $\mathrm{SBR}_1(\mathbf{S}_{-1}) \cap \mathbf{B}(\mathbf{S})_{-1} = \emptyset$. As a consequence

$$\mathrm{SBR}_1(\mathbf{S}_{-1}) \subseteq \mathrm{SBR}_1((\mathbf{B}(\mathbf{S}))_{-1}).$$

Suppose we proved that $(\mathbf{B}(\mathbf{S}))_j \subseteq (\mathbf{B}^2(\mathbf{S}))_j$ for $j = 1, ..., i$ ($i < n$). We now prove that $(\mathbf{B}(\mathbf{S}))_{i+1} \subseteq (\mathbf{B}^2(\mathbf{S}))_{i+1}$, and it will conclude the proof. We have

$$((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_i, \mathbf{S}_{i+2}, ..., \mathbf{S}_I) \subseteq ((\mathbf{B}^2(\mathbf{S}))_1, ..., (\mathbf{B}^2(\mathbf{S}))_i, (\mathbf{B}(\mathbf{S}))_{i+2}, ..., (\mathbf{B}(\mathbf{S}))_I)$$

and $\text{SBR}_{i+1}\left(((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_i, \mathbf{S}_{i+2}, ..., \mathbf{S}_I)\right)$ does not intersect the set $\mathbf{B}^2(\mathbf{S}))_1 \cup ... \cup \mathbf{B}^2(\mathbf{S}))_i \cup (\mathbf{B}(\mathbf{S}))_{i+2} \cup ... \cup (\mathbf{B}(\mathbf{S}))_I$. Consequently it is contained in

$$Br_{i+1}\left((\mathbf{B}^2(\mathbf{S}))_1, ..., (\mathbf{B}^2(\mathbf{S}))_i, (\mathbf{B}(\mathbf{S}))_{i+2}, ..., (\mathbf{B}(\mathbf{S}))_I\right).$$

In other terms $(\mathbf{B}(\mathbf{S}))_{i+1} \subseteq (\mathbf{B}^2(\mathbf{S}))_{i+1}$, and the proof is complete. When $\mathbf{S} \in \mathcal{S}_u$ this follows from Lemma A.2. □

Proof of Theorem 1. The sequence $(\mathbf{B}^k(\emptyset))_k$ is non-decreasing and bounded above in a finite set. Thus there exist $\underline{\mathbf{S}}^*$ and an integer K such that $\mathbf{B}^K(\emptyset) = \underline{\mathbf{S}}^*$. Let $\mathbf{S}^*$ be a strict Nash equilibrium. We need to show that $\underline{\mathbf{S}}^* \subseteq \mathbf{S}^*$ and the proof will be complete. Both $\emptyset$ and $\mathbf{S}^*$ are under-informed. Thus $\mathbf{B}^k(\emptyset) \subseteq \mathbf{B}^k(\mathbf{S}^*) = \mathbf{S}^*$ for any k by Lemma A.1.

We now prove the existence of a maximum SNE. Let $\{\mathbf{S}^*(k)\}_{k=1,...,K}$ be the set of SNEs, and consider a profile $\mathbf{S}$ such that $\mathbf{S}_i \cap \mathbf{S}_{-i} = \emptyset$, and satisfying the following properties:

$$\mathbf{S}_i^*(k) \subseteq \mathbf{S}_i \subseteq \mathbf{S}_i^*(1) \cup ... \cup \mathbf{S}_i^*(K)\ \forall i,\ \forall k; \quad \mathbf{S} = \mathbf{S}^*(1) \cup ... \cup \mathbf{S}^*(K).$$

By the monotonicity property, for $k = 1, ..., K$, we have

$$\mathbf{S}_i^*(k) = SBR_i(\mathbf{S}_{-i}^*(k)) \subseteq SBR_i(\mathbf{S}_{-i}) \cup \mathbf{S}_{-i}.$$

Consequently, since $\mathbf{S}_i \cap \mathbf{S}_{-i} = \emptyset$, we have

$$\mathbf{S}_i \subseteq SBR_i(\mathbf{S}_{-i}).$$

In other terms, $\mathbf{S}$ is under-informed. The sequence $\mathbf{B}^k(\mathbf{S})$ is non-decreasing and therefore converges to a SNE $\overline{\mathbf{S}}^*$ such that $\mathbf{S}^*(k) \subseteq \overline{\mathbf{S}}^*$, for all k. This concludes the proof. □

Proof of Proposition 2. We show that if $\mathbf{S}^* \in SNE$ and $\mathbf{S}^* \subseteq \overline{\mathbf{S}}$ then $\pi_i(\mathbf{S}^*) \geq \pi_i(\overline{\mathbf{S}})$; Therefore, any SNE Pareto-dominates any SNE with a larger set of informed agents. Let $\mathbf{D} = \overline{\mathbf{S}} \setminus \mathbf{S}^*$. We have

$$\pi_i(\overline{\mathbf{S}}) - u_i = \frac{m(\mathbf{S}^*)}{m(\overline{\mathbf{S}})}(\pi_i(\mathbf{S}^*) - u_i) + \frac{1}{m(\overline{\mathbf{S}})} \sum_{d \in \mathbf{D}} e_{d,i}.$$

However $e_{d,i} \leq \pi_i(\mathbf{S}^*) - u_i,\ \forall d \in \mathbf{D}$ because $\mathbf{S}^*$ is strict. Hence $\pi(\overline{\mathbf{S}}) \leq \pi(\mathbf{S}^*)$. □

Proof of Remark 2. Consider the extensive-form game whose set of players is $\mathcal{I}$ and whose associated tree $\mathcal{T}$ is defined by the set of nodes $\{(t, i, \mathbf{S})\}_{t \leq T, i \in \mathcal{I}, \mathbf{S} \subseteq \mathcal{J}}$ - where $T = J + 1$ - with the following structure:

- the *root* is $(1, 1, \emptyset)$
- for $t \leq T - 1$, $\mathbf{S} \subseteq \mathcal{J}$, $i < I$, the set of successors of node $(t, i, \mathbf{S})$ is $\{(t, i + 1, \mathbf{S}') : \mathbf{S} \subseteq \mathbf{S}' \}$
- for $t \leq T - 1$ and $\mathbf{S} \subseteq \mathcal{J}$, the set of successors of node $(t, I, \mathbf{S})$ is$\{(t + 1, 1, \mathbf{S}') : \mathbf{S} \subseteq \mathbf{S}' \subseteq \mathcal{N} \setminus \mathcal{J} \}$;
- $(T, 1, \mathbf{S})$ is a terminal node with payoff $(\pi_i(\mathbf{S}))_{i \in \mathcal{N}}$.

The statement of Remark 2 can be rephrased as follows: *"an action profile is a sub-game perfect equilibrium if and only if the associated set of informed agents is $\underline{\mathbf{S}}$."*
Since any profile associated to $\underline{\mathbf{S}}$ Pareto-dominates any other Nash equilibrium of the communication game, it is immediate to conclude that any profile associated to $\underline{\mathbf{S}}$ is subgame-perfect, since we reach a terminal node only after all players decide not changing the set of informed agent.
Consider an action profile such that $\mathbf{S}$ is not contained in $\underline{\mathbf{S}}$, and let $(\hat{t}, \hat{i}, \hat{\mathbf{S}})$ be the first node in the path with the property that $\hat{\mathbf{S}} \not\subseteq \underline{\mathbf{S}}$. In the sub-game associated with this initial node, the induced action profile associated with $\mathbf{S}$ can not correspond to a Nash equilibrium since any Nash equilibrium of the normal-form game is Pareto dominated by the minimum SNE. $\square$

Proof of Theorem 2. Let $\mathbf{S}^*$ be an SNE. Then there exists $t \in \mathcal{J}$ such that $\mathbf{S}^* = \{j \in \mathcal{J} : j \leq t\}$, because any strict best-response of player i is of the form $\{j \in \mathcal{J} : j \leq t_i\}$. Suppose that there exists another SNE $\hat{\mathbf{S}}^*$, such that $\hat{\mathbf{S}}^* = \{j \in \mathcal{J} : j \leq \hat{t}\}$, with $\hat{t} > t$. Then there exists some $i \in \mathcal{I}$ such that $\hat{t} \in \hat{\mathbf{S}}^*_i$, Thus we may assume without loss of generality that $\hat{\mathbf{S}}^*_i = \hat{\mathbf{S}}^*$ and $\hat{\mathbf{S}}^*_{-i} = \emptyset$ (i informs everyone up to regular agent $\hat{t}$). Since $\pi_i(\hat{\mathbf{S}}^*_i, \emptyset) = \pi_i(\hat{\mathbf{S}}^*) < \pi_i(\mathbf{S}^*) = \pi_i(\mathbf{S}^*, \emptyset)$, player i has a profitable deviation, and it contradicts the fact that $\hat{\mathbf{S}}^*$ is a SNE. $\square$

Proof of Proposition 4. Let $\mathbf{S}^*$ be a SNE, and suppose that i, j are such that $\mathcal{N}_i \cap (\mathcal{I} \cup \mathbf{S}^*_{-i}) \neq \emptyset$, $\mathbf{S}^*_i = \emptyset$ and $\mathbf{S}^*_j \neq \emptyset$. Then

$$e_i \leq \pi^*_i = \frac{e_{ii} + e_i |\mathcal{N}_i \cap (\mathcal{I} \cup \mathbf{S}^*_{-i})|}{I + |\mathbf{S}^*|} \text{ and } e_j > \pi^*_j = \frac{e_{jj} + e_j |\mathcal{N}_j|}{I + |\mathbf{S}^*|}.$$

Hence

$$\eta_i \geq \frac{e_{ii}}{e_i} + |\mathcal{N}_i \cap (\mathcal{I} \cup \mathbf{S}^*_{-i})| \geq I + |\mathbf{S}^*| > \frac{e_{jj}}{e_j} + d_j = \eta_j$$

This proves the first point of the proposition.

Now let $i^* \in \{0, ..., I\}$, and consider a profile $\mathbf{S}^*$ such that

$$\mathbf{S}^*_i = \mathcal{N}_i \setminus (\mathcal{I} \cup \mathbf{S}^*_{-i}), \ \forall i \leq i^*, \text{ while } \mathbf{S}^*_i = \emptyset \ \forall i > i^*.$$

Note that

$$|M(i^*)| = I + |\mathbf{S}^*|, \text{ and } \xi(i^*) = |M(i^*)| \min_{i > i^*} \frac{\pi_i(\mathbf{S}^*)}{e_i}.$$

Observing that $\pi(\mathbf{S}^*) = \frac{e_{ii} + e_i d_i}{|M(i^*)|}$ when $i \leq i^*$, the profile $\mathbf{S}^*$ is a SNE if

$$\min_{i > i^*} \frac{\pi_i(\mathbf{S}^*)}{e_i} \geq 1, \text{ i.e. } \xi(i^*) \geq |M(i^*)| \text{ and } \max_{i \leq i^*} \frac{\pi_i(\mathbf{S}^*)}{e_i} < 1, \text{ i.e. } \eta_{i^*} < |M(i^*)|$$

Consequently, if $\eta^* \geq 0$ is such that $\eta^* < m(\eta^*) \leq \xi(\eta^*)$, $\mathbf{S}^*$ is a SNE. □

Proof of Proposition 5. We show that the communication threshold is lowered for externality matrix $\mathbf{E}'$. Formally, let $j \notin \mathcal{M} = \mathcal{I} \cup \mathbf{S}_{-i}$, and assume that

$$e_{ji} > \frac{1}{m}\left(e_{ii} + \sum_{l \in \mathcal{M} \setminus \{i\}} e_{li}\right)$$

Then

$$\begin{aligned} e'_{ji} = \varphi_i(e_{ji}) &> \varphi_i\left(\frac{1}{m}\left(e_{ii} + \sum_{l \in \mathcal{M} \setminus \{i\}} e_{li}\right)\right) \\ &\geq \frac{1}{m}\left(\varphi_i(e_{ii}) + \sum_{l \in \mathcal{M} \setminus \{i\}} \varphi_i(e_{li})\right) \\ &\geq \frac{1}{m}\left(e'_{ii} + \sum_{l \in \mathcal{M} \setminus \{i\}} e'_{li}\right) \end{aligned}$$

Hence the strict best-response cannot be reduced under $\mathbf{E}'$: $SBR_i(\mathbf{S}_{-i} \mid \mathbf{E}) \subseteq SBR_i(\mathbf{S}_{-i} \mid \mathbf{E}')$. We now prove the last point: first note that, if $\mathbf{T}_{-i} \subseteq \mathbf{S}_{-i}$ then

$$\text{SBR}_i(\mathbf{T}_{-i} \mid \mathbf{E}) \cup \mathbf{T}_{-i} \subseteq \text{SBR}_i(\mathbf{S}_{-i} \mid \mathbf{E}) \cup \mathbf{S}_{-i} \subseteq \text{SBR}_i(\mathbf{S}_{-i} \mid \mathbf{E}') \cup \mathbf{S}_{-i}.$$

Consequently, for any $k \in \mathbb{N}^*$, we have that $\mathbf{B}^k(\emptyset) \subseteq (\mathbf{B}')^k(\emptyset)$, which proves that $\underline{\mathbf{S}}(\mathbf{E}) \subseteq \underline{\mathbf{S}}(\mathbf{E}')$. □

Appendix B: Proof of the statements of Section 2.2

We provide a more formal framework for Section 2.2. Agents $i = 1, ..., n$ are differentiated by a trait/type $a_i^0 \in \mathbb{R}_+$. Depending on the economic model, the trait can be a production cost, the quality of a produced good, or an initial wealth endowment. Suppose that, for any type profile $\mathbf{a}^0 \in \mathbb{R}_+^n$, the game $\mathcal{G}(\mathbf{a}^0) := \left(\mathcal{N} = \{1, ..., n\}, \mathbf{X}_i, v_i(\cdot \mid \mathbf{a}^0)\right)$ has a unique Nash equilibrium $\mathbf{x}^*(\mathbf{a}^0)$, with associated payoff vector $v_i^*(\mathbf{a}^0) := v_i(\mathbf{x}^*(\mathbf{a}^0) \mid \mathbf{a}^0)$.

We assume that the equilibrium payoff is increasing in own trait: $a_i^0 \mapsto v_i^*(\mathbf{a}^0)$ increasing. Prior to playing the underlying game $\mathcal{G}(\mathbf{a}^0)$, agents can compete for an opportunity captured by an improvement of their own trait by an amount $\gamma > 0$. Agents $i \in \mathcal{I} \subseteq \mathcal{N}$ are initially aware of this opportunity, and can diffuse the information to other agents. Given a communication profile $\mathbf{S}$, and recalling that $\mathcal{M}(\mathbf{S})$ is the set of informed agents after communication, the payoff of player i can then be written[33]

$$\pi_i(\mathbf{S}) = \frac{1}{m(\mathbf{S})} \sum_{j \in \mathcal{M}(\mathbf{S})} v_i^*(\mathbf{a}^0 + \gamma \mathbf{h}^j)$$

This corresponds to our setting with externality matrix $\mathbf{E}$ such that $e_{ji} = v_i^*(\mathbf{a}^0 + \gamma \mathbf{h}^j) - v_i^*(\mathbf{a}^0)$.

B.1. Cost-reducing innovation in horizontally differentiated oligopolies

Proof of Observation 1. If the cost profile is $\mathbf{c} \in \mathbb{R}_+^n$, the payoff of agent i is given by

$$v_i(\mathbf{q} \mid \mathbf{c}) = q_i \left(\alpha - c_i - q_i - (\mathbf{B}\mathbf{q})_i\right),$$

and the equilibrium profile is

$$\mathbf{q}^*(\mathbf{c}) = \alpha \left(2\mathbf{I}_n + \mathbf{B}\right)^{-1} \mathbf{1} - \left(2\mathbf{I}_n + \mathbf{B}\right)^{-1} \mathbf{c}.$$

Consequently, if agent j wins the contest the equilibrium profile is

$$\mathbf{q}^*(\mathbf{c}^0 - \gamma \mathbf{h}^j) = \mathbf{q}^*(\mathbf{c}^0) + \gamma \left(2\mathbf{I}_n + \mathbf{B}\right)^{-1} \mathbf{h}^j.$$

We then have

$$q_i^*(\mathbf{c}^\mathbf{0} - \gamma \mathbf{h}^j) = q_i^*(\mathbf{c}^0) + \gamma m_{ij}$$

33. $\mathbf{h}_k^j := 0$ if $k \neq j$ and $\mathbf{h}_j^j = 1$.

Hence

$$e_{ji} = \left(q_i^*(\mathbf{c}^0) + \gamma m_{ij}\right)^2 - \left(q_i^*(\mathbf{c}^0)\right)^2.$$

□

B.2. Quality-improving innovation in vertically differentiated oligopolies

Proof of Observation 2. The utility of firm i, when the quality profile is $\mathbf{a}$, is then

$$v_i(\mathbf{q} \mid \mathbf{a}) = q_i\left(\alpha - c_i - 2\frac{q_i}{a_i^2} - \frac{2\sigma}{a_i}\sum_{j\neq i}\frac{q_j}{a_j}\right)$$

Hence, the equilibrium profile is then

$$\mathbf{q}^*(\mathbf{a}) = \frac{1}{2}\mathbf{M}(\mathbf{a})\mathbf{1}.$$

Since $v_i(\mathbf{q}^*(\mathbf{a}) \mid \mathbf{a}) = \frac{1}{a_i^2}q_i^*(\mathbf{a})^2$, we get the result. □

B.3. Job offer opportunity in a local public good game

Proof of Observation 3. If agent j wins the contest, player i's equilibrium payoff is

$$v_i\left(\mathbf{g}^*(\mathbf{w}^0 + \gamma\mathbf{h}^j) \mid \mathbf{w}^0 + \gamma\mathbf{h}^j\right) = \varphi_i\left((w_i^0 - (\mathbf{M}\mathbf{w}^0)_i - \gamma m_{ij}\right)$$

Hence

$$e_{ji} = \varphi_i\left(\overline{\mathbf{w}}_i^0 - \gamma m_{ij}\right) - \varphi_i\left(\overline{\mathbf{w}}_i^0\right), \text{ if } i \neq j;\; e_{ii} = \varphi_i\left(\overline{\mathbf{w}}_i^0 - \gamma m_{ii} + \gamma\right) - \varphi_i\left(\overline{\mathbf{w}}_i^0\right).$$

□

B.4. Job offer opportunity in an altruist network with transfers

By Theorem 1 in Bourlès et al. (2017), the consumption profile is unique at equilibrium, so that $v_i^*(\mathbf{y}^0)$ is well-defined.

Proof of Observation 4. If the prize is high enough then, at equilibrium, money flows from the winner of the contest, say j. Proposition 4 in Bourlès et al. (2021) entails the following characterization: for any $l \neq j$, we have

$$y_j = y_l - \frac{1}{A}\ln(\hat{\alpha}_{jl}).$$

Summing over l and normalizing by n, we get

$$y_j = y^0 + \frac{\gamma}{n} - \frac{1}{nA}\ln\left(\prod_{l=1}^{n}\hat{\alpha}_{jl}\right).$$

Thus, denoting $C := \frac{e^{-A(y^0+\gamma/n)}}{A} - \frac{e^{-Ay^0}}{A}$, we have

$$w_j(y_j) = -C\left(\prod_{l=1}^{n}\hat{\alpha}_{jl}\right)^{1/n} \quad \text{and} \quad w_i(y_i) = -C\frac{\left(\prod_{i=1}^{n}\hat{\alpha}_{jl}\right)^{1/n}}{\hat{\alpha}_{ji}} \quad \text{for } i \neq j.$$

and agent i's social utility, given that agent j wins the contest, is given by

$$-C\left[\frac{\hat{\alpha}_j}{\hat{\alpha}_{ji}} + \sum_{k\neq i}\alpha_{ik}\frac{\hat{\alpha}_j}{\hat{\alpha}_{jk}}\right] = -C\sum_{k\in\mathcal{N}}\alpha_{ik}\frac{\hat{\alpha}_j}{\hat{\alpha}_{jk}} = -C\hat{\alpha}_j\sum_{k}\frac{\alpha_{ik}}{\hat{\alpha}_{jk}}.$$

Note that the social utility experienced by agent i encompasses not only agent j's utility from winning through altruism, but also all private utility variations of her neighbors that are issued from the transfers originated by the reward to agent j. □

B.5. Winning a prize in an altruist network without transfers

Assume now a society with an altruist network and without transfers. In this world, agents enjoy the utility level reached by their neighbors but do not optimize their benefit through transfers.

Assume for simplicity that $\alpha_{ij} = \alpha g_{ij}$. We will see that the game generates binary input externality matrices. We also explore further the case in which agents have identical initial endowments. In particular, we can show that, in an altruist network without transfers, only those players of lower degrees inform others.

The proof is as follows. Under identical endowment, define $u_{\text{other}} = \alpha v_0^+ + v_0 + \alpha(d_i - 1)v_0$ and $u_{\text{own}} = (v_0^+ + \alpha d_i v_0)$ as respectively the utility when a neighbor wins and the utility of a player when she wins. The condition to inform at an equilibrium of size m is given by

$$u_{\text{other}} > \frac{u_{\text{own}} + d_i u_{\text{other}}}{m}$$

which boils down to a condition on an order-2 polynoma $ad_i^2 + bd_i + c \geq 0$, with $a = \alpha^2 v_0, b = \alpha v_0^+ - v_0(\alpha m - 1), c = v_0^+ - m(\alpha v_0^+ + (1-\alpha)v_0)$. To complete the proof, it is sufficient to show that there cannot be two positive roots. Indeed, if $b > 0$ there is one negative root; if $b < 0$ two positive roots requires $c > 0$. Note then that if $v_0^+ \leq mv_0$, otherwise $b > 0$. But $c < b$ is equivalent to $v_0^+ \leq mv_0$, which forbids that $b < 0 < c$. □

Appendix C: Multi-level local externality matrices

In this Appendix, we present two-level-local externality matrices, whose analysis straightforwardly generalizes to multi-level-local externality matrices.

Consider externality matrices such that $e_{ii} = e_0$ for all i, and $e_{ij} \in \{g_{ij} \cdot \underline{e}, g_{ij} \cdot \overline{e}\}$ for all $i, j \neq i$, and with $g_{ij} \in \{0, 1\}$; this class generates a communication game of parameters represented by the quadruplet $(e_0, \underline{e}, \overline{e}, \mathbf{G})$. This class of matrix, although stylized, induces more complexity than binary input externality matrices. Some interesting insights from the analysis of two-level-local externality matrices are that partial communication can emerge, and that degree centrality is not sufficient to describe incentives: distance-two neighbors matter.

It is easily seen that there are two kinds of informers at equilibrium. We associate an index to each player, that is increasing in both the degree and the proportion of high-externality neighbors. Then we can partition adequately those players with low index (i.e., with low degree and preferably linked to low-externality neighbors), and those players with high index (i.e., with higher degree and more linked to high-externality neighbors). Players with a low index inform all uninformed neighbors, players with high index inform high-externality uninformed neighbors only if low-externality regular neighbors are preferably linked to high-index players.

To see this formally, consider an equilibrium with m^* informed agents. A basic observation is that, at equilibrium, if a player i informs a low-externality neighbor, then player i informs all uninformed neighbors; and if a player informs one high-externality neighbor, she informs all uninformed high-externality neighbors. Defining by β_i the proportion of player i's high-externality neighbors, the incentives to inform a low-externality neighbor, for player i, are then given by the condition

$$m^* > \mu_i = \frac{e_0}{\underline{e}} + \Big(\beta_i \frac{\overline{e}}{\underline{e}} + (1 - \beta_i)\Big) d_i$$

Note that index μ_i only depends on primitives of the model. For those players of index lower than m^*, they inform all neighbors.[34] Consider now those players whose index is large enough to violate the above condition. They still find profitable to inform a

34. Then, for two arbitrary equilibriums, the respective subsets of players informing all uninformed neighbors in each equilibrium are nested.

high-externality neighbor under the condition

$$m^* > \nu_i(m^*) = \frac{e_0}{\overline{e}} + \beta_i d_i + \frac{e}{\overline{e}} \sum_{k \in \mathcal{N}_i \cap \mathbf{S}_{-i}} g_{ik} \cdot I\left[\min_{p \in \mathcal{N}_k \cap \mathcal{I}} \mu_p < m^*\right]$$

Hence, informing a high-externality neighbor is more likely to be valuable when the low-externality regular neighbors of player i are connected to a smaller set of low-index players. Note that, in opposite to the former condition, the RHS depends now on m^*; a simple algorithm allows to find the exact set of informers.

To summarize, to know whether a given configuration with m informed agents is an equilibrium, one has to check that:

(i) All players i of index $\mu_i < m$ inform all uninformed neighbors;

(ii) All players i such that $\mu_i \geq m$ and $\nu_i(m) < m$, inform all high-externality neighbors and only them;

(iii) All players i such that $\mu_i \geq m$ and $\nu_i(m) \geq m$ don't communicate.

Importantly, whereas the index μ does not depend on communication strategies, the index $\nu_i(m)$ does.[35]

This analysis is easily extended to multi-level externality matrices. For externality matrices with $k+1$ possible externality levels (including the zero level), and by a direct generalization of the above case corresponding to $k = 2$, the set of conditions allowing to identify the equilibrium communication of a given player require to incorporate indexes of players at distance up to $2k$ from that player.

35. Such partial characterization may be complemented by algorithmic computation.