Sharing Opportunities under Externalities

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Abstract
A subset of economic agents in a society is aware of the existence of an economic opportunity, and compete for exploiting the opportunity. We study incentives to communicate about the existence of this economic opportunity when the exploitation of the opportunity by the winner generates externalities to other agents. We characterize the equilibria of the communication game and identify conditions under which more externalities generates more communication. (JEL: C72; D83; D85)

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1. Introduction

The discovery and exploitation of economic opportunities are essential to economic activity, requiring both the ability to identify them and the right technology for effective utilization. When these opportunities are not publicly disclosed or when the technology remains privately owned, economic agents often rely on their peers for access. These peers might be motivated to share such opportunities if they anticipate significant positive externalities, despite the competitive nature of these opportunities potentially limiting incentives to share information. There are situations in which incentives to share information are strong. For instance, in the context of grants in academic research, researchers knowing the existence of a grant may inform colleagues, expecting benefits derived from the resources obtained by the grant holder; In Research and Development (R&D) activity, companies aware of a potential innovation may foster competitive dynamics in the innovation race to disadvantage other competitors;[1] In the context of job searches, individuals frequently share information about job openings with their social contacts,[2] driven by altruism[3] or career concerns. Another example is public goods. Some people spend a higher percentage of their income than others on charity/public goods, and opportunities for these people potentially generate larger benefits for others. Therefore, understanding the diverse motivations for sharing information about economic opportunities is critical, reflecting its significant influence on innovation and economic activity.

This paper explores the diffusion of information regarding economic opportunities across social and economic contacts, building on the illustrative examples provided earlier. We propose a novel framework to understand the underlying mechanisms incentivizing the communication of such opportunities. Central to our analysis are

1. R&D partnerships are now a widespread activity in industries, especially those with rapid technological change, such as the IT or the pharmaceutical industry; see Hagedoorn (2002), Powell et al. (2005), or Hagedoorn (2006). Such partnerships can take various forms, including crowdsourcing platforms and open innovation. A company facing a specific technical challenge might find it advantageous to share this challenge on an open innovation platform, inviting external innovators to propose solutions. Not only can this lead to creative and effective solutions, but the company can also establish relationships with external talent and potentially discover new opportunities for collaboration.

2. It is well-known that a huge proportion of job offers are transmitted by social contacts. See for instance Calvo-Armengol and Jackson (2004), Calvo-Armengol and Jackson (2007).

3. There is an experimental literature showing evidence of prosocial behaviors by winners of contests; see for instance Engelman and Strobel (2008), or Binzel and Fehr (2013).

three key elements: the competitive nature of exploiting rival opportunities, the division within society between those aware and those unaware of these opportunities, and the externalities arising from their exploitation. Our model suggests that agents may find it advantageous to inform others about an opportunity when they stand to gain from the externalities ensuing from its utilization by the informed party. The drive to inform is fundamentally rooted in these externalities, which exhibit diverse characteristics across various economic settings. The aim of this study is to explore how the structure of externalities influences the incentives to share information about economic opportunities.

We model this question through a simple normal-form communication game. To ground our abstract model in a concrete context, our leading application throughout the paper will be research grants in academics. We consider a society partitioned into a set of agents, called players, who are initially informed about the existence of the contest, and those who are not. We consider a matrix of externalities, whose i, j entry represents the externality received by agent j from agent i if agents i wins the contest. The matrix of externalities is common knowledge among players. Players then simultaneously choose to inform a set of uninformed agents. All informed agents enter the contest, the winner is selected with uniform probability, and the payoff of a player is given by the average externality she receives from all informed agents. Some aspects of the specification in the model are key to our analysis: the matrix of externalities is exogenous to communication behaviors, payoffs are additively separable in the received externalities, and there is a single communication round – we do not consider communication made by those informed by players.[4]

We present our results in several stages. We first provide some general properties of the game, available for any externality matrix, and then turn to more specified cases in line with our applications. Regarding general results, we study the Nash equilibria of the communication game and analyze the impact of a change in externalities. In this model, players inform agents providing highest externalities, until a threshold

4. Beyond information transmission, our model also covers technology transfer contexts. To give a flavor, consider a primitive society in which people share what they get from from hunting. Then, transferring the technology necessary to hunt efficiently to social contacts can be valuable. Many ethnographic studies document how skills are transmitted within traditional societies. For example, studies of Indigenous communities often describe how hunting techniques, tool-making skills, and other practical knowledge are passed down through generations. The works of anthropologists like Claude Lévi-Strauss and Margaret Mead include observations of such processes.

which is given by the average externality obtained from all informed agents including those informed by players. Our first result pertains to equilibrium existence. While the model does not exhibit strategic complementarity, we show that a player's best-response can only be increased (in group inclusion sense) when others communicate more. The reason is that, if player i prefers not informing an agent k, the fact that agent k gets recommended by player j can only lower agent i's communication threshold, which fosters incentives. We derive from this fundamental monotonicity property that the game admits a minimum equilibrium and a maximum equilibrium in terms of the set of informed agents, a standard property of games with strategic complementarities. Hence, equilibria are partially ordered. Moreover, the minimum equilibrium Pareto-dominates all other equilibria over the set of players – symmetrically, the maximum equilibrium is Pareto-dominated by all other equilibria over the set of players. This sharp result holds for any set of players, and any externality matrix.

Turning to comparative statics, we examine whether larger externalities systematically enhances communication. Actually it may not because larger externalities also means higher communication thresholds. However, we identify conditions on the inflation of externalities which can only foster communication for any set of players. This necessarily happens when externalities are subject to an increasing and concave transformation. In research grant context for instance, when the externality is an increasing function of the money transferred from the grant holder to a recipient, a concave increase arises under decreasing returns to money transfer. In such situations, increasing externalities enhances communication unambiguously from any equilibrium. Moreover, we obtain such a positive result without concavity requirement when externality matrices are bilaterally symmetric and row-stochastic, which can correspond to the situation where the winner of the contest shares the prize with people in the society.

To better understand how the structure of externalities shapes who informs whom, we examine polar classes of externality matrices. Firstly, we focus on a class we call *common-preference matrices*. In this class, all agents have the same ordinal ranking in preference over other agents in terms of received externalities; for instance, in the context of researchers competing for grants and exchanging information about the existence of the grant, these stylized matrices capture situations in which the global impact of researchers in the community shape externalities;[5] or, in the context of job offers, this means that some social contacts provide more job offers than others (or job offers of higher quality). We show that, for any externality matrix in this class and any set of players, there is a unique equilibrium, and players inform those with highest impact first.

We then introduce networks into the analysis. Indeed, people being generally embedded in a network of social contacts, externalities are generally constrained to be local. Networks bring multiplicity. Preferences regarding the ranking of externalities are then no longer common in general, which leads to break the uniqueness result established for common-preference matrices. We focus on two polar cases. In the first polar case, agents receive the same externalities from all their neighbors (an agent is said to be neighbor of a given agent when the externality is non-null). For instance, this can arise when the use by a partner of a grant holder of a given transfer depends on the impact of that partner. Equilibrium communication strategies are bang-bang: players either don't inform anyone, or they inform all (uninformed) neighbors. This simple equilibrium property allows us to identify an individual index, decreasing in the received externality and increasing in the number of neighbors, that proves key to equilibrium characterization: players who communicate are those with lower index. In the context of research grants, communication emerges from those initially informed researchers who are typically less connected and with less impact. Importantly, and in contrast to the case of common-preferences, researchers with highest impact need not be informed at equilibrium. In the second polar case, each agent generates same externalities to all their neighbors. For instance, a researcher with high impact may apply to grants of larger amounts. In contrast to the first polar case, partial communication at the individual level can emerge, and the number of neighbors is not sufficient to describe incentives in general. Yet, here again, by the network aspect, the agents generating the highest externalities need not be informed at equilibrium.

Finally, our main insights hold when the model is extended to heterogeneous probabilities to win the contest, to communication-related transaction costs, and to a probability to win the contest depending on the number of informed agents.

5. This impact can partly be measured by an objective measure of past performance, like publication records. For instance, the amount of the grant often depends on the researcher's impact.

Related literature. This paper is contributes to distinct strands of literature. First, this paper contributes to the literature in which one agent chooses to tell others about an opportunity. Such communication arises in the literature about informational concerns of inventors in the absence of intellectual property rights (see for instance Anton and Yao (1994) or Baccara and Razin (2007), where the risk of expropriation creates a potential concern about informing a partner about an innovation. While Anton and Yao focus on a bargaining process among one inventor and two firms, Baccara and Razin consider multiple inventors subject to information leakage in the course of a sequential process of team formation. Our model grasps the information transmission aspect from that literature. The rival nature of the opportunity gives rise to hold-up problem in that literature which we don't, rather focusing on the competition issue to exploit the opportunity.

Second, this paper also adds to the literature on information acquisition through peers. Galeotti and Goyal (2010) model information acquisition about a public good through social networks, to explain the empirical observation that individuals acquire information from a small subset of their social contacts. Herskovic and Ramos (2021) model information acquisition from peers in a beauty contest setting, in which agents form connections to acquire information. In both models, there is no strategic communication consideration, because connecting to another agent allows to observe her signal. We contribute to that literature by incorporating strategic communication, focusing rather on contexts in which accessing information requires the consent of the information provider.

Third, this paper also contributes to literatures in which the information on an economic opportunity is private. In that respect, our paper adds to the literature on innovation in industries. Goyal and Joshi (2003) and Goyal and Moraga-González (2006) model the formation of R&D partnerships among rival firms. In their setting, partnerships lead to innovation-processes of the partners. We complement that literature by considering situations in which firms may find profitable to include other firms into a race to innovation without merging R&D effort. In the same spirit, our paper also adds to the literature on job search through social contacts. Founding their study on the well-known fact that a huge proportion of job offers are transmitted by social contacts, Calvo-Armengol and Jackson (2004), Calvo-Armengol and Jackson (2007) explore unemployment dynamics when social contacts transmit job offers. While, in these models, information transmission is non-strategic, we provide a rationale for strategic information transmission.

Fourth, this paper echoes the literature addressing strategic communication through social contacts. In that literature, the need for communication comes from seeking to influence others' actions under differentiated individual preferences and, in some contexts, coordination issues. Recent extensions to networks include Hagenbach and Koessler (2010), Galeotti et al. (2013), Calvó-Armengol et al. (2015). The two former focus on costless, non-verifiable information (cheap talk model as in Crawford and Sobel (1982)), whereas the latter models the endogenous acquisition of a communication technology under costly and verifiable information. The main focus of that literature is on organizational economics (for decentralized decisions making within organizations, see Dessein and Santos (2006), Alonso et al. (2008), or Rantakari (2008)); or on political economy (See Dewan and Myatt (2008) for a study related to political parties). Focusing rather on social networks, Bloch et al. (2018) examine the strategic spread of rumors in a model in which agents can decide whether to pass on the received information, and find that, when agents, say partisans, diffuse false information, other agents can block messages coming from parts of the network with many partisans.[6] Our main contribution to that literature is to propose a new rationale for strategic communication, by identifying incentives to communicate about the existence of a rival opportunity in presence of externalities.

Last, there is a literature on strategic experimentation and social learning (Keller et al. (2005)). Heidhues et al. (2015) introduce privacy of payoffs, and agents can communicate via cheap-talk messages. Marlats and Ménager (2021) introduce strategic costly observation of actions and outcomes. In contrast to that literature, we suppose that the value of the opportunity is known with certainty.

The paper is organized as follows. The communication game is exposed in Section 2, the characterization of the equilibria of he communication game, as well as the comparative statics, are presented in Section 3. Results under polar externality matrices are presented in Section 4. Section 5 presents a series of extensions, and Section 6 concludes. All proofs are relegated in Appendix A, Appendix B studies equilibria on the class of multi-level externality matrices.

6. Merlino et al. (2023) introduce incentives to verify information status when false information spreads in networks.

2. The communication game

In this section, we present the model, discuss its possible interpretations, and provide a few possible applications.

A society is given by a set of agents $\mathcal{N} = \{1, 2, \cdots, n\}$. Agents compete for an *opportunity*. An agent is initially either aware of the existence of the opportunity, or not. Hence, the society is partitioned as follows: $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$ where $\mathcal{I}$, of cardinal I, is the set of agents informed before the communication stage (called *players*), and $\mathcal{J}$ is the set of agents who are not informed before communication stage (called *regular* agents). Thereafter we will speak about information transmission, but another interpretation is that the informed agents are those who have the adequate technology to exploit the opportunity. Informed agents compete for the opportunity (they don't incur a cost to compete), and the single winner is selected from a stochastic rule. Each competitor wins with a uniform probability in the paper; the model is straightforwardly extended to heterogeneous probabilities of winning (see Section 5 thereafter).

The exploitation of the opportunity by the winner generates externalities, which are represented by an n-square matrix $\mathbf{E} = (e_{ij})_{i,j \in \mathcal{N}}$, where entry $e_{ij} \in \mathbb{R}$ is the utility of agent j when agent i wins the contest. We refer to this matrix as the *externality matrix*. At this level of abstraction, externalities can be either positive or negative. In some specific contexts, it may be legitimate to focus on non-negative entries, or even row-stochastic matrices $\mathbf{E}$. In particular, diagonal entries need not be positive to rationalize entry in the contest and communication.

Given a externality matrix $\mathbf{E}$ as well as a subset of initially informed agents $\mathcal{I}$, we define a normal-form game $(\mathcal{I}; (\mathcal{S}_i)_{i \in \mathcal{I}}; (\pi_i)_{i \in \mathcal{I}})$ as follows: agent i chooses a set $\mathbf{S}_i \in \mathcal{S}_i := \mathcal{P}(\mathcal{J})$ of regular agents to inform. Let $\mathbf{S} := (\mathbf{S}_i)_{i \in \mathcal{I}}$ be an action profile. For simplicity, we also denote by $\mathbf{S}$ the set $\underset{i \in \mathcal{I}}{\cup} \mathbf{S}_i$, i.e. the set of agents who have been informed of the opportunity through communication. Adding communication costs or rewards does not significantly alter the analysis (see Section 5). We let $\mathcal{M}(\mathbf{S}) := \mathcal{I} \cup \mathbf{S}$ and $m(\mathbf{S}) := |\mathcal{M}(\mathbf{S})|$. Let $\mathbf{S}_{-i} := (\mathbf{S}_j)_{j \neq i}$ be the profile of actions of all players, except for i.[7] Then, player i's payoff is given by

$$\pi_i(\mathbf{S}_i, \mathbf{S}_{-i}) = \frac{1}{m(\mathbf{S})} \sum_{k \in \mathcal{M}(\mathbf{S})} e_{ki}$$

The quantity $\frac{1}{m(\mathbf{S})} \sum_{k \in \mathcal{M}(\mathbf{S})} e_{ki}$ is the expected externality that player i obtains over all informed agents, including herself, after the communication phase.

Note that, in this model, the matrix of externalities is exogenous to communication behaviors, payoffs are additively separable in the received externalities, and there is a single communication round.

3. General results

In this section, we study the existence of equilibria of the communication game, discuss uniqueness, then we address comparative statics, and finally we examine welfare implications.

3.1. Equilibria

Best-responses. We analyze the best-responses of the communication game. A best-response $\mathbf{S}_i$ to a profile of actions of other players $\mathbf{S}_{-i}$ is such that $\pi_i(\mathbf{S}_i, \mathbf{S}_{-i}) \geq \pi_i(\mathbf{S}'_i, \mathbf{S}_{-i})$ for all $\mathbf{S}'_i$. Given that the expected payoff is the expected externality among informed agents, it is profitable for agent i to inform a regular agent j whenever e_{ji} exceeds the average externality obtained among already informed agents: given $i \in \mathcal{I}$ and $\mathbf{S}_{-i} \in \mathcal{S}_{-i}$, $\mathbf{S}_i$ is a best-response against $\mathbf{S}_{-i}$ iff[8]

$$e_{ji} \geq \frac{1}{m(\mathbf{S})} \sum_{l \in \mathcal{M}(\mathbf{S})} e_{li} \geq e_{ki}, \;\; \forall j \in \mathbf{S}_i \setminus \mathbf{S}_{-i}, \;\; \forall k \in \mathcal{J} \setminus \mathbf{S}.$$

If j is not informed by other players, but is informed by player i, then it must be the case that the externality e_{ji} is larger than player i's payoff. The current payoff is a

7. From the point of view of player i, all that matters in this game is the set of agents to which other players transmitted their knowledge. We characterize equilibria in terms of their set of informed agents. However, there can be many equilibrium strategies generating a given set of informed agents (through appropriate permutations on the label of the informer of a given informed agent). We disregard those permutations in the paper.

8. For convenience $\mathbf{S}_{-i}$ also denotes the set $\bigcup_{j \in \mathcal{I}, j \neq i} \mathbf{S}_j$.

threshold above which externalities entail profitable communication. This threshold is endogenous to the agents' communication strategies.

The set of best-responses is never empty. However it is typically not a singleton, because if one player informs an agent then any other player is indifferent between informing this agent or not. As a consequence, the set $Br_i(\mathbf{S}_{-i})$ is stable by intersection: if $\mathbf{S}_i$ and $\mathbf{S}'_i$ both belong to $Br_i(\mathbf{S}_{-i})$ then the same holds for $\mathbf{S}_i \cap \mathbf{S}'_i$. Moreover, there is indifference for player i between informing agent j or not when e_{ji} is equal to agent i's payoff. These observations motivate the following refinement:

DEFINITION 1 (*Tight best-response*). We say that $\mathbf{S}_i \in Br_i(\mathbf{S}_{-i})$ is a *tight best-response* against $\mathbf{S}_{-i}$ if, for any $\mathbf{T}_i \subseteq \mathbf{S}_i$ such that $\mathbf{T}_i \neq \mathbf{S}_i$, we have

$$\pi_i(\mathbf{T}_i, \mathbf{S}_{-i}) < \pi_i(\mathbf{S}_i, \mathbf{S}_{-i})$$

Tightness refinement is devoted to the treatment of indifference between communication strategies, by forcing the minimal communication. In particular, when two or more (uninformed) agents deliver each an identical externality level that happens to be equal to the player's current payoff, not informing those agents is the tight strategy. In short, a best-response is tight if none of the current communications of an agent to a set of neighbors can be cut without strictly penalizing the agent's payoff. Note that, if the empty set is a best-response, it is tight by definition. Moreover, since the best-response set is stable by intersection, the tight best-response is the intersection of all best-responses and is therefore unique. We denote this set $\mathrm{TBR}_i(\mathbf{S}_{-i})$. Formally, let $\mathcal{J} \setminus \mathbf{S}_{-i} = \{j_1, j_2, ...j_L\}$ be such that $e_{j_1,i} \geq ... \geq e_{j_L,i}$. Then $\mathrm{TBR}_i(\mathbf{S}_{-i}) = \{j_1, ...j_l\}$ iff[9]

$$e_{j_l i} > Mean\{e_{ji} : j \in \{j_1, ..., j_l\} \cup \mathbf{S}_{-i} \cup \mathcal{I}\} \geq e_{j_{l+1},i}$$

The proof is in Section A. Player i's tight best-response is easily identified: player i ranks the externalities obtained from all uninformed agents in the society. Then, she examines the profitability of informing the agent with the highest externality in that pool, say agent 1. If informing this agent is not strictly profitable, the empty set is the tight best-response. Otherwise, agent i should inform agent 1. Then, agent i examines the possibility of informing the agent with the second largest externality in the pool,

9. If R is a set of real numbers, $Mean\,R$ denotes the average value of this set.

say agent 2. If this is not profitable (that is, if the incoming externality from agent 2 is not strictly larger than agent i's payoff including communication with agent 1), the tight best-response consists in informing agent 1. Otherwise, agent i should inform agent 2. Etc. The process involves no more than $n-1$ steps. To sum up, at every stage of this process, agent i's payoff is strictly increasing. When the process stops, all externalities obtained from informing agents, and only these externalities, exceed player i's payoff at the tight best-response. The tight best-response map

$$\text{TBR} : \mathcal{J}^n \to \mathcal{J}^n, \ \text{TBR}(\mathbf{S}) = (\text{TBR}_1(\mathbf{S}_{-1}), ..., \text{TBR}_n(\mathbf{S}_{-n}))\,.$$

is well defined and one-to-one. A tight Nash equilibrium (or TNE) is a fixed point of the tight best-response map.

Equilibria. We start by illustrating how strategic communication can emerge and lead to multiple equilibria. This simple example is at the hart of the general analysis that follows.

EXAMPLE 1. Consider Figure 1. The configuration in which both players inform nobody is a TNE. There is another TNE, where agent 4 is informed by player 2, while player 3 is informed by player 1. Multiplicity comes from increased return to communication when the other player decides to inform. Note finally that there is a minimum equilibrium and a maximum equilibrium in terms of informed agents, and that every player is better off in the smallest equilibrium compared to the larger equilibrium. ■

Incentives to communicate depend on both the externality that the informer can get from the information receiver and the communication threshold which is the average externality got from informed agents. In that respect, we provide a simple characterization of any equilibrium in communication:

PROPOSITION 1. *Consider any externality matrix $\mathbf{E}$ and any set of players $\mathcal{I}$. Every equilibrium $\mathbf{S}^*$ is such that:*

$$j \in \mathbf{S}_i^* \Leftrightarrow e_{ji} > \frac{1}{m(\mathbf{S}^*)} \sum_{k \in \mathcal{M}(\mathbf{S}^*)} e_{ki} = \pi_i(\mathbf{S}^*)$$

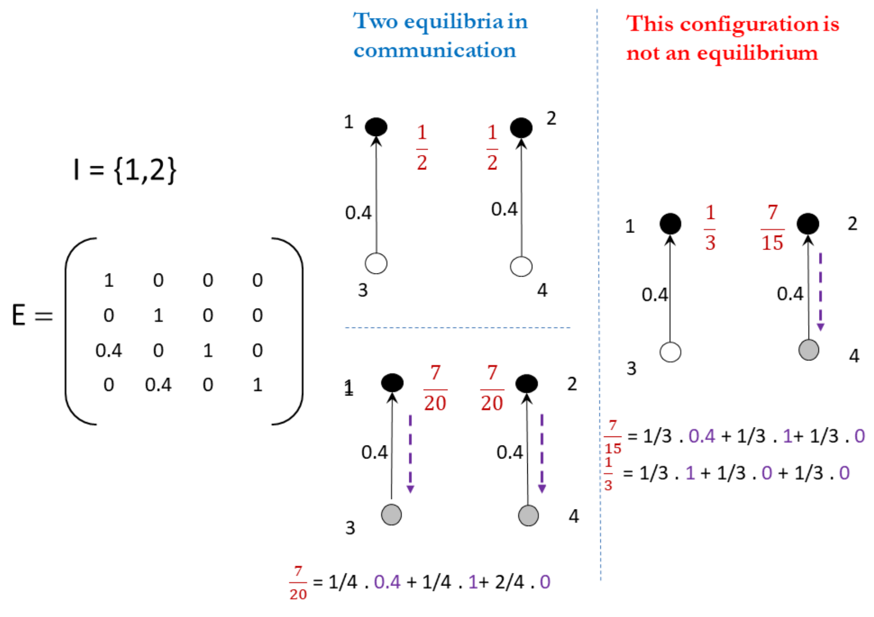

FIGURE 1. Multiple equilibria with $n = 4$. Black agents are players, grey agents are informed by players. Dotted arrows represent information flux, dash arrows represent the direction of externalities. The numbers near the arrows indicates the values of the externalities, the numbers in black near nodes represent agents' labels, the numbers in red near nodes represent players' payoffs.

That is, at equilibrium, information receivers, and only them, generate a larger externality to the informer than the informer's equilibrium payoff.[10] This characterization indicates that incentives to inform are shaped by two factors: (i) the receiver of the information should provide a sufficiently high externality level to the informer, and (ii) the informer's payoff is sufficiently low.

As said earlier, incentives to communicate are higher when the externality obtained from communicating is larger and when the average externality from informed agents is lower. Hence, it is possible to have an equilibrium in which an agent generating a high externality level is not informed, while an agent generating a lower externality is. To illustrate, consider the following example.

10. Note that the condition $e_{ji} > \pi_i(\mathbf{S}^*)$ is equivalent to the condition $e_{ji} > \pi_i(\mathbf{S}^* \setminus \{j\})$, meaning that condition on incentives, which says that benefit from communication exceeds the before-communication payoff, can also be expressed in terms of the after-communication payoff.

EXAMPLE 2. Consider the externality matrix

$$\mathbf{E} = \begin{bmatrix} e_0 & 0 & 0 & 0 & 0 \\ e_1 & e_0 & 0 & 0 & 0 \\ e_1 & 0 & e_0 & 0 & 0 \\ 0 & e_2 & 0 & e_0 & 0 \\ e_1 & 0 & 0 & 0 & e_0 \end{bmatrix}$$

with $\frac{e_0}{3} < e_2 < e_1 < \frac{e_0}{2}$ and assume $\mathcal{I} = \{1, 2, 3\}$. Here, the communication strategy profile $\mathbf{S} = \{\emptyset, \{4\}, \emptyset\}$, in which player 2 informs agent 4 while neither player 1 nor player 3 communicate, is an equilibrium. This is because player 1's communication threshold $\left(\frac{e_0+2e_1}{4}\right)$ is larger than player 2's threshold $\left(\frac{e_0}{3}\right)$. In this case, incentives to communicate are not aligned with the ranking of externalities ($e_1 > e_2$). ■

We turn to the existence of an equilibrium in communication. To show existence, a key property of tight best-responses is that, for any player i, TBR$_i$ is increasing in the following sense:[11]

LEMMA 1 (Monotonicity). *For any player i and any $\mathbf{S}_{-i}, \mathbf{S}'_{-i}$ such that $\mathbf{S}_{-i} \subseteq \mathbf{S}'_{-i}$, we have TBR$_i(\mathbf{S}_{-i}) \subseteq$ TBR$_i(\mathbf{S}'_{-i}) \cup \mathbf{S}'_{-i}$.*

Example 2 illustrates this monotonicity property: if player 2 finds it best to inform agent 3 when player 1 does not inform agent 5, she still prefers to inform agent 3, when agent 5 is informed by player 1. The reason why Lemma 1 holds is that, at the tight best-response, the arrival of a new informed agent does not increase the current payoff of the player. Indeed, the very fact that the new informed agent was not informed by player i means that her externality is lower than the average externality that player i experiences from other informed agents; and thus informing that agent can only lower player i's payoff. One important consequence of Lemma 1 is the existence of a minimum and a maximum TNE.

11. Note that simultaneous best-responses TBR := (TBR$_1$, ..., TBR$_i$) may not be increasing: we might have $\mathbf{S}_i \subseteq \mathbf{S}'_i$ $\forall i$, but TBR($\mathbf{S}$) $\not\subseteq$ TBR($\mathbf{S}'$).

THEOREM 1. *There exist two tight Nash equilibria* $\underline{\mathbf{S}}^*, \overline{\mathbf{S}}^*$ *with the respective properties: for any TNE* $\mathbf{S}^*$*, we have* $\underline{\mathbf{S}}^* \subseteq \mathbf{S}^* \subseteq \overline{\mathbf{S}}^*$*. We call* $\underline{\mathbf{S}}^*$ *the minimum TNE and we call* $\overline{\mathbf{S}}^*$ *the maximum TNE.*[12]

The proof takes care that communication strategies are discrete and that the monotonicity property only holds over tight best-responses. We introduce a sequential best-response map, and show that, starting from the empty strategy set, the iteration of the map converges to a minimum TNE, $\underline{\mathbf{S}}$ (for the maximum TNE, we use a similar argument, with different initial conditions). This result echoes supermodular games, through the monotonicity property of tight best-responses, although the game is not supermodular, because the payoffs are not supermodular on the partially ordered spaces of actions.[13]

Having shown the existence of a minimum TNE has a major welfare implication in terms of Pareto-dominance:[14]

PROPOSITION 2. *The minimum TNE strictly Pareto-dominates all other TNEs (over the set of players).*

Proposition 2 follows from a simple observation: by construction of best-responses, for any equilibrium with a set of informed agents larger than $\underline{\mathbf{S}}^*$, the expected externality from those informed agents in the larger TNE who are not in set $\underline{\mathbf{S}}^*$ is lower than the expected externality got from agents in set $\underline{\mathbf{S}}^*$. Note that

12. Formally, $\underline{\mathbf{S}}^*, \overline{\mathbf{S}}^*$ are not unique in terms of action profile. They are unique in terms of set of informed agents.

13. Indeed, this is not a game with strategic complements. For example, assume $n = 4, \mathcal{I} = \{1, 2\}$, and

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Then,

$$\pi_1(\{3\}, \emptyset) - \pi_1(\emptyset, \emptyset) = \frac{3}{3} - \frac{1}{2} = \frac{1}{2} > \pi_1(\{3\}, \{4\}) - \pi_1(\emptyset, \{4\}) = \frac{4}{4} - \frac{2}{3} = \frac{1}{3}$$

That is, for player 1, it is less valuable to inform agent 3 when agent 4 is informed by player 2 than when agent 4 is not informed.

14. A communication profile $\mathbf{S}'$ Pareto dominates $\mathbf{S}$ if $\pi_i(\mathbf{S}') \geq \pi_i(\mathbf{S})$, for all $i \in \mathcal{I}$.

Pareto-dominance applies here to players only, and that regular agents can be better off in larger equilibria.

3.2. Comparative statics

We now investigate comparative statics over the externality matrices. A preliminary observation is that, for a given set of players, increasing externalities among players can only reduce communication at the minimum and the maximum equilibria because communication thresholds can only be increased. For that same reason, it is clear that increasing the diagonal entries of a externality matrix can only reduce communication. Second, it might seem at first glance that communication at the minimum equilibrium can only increase with the externalities that players can obtain from regular agents. This is not true because increasing externalities also increases the threshold levels triggering communication.

However we can identify some situations in which externality increases cannot be detrimental to communication. Consider a single increase, say e_{ji}, in the externality matrix. Even though agent j becomes more interesting for i, a possible adverse effect is that, if agent j was not informed before the increase and is informed after, player i's best-response communication can actually be reduced, because her communication threshold has increased. Now suppose that this does not happen, i.e. player i's best-response is not reduced after the inflation. A direct implication of the monotonicity property is that no other player would be better off reducing communication after the increase. Let $\Delta^{(i)}$ denote a matrix such that,$\forall k$, $\Delta^{(i)}_{ki} \geq 0$, and $\Delta^{(i)}_{kj} = 0,\ \forall j \neq i$; We get:

LEMMA 2. *If,* $\forall i = 1, \cdots, n$, $\forall \mathbf{S}_{-i}$, $TBR_i(\mathbf{S}_{-i} \mid \mathbf{E}) \subseteq TBR_i(\mathbf{S}_{-i} \mid \mathbf{E} + \Delta^{(i)})$, *then*

$$\underline{\mathbf{S}}(\mathbf{E}) \subseteq \underline{\mathbf{S}}(\mathbf{E} + \Delta^{(i)}).$$

The proof is omitted and directly related to Lemma 1. Said differently, a decrease in communication following an increase in column i can only be driven by a decrease of communication of player i herself.

Which condition on inflation $\Delta^{(i)}$ should be imposed to ensure that player i's optimal communication will increase? Consider the set of agents informed by player i under externality matrix $\mathbf{E}$. It should be that the inflation $\Delta^{(i)}$ puts the communication threshold below the minimum externality generated by agents in this set after the inflation. This condition depends on the communication pattern at the considered equilibrium. A sufficient condition, for any set of players, consists in concave transformations:[15]

THEOREM 2. *Given two externality matrices* $\mathbf{E}$ *and* $\mathbf{E}'$ *such that* $e'_{ij} = \varphi_i(e_{ij})$ *for all* $i \neq j$*; and* $e'_{ii} \leq \varphi_i(e_{ii})$*, for some concave and strictly increasing* φ_i*. Then*

$$\mathit{TBR}_i(\mathbf{S}_{-i} \mid \mathbf{E}) \subseteq \mathit{TBR}_i(\mathbf{S}_{-i} \mid \mathbf{E}'),\ \forall \mathbf{S}_{-i},\ \textit{and}\ \ \underline{\mathbf{S}}^*(\mathbf{E}) \subseteq \underline{\mathbf{S}}^*(\mathbf{E}').$$

The intuition is as follows: First, by concavity of function φ_i, player i's communication threshold, which is given by averages of externalities, can only be lowered when passing from $\mathbf{E}$ to $\mathbf{E}'$. Second, on top of the concavity transformation, another positive effect occurs when diagonal entries are lowered, as said earlier. This effect is favorable to communication because diagonal entries are necessarily counted to compute the average externality received over all informed agents winning the contest; reducing this entry can thus only decrease the communication threshold. In research grant context for instance, when the externality is an increasing function of the money transferred from the grant holder to a recipient, a concave increase arises under decreasing returns to money transfer, meaning that higher grants tend to reduce the amount of externality per transferred dollar.

We provide now another comparative statics result, where we can relax on the concavity aspect of the increase for more specific matrices. To relax on concavity, we need externality matrices to be bilaterally symmetric and row-stochastic (or more generally of same entry summation across lines; That is, $\mathbf{E}^T = \mathbf{E}$ and $\mathbf{E}\mathbf{1} = \kappa\mathbf{1}$ for $\kappa \in \mathbb{R}^*$). For instance, this class of externality matrices can fit with the situation in which the winner of the contest shares the prize with members of the society. We obtain:

PROPOSITION 3. *Consider two symmetric and row-stochastic externality matrices* $\mathbf{E}$ *and* $\mathbf{E}'$ *such that* $e_{ij} \geq e'_{ij}$ *for all* $i \neq j$*. Then* $\mathit{TBR}'_i(\mathbf{S}_{-i}) \subseteq \mathit{TBR}_i(\mathbf{S}_{-i}),\ \forall \mathbf{S}_{-i}$*, and* $\underline{\mathbf{S}}(\mathbf{E}') \subseteq \underline{\mathbf{S}}(\mathbf{E})$.

By symmetry, the sum of externalities at a given equilibrium is equal to the sum of externalities given to others, and the latter is not larger than one minus the agent's

15. Note that the following neutrality result is direct: given two externality matrices $\mathbf{E}$ and $\mathbf{E}'$ such that $\mathbf{E}' = a\mathbf{E} + b\mathbf{J}$ (with $\mathbf{J}$ the matrix of ones), with $a \neq 0$, then $\underline{\mathbf{S}}(\mathbf{E}') = \underline{\mathbf{S}}(\mathbf{E})$.

internality. Combined with row-stochasticity, this means that, considering the set of agents informed by players at any equilibrium under matrix $\mathbf{E}$, the threshold increases under matrix $\mathbf{E}'$. This comparative statics encompasses link addition. For example, if two researchers happen to develop a bilateral social contact in a world where the grant owner shares the price with partners, communication cannot be reduced.

4. Who informs who? Some polar cases

In this section, we investigate some polar class of matrices that help understanding the impact of the structure of externalities. We first consider the case of common ordinal ranking in the externality levels. Then, we introduce networks to bring the idea that agents receive externalities from a subset of the other agents only. Denoting by $\mathbf{G}$ the network conveying externalities, we explore two cases: the case $e_{ij} = g_{ij} \cdot e_j$ where externalities are only related to the recipient's characteristics, and the case $e_{ij} = g_{ij} \cdot e_i$ where externalities are only related to the provider's characteristics. As it will be clear thereafter, whether externalities are related to the characteristics of the provider or the recipient of the externality makes a difference.

Consider for instance research communities involved in project building eligible to research grants, where an externality matrix reflects various transfers occurring from a grant holder to partners. This environment naturally invites to relate the intensity of externalities to the impact of the generator and the recipient of that externality (e.g. through an observable measure of individual past performance), and to integrate the network of colleagues that imposes constraints on the flow of externalities.

4.1. Common preference

In many circumstances, externalities generated are ranked in the same way by externality receivers. For instance, for research grants, more productive researchers might deliver higher externality levels than less productive researchers; for instance, when the amount of the grant depends on the researcher's productivity, the transfers to partners of the grant holder are potentially larger. On the other hand, more productive researchers may also benefit more from a grant obtained by a colleague; for instance, more performing researchers participate to a higher number of costly activities such as workshops, conferences, social events, professional visits, etc.

To capture this idea, we consider the set of externality matrices where $e_{ij} \geq e_{i+1,j}$ for all $i < n, j \neq i$ (in our arbitrary convention, externality levels are decreasing in the indexes of provider labels). Clearly for such matrices all agents have the same ordinal ranking on received externalities: if researcher i provides higher externalities to researcher j than researcher k does, then researcher i provides higher externalities than researcher k does to any other researcher j'. These matrices are called the *common-preference matrices*.

It is straightforward that, in all non-empty equilibria, the informesd agents are selected following successive (decreasing) ranking, starting from the agent originating the highest externality: if agent j is informed by a given player, all agents providing higher externality levels are also informed. This means that all equilibria are nested. Nestedness does not imply uniqueness. Yet:

PROPOSITION 4. *For any common-preference matrix, and any set of players, there is a unique TNE.*

By Proposition 4, as soon as winners generate a common ordinal ranking in externalities, and thus a common ranking of preferences in the society, there is a unique equilibrium. The equilibrium can easily be determined. An equilibrium is characterized by a threshold t^* such that only regular agents of index lower than or equal to t^* are informed. This index satisfies

$$\min_{i \in \mathcal{I}} e_{t^*,i} - \pi_i^* > 0 \text{ and } \max_{i \in \mathcal{I}} e_{t^*+1,i} - \pi_i^* < 0$$

By Proposition 4, the common preference context plays as a powerful equilibrium selection device, forging uniqueness. Furthermore, as intuition would suggest, communication is aligned with the generated externality levels. A main take-away for the common preference case is that players inform those agents providing the highest externalities first.

Equilibrium uniqueness gives a particular interest to a comparative statics over the externality matrix in the world of common preference. By Theorem 2, we know that only concave increases of the externality matrix guarantee fostered communication. By contrast, Figure 2 presents a case in which an increase of externalities reduces communication. In that example, the non-homogeneous increase of externalities highers agent 1's threshold in a way that reduces her communication strategy.

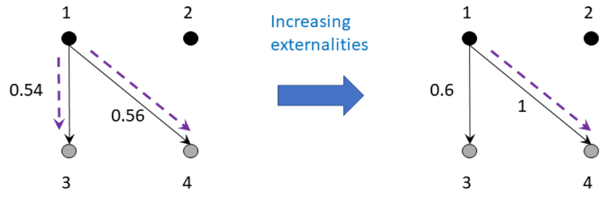

FIGURE 2. Increasing externalities leads to reduced communication. In Left panel, $e_3 = 0.54, e_4 = 0.56$; In Right panel, $e'_3 = 0.6, e'_4 = 1$. In both panels, $e_{ii} = 1$ for $i = 1, 2$.

4.2. *Local externalities*

In real world, people are embedded in a network of professional and social relationships, and externalities only flow through social contacts. In such contexts, externality are local, conveyed by the social network.[16] Then, preferences regarding the ranking of externalities are no longer common in general.

We can observe from the simple example presented in Figure 1 that networks break equilibrium uniqueness. To understand deeper the impact of network structure of communication, we consider two polar cases; either recipients receive the same externalities from all externality providers they are linked with, or an externality provider generates same externalities to all neighbors.

Receiving same externalities from neighbors. This case is likely to arise when the externality levels are related to a characteristics of the externality receiver; for instance, the use by a partner of a grant holder of a given transfer might depend on the impact of that partner. We call *inward-local* the matrices such that $e_{ij} = g_{ij} \cdot e_j$ for

16. In some contexts, the network could be endogenous to externality levels. This issue is not considered in this paper.

all $i \neq j$, $g_{ij} \in \{0,1\}$. Network $\mathbf{G} = (g_{ij})$ captures the network structure conveying the externalities, with the convention that $g_{ii} = 1$. Let $\mathcal{N}_i := \{j \neq i : g_{ji} = 1\}$ be the set of neighbors of i, and $d_i := |\mathcal{N}_i|$. Externality matrices are then shaped by three inputs: the vector $(e_{ii})_{i \leq n}$, the vector $(e_i)_{i \leq n}$, and the network $\mathbf{G}$. Since a player can only receive the same externality level from neighbors, it is immediate that a profitable communication to one neighbor induces a profitable communication to all neighbors. That is, partial communication is never individually optimal. However, this is not incompatible with partial communication at the society level, when only a subset of players communicates. In the extreme case where the network is complete, there is a unique equilibrium, in which a player i informs all neighbors if $e_i > e_{ii}$, and a player i does not inform at all if $e_i \leq e_{ii}$. Now for general network structure, it can be that only a subset of players communicate, and there can be equilibrium multiplicity.

In order to explore equilibria further, we need to introduce few notation. Define, for any player $i \in \mathcal{I}$, the individual index

$$\eta_i := \frac{e_{ii}}{e_i} + d_i$$

This simple index, which increases with degree and decreases with received externality, embodies the two-dimensional aspect of incentives. Note that a larger degree and/or a lower received externality are both detrimental to communication and tend to increase the index. Assume without loss of generality that $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_I$ and, given $i_0 \in \{0, 1, ..., I\}$ define

$$M(\eta_{i_0}) := \left(\bigcup_{i \leq i_0} \mathcal{N}_i \right) \cup \mathcal{I}, \quad \text{and} \quad \xi(\eta_{i_0}) := \min_{i > i_0} \left\{ \frac{e_{ii}}{e_i} + |\mathcal{N}_i \cap M(i_0)| \right\},$$

with the convention that $\eta_0 = 0$ and $\xi(n) = +\infty$. Note that $M(0) = \mathcal{I}$ and $\xi(0) = \min_{i \in \mathcal{I}} \left\{ \frac{e_{ii}}{e_i} + |\mathcal{N}_i \cap \mathcal{I}| \right\}$. The next proposition provides a detailed description of equilibria in this class of externality matrices:

PROPOSITION 5. *Consider an inward-local communication game. To any TNE $\mathbf{S}^*$, we can associate a threshold $\eta^* \in \{\eta_1, ..., \eta_{\mathcal{I}}\}$ such that*

$$\mathbf{S}^*_i = \mathcal{N}_i \setminus (\mathcal{I} \cup \mathbf{S}^*_{-i}) \ \forall i: \ \eta_i \leq \eta^*, \textit{ while } \ \mathbf{S}^*_i = \emptyset \ \forall i: \ \eta_i > \eta^*.$$

Furthermore a profile such that only agents i with index $\eta_i \leq \eta^*$ communicate is a TNE as soon as $\eta^* < |M(\eta^*)| \leq \xi(\eta^*)$.[17] *All in all, equilibria are nested and communication is driven by low-index players.*

Proposition 5 formally expresses that any TNE is associated to an index η^* with the property that (i) players of index lower than η^* inform all neighbors and (ii) players of index strictly larger than η^* do not inform anyone. Proposition 5 is useful to understand equilibria. Note first that all equilibria are nested. Then the main message is that communication is driven by low-index players. This index embodies the two factors, received externality and threshold, shaping incentives to communicate. In particular, the lower the degree and/or the higher the received externality, the lower the index. More precisely, we can check whether an index η^* corresponds to an equilibrium by applying the following procedure. First, compute the value $|M(\eta^*)|$ which is the number of informed agents once all players of index not larger than η^* inform their neighbors, and then check that $\eta^* < |M(\eta^*)|$; Second, among all players of index η strictly larger than η^*, identify the one with smallest index ξ given that all players of index η^* or less communicate, and check whether that value, $\xi(\eta^*)$, satisfies $|M(\eta^*)| \leq \xi(\eta^*)$. Therefore, the main take-away from this case is is that communication is bang-bang (partial communication at individual level never emerges), and the network aspect is captured by degree centrality in the analysis. Hence, in contrast to the common-preference case, informed agents are not necessarily the ones providing the highest externality.

An interesting subclass is the set of externality matrices in which all neighbors generate the same externality level. Consider the subclass such that $e_{ii} = e_0$ for all i, and $e_{ij} = g_{ij} \cdot e$ for all $i, j \neq i$. This class generates a game of parameters represented by the triplet $(e_0, e, \mathbf{G})$. In this context, the ratio $\frac{e_{ii}}{e_i} = \frac{e_0}{e}$ being identical for all players, the index η_i is only differentiated through degrees. Hence, by Proposition 5, for any externality matrix represented by the triplet $(e_0, e, \mathbf{G})$, at any equilibrium communication is triggered by agents with smaller degrees.

Generating same externalities to neighbors. This case is likely to arise when the externality levels are related to a characteristics of the externality provider; for

17. The no-communication strategy profile $\mathbf{S} = \emptyset$ is a TNE as soon as $I \leq \xi(0)$. Also a full communication profile $\mathbf{S} = \mathcal{J}$ is a TNE if $\eta_I < n$.

instance, a researcher with high impact may apply to grants of larger amounts. This situation is captured by the stylized class of matrices, where $e_{ij} = g_{ij} \cdot e_i$. We focus on a world with only two productivity levels for the sake of simplicity, meaning that we examine two-level-local matrices. The analysis is straightforwardly generalized to multi-level-local matrices (see Appendix B). Consider indeed externality matrices such that $e_{ii} = e_0$ for all i, and $e_i \in \{g_{ij} \cdot \underline{e}, g_{ij} \cdot \overline{e}\}$ for all $i, j \neq i$, and with $g_{ij} \in \{0, 1\}$; this class generates a game of parameters represented by the quadruplet $(e_0, \underline{e}, \overline{e}, \mathbf{G})$, and embodies the requirement that $e_{ij} = e_i$ for all i, j.

It is easily seen that there are two kinds of players at equilibrium. We associate an index to each player, that is increasing in both the degree and the proportion of high-externality neighbors. Then we can partition adequately those players with low index (i.e., with low degree and preferably linked to low-externality neighbors), and those players with high index (i.e., with higher degree and more linked to high-externality neighbors). Players with a low index inform all neighbors, while players with high index inform high-externality neighbors only if low-externality neighbors are preferably linked to high-index players.

To see this formally, consider an equilibrium with m^* informed agents. A basic observation is that, at equilibrium, if a player i informs a low-externality neighbor, then player i informs all uninformed neighbors; and if a player informs one high-externality neighbor, she informs all high-externality neighbors. Defining by β_i the proportion of player i's high-externality neighbors, the incentives to inform a low-externality neighbor, for player i, are then given by the condition

$$m > \mu_i = \frac{e_0}{\underline{e}} + \Big(\beta_i \frac{\overline{e}}{\underline{e}} + (1 - \beta_i)\Big) d_i$$

Index μ_i only depends on primitives of the model. For those players of index lower than m^*, they inform all neighbors.[18] Consider now those players whose index is large enough to violate the above condition. They still find profitable to inform a high-externality neighbor under the condition

$$m > \nu_i(m) = \frac{e_0}{\overline{e}} + \beta_i d_i + \frac{\underline{e}}{\overline{e}} \sum_{k \in \mathcal{N}_i \cap \mathbf{S}_{-i}} g_{ik} \cdot I\left[\min_{p \in \mathcal{N}_k \cap \mathcal{I}} \mu_p < m\right]$$

Hence, informing a high-externality neighbor is more likely to be valuable when the low-externality neighbors of player i are connected to a smaller set of low-index

18. Then, for two arbitrary equilibria, the respective subsets of players informing all uninformed neighbors in each equilibrium are nested.

players. Note that, in opposite to the former condition, the RHS depends now on m. A simple algorithm allows to find the exact set of informers:

PROPOSITION 6. *Consider a two-level-local matrix. A given configuration with m informed agents is an equilibrium if and only if it satisfies the following algorithmic procedure:*

(i) *All players i of index $\mu_i < m$ inform all uninformed neighbors;*
(ii) *All players i such that $\mu_i \geq m$ and $\nu_i(m) < m$, inform all high-externality neighbors and only them;*
(iii) *All players i such that $\mu_i \geq m$ and $\nu_i(m) \geq m$ don't inform anyone.*

Importantly, whereas the index μ does not depend on players' strategies, the index $\nu_i(m)$ does. This endogeneity of index $\nu_i(m)$ stands in contrast with Proposition 5 in the preceding case study, and makes hardly achievable to identify informed agents from primitives of the model only. That is, Proposition 6 allows a check whether a given communication profile is an equilibrium. This proposition would be usefully complemented by algorithm computation in order to find all equilibria.

Hence, the main take-away from this case study is is that partial communication can emerge, and degree centrality is not sufficient to describe incentives: distance-two neighbors matter in this binary externality case.[19] Here again, the agents with highest impact need not be informed at equilibrium.

5. Extensions

This section examines three extensions. We envisage, in the order, heterogeneous probabilities to win the contest, communication-related transaction cost, and a probability to win the contest that depends on the number of informed agents.

Heterogeneous probabilities to win the contest. Lemma 1 extends to heterogeneous probabilities to win the contest. Assume that agents have an individual technology to compete, captured by the vector of characteristics $\theta = (\theta_i)_{i \in \mathcal{N}}$. Then, assume that the probability for agent i to win the contest is given by the ratio $\frac{\theta_i}{\sum_{k \in \mathcal{M}} \theta_k}$.

19. By extending this analysis to k levels of externalities, distance-$2k$ neighbors would matter.

In that case, player i wishes to inform agent j under strategy profile $(\mathbf{S})$ whenever

$$e_{ji} > \frac{\sum\limits_{k\in\mathcal{M}(\mathbf{S})} \theta_k e_{ki}}{\sum\limits_{k\in\mathcal{M}(\mathbf{S})} \theta_k}$$

Therefore, best-responses have the same structure as in the benchmark case. In particular, to see that the monotonicity property holds, it is easily checked that player i's threshold can only be decreased when agent j is sponsored by a third party.[20]

Communication-related transaction cost. In general, communication can generate transaction costs depending on the context. We represent such friction through matrix $\mathbf{F} = (f_{ij})$, with $f_{ij} \in \mathbb{R}$; for instance, $f_{ij} < 0$ can represent a cost to player i induced from informing agent j or for transmitting the technology necessary to compete in the contest. Alternatively, $f_{ij} > 0$ could represent a reward, like warm-glow effect (i.e. the satisfaction of sponsoring), or a fixed price, etc, that is auxiliary to the communication from player i to agent j. It is straightforward to see that the monotonicity property holds for any matrix $\mathbf{F}$, irrespective of its sign. Indeed, because f_{ij} does not affect player i's threshold[21], if player i finds it profitable to inform an agent, further communication can only lower the threshold.

A probability to win the contest that depends on the number of informed agents. Assume that the expected externality when there are m informed agents is of the form $\frac{1}{f(m)} \cdot e$, with function f positive, increasing and concave; in the benchmark, function f is the identity function. It can be checked that monotonicity property given in Lemma 1 holds.[22]

20. In the model, the probability to win the contest is exogenous. Rather introducing individual costly efforts to win the contest brings a new motive shaping incentives to inform. This new motive is related to the interaction among the efforts of the participants of the contest: agents may want to communicate in the purpose of influencing - typically decreasing - the efforts of competitors. This explains the possible failure of the monotonicity property.

21. Every equilibrium $\mathbf{S}^*$ is such that: $j \in \mathbf{S}_i^* \Leftrightarrow e_{ji} + f_{ij} > \frac{1}{m(\mathbf{S}^*)} \sum_{k\in\mathcal{M}(\mathbf{S}^*)} e_{ki}$.

22. We need to check that, if agent i finds it profitable to inform agent k but not agent l, then if agent l is informed by a third party, agent i still finds it profitable to inform agent k. Formally, the first statement is written $\frac{e_{ki}}{f(m+1)} + \sum\limits_{j\in\mathcal{M}} \frac{e_{ji}}{f(m+1)} \geq \sum\limits_{j\in\mathcal{M}} \frac{e_{ji}}{f(m)} > \frac{e_{li}}{f(m+1)} + \sum\limits_{j\in\mathcal{M}} \frac{e_{ji}}{f(m+1)}$. Equivalently, denoting $H(m) = \frac{f(m+1)-f(m)}{f(m)}$, this means $e_{ki} \geq H(m) \sum e_{ji} > e_{li}$. We then show that this implies $e_{ki} \geq$

Players not participating to the contest. In the interpretation of the model in which sponsoring means giving access to a contest, we assumed that players participate to the contest. However, in some situations, like the information transmission of a job offer through social contact, the informer is not willing to apply for the job. The model fully accommodates such a possibility, where a subset of players do not compete for the contest, and our results extend directly. Technically, payoffs do not incorporate the diagonal entry of the externality matrix. Neither monotonicity property nor the comparative statics are unaffected by such a modification (but the discussion about efficiency can of course be affected).

6. Concluding remarks

In this paper, we investigated the private incentives to share information about the existence of competitive opportunities, within a context where an agent's exploitation of such an opportunity generates externalities for others. We found that in this environment, private incentives for communication are amplified when others also engage in sharing information, leading to the emergence of a minimal communication equilibrium that Pareto-dominates all other equilibria among the subgroup of initially informed agents. Our analysis further reveals conditions under which larger externalities positively influence communication.

Despite the highly stylized nature of our model, exploring deeper the public policy implications of this communication framework in real-world scenarios remains a compelling avenue for future research. Firstly, policy measures could be designed to influence individual communication incentives by adjusting the externalities involved. For example, in the context of public goods, policymakers could fund enhancements to public service quality; similarly, in the job market, interventions could target wage adjustments to influence job-related information sharing. Secondly, it might be worth exploring policies that strategically expand the circle of agents aware of opportunities, ensuring that this expansion aligns with social welfare goals.

$H(m+1)\sum_{j\in\mathcal{M}\cup\{l\}} e_{ji}$. An immediate sufficient condition is then $H(m) > H(m+1)\big(1+H(m)\big)$. In terms of function f, we get after few development $f(m+1) > \frac{f(m)+f(m+2)}{2}$, which means that function f is concave.

References

Alonso, R., Dessein, W., and Matouschek, N. (2008). When does coordination require centralization? American Economic Review, 98:145–179.

Anton, J. and Yao, D. (1994). Expropriation and inventions: Appropriable rents in the absence of property rights. American Economic Review, 84(1):190–209.

Baccara, M. and Razin, R. (2007). argaining over new ideas: Rent distribution and stability of innovative firms. Journal of the European Economic Association, 5(6):1095–1129.

Binzel, C. and Fehr, D. (2013). Giving and sorting among friends: Evidence from a lab-in-the-field experiment. Economics Letters, 121(2):214–217.

Bloch, F., Demange, G., and Kranton, R. (2018). Rumors and social networks. International Economic Review, 59:421–448.

Calvó-Armengol, A., De Martí, J., and Prat, A. (2015). Communication and influence. Theoretical Economics, 10(2):649–690.

Calvo-Armengol, A. and Jackson, M. (2004). The effects of social networks on employment and inequality. American Economic Review, 94(3):426–454.

Calvo-Armengol, A. and Jackson, M. (2007). Networks in labor markets: wage and employment dynamics and inequality. Journal of Economic Theory, 132(1):27–46.

Crawford, V. P. and Sobel, J. (1982). Strategic information transmission. Econometrica, pages 1431–1451.

Dessein, W. and Santos, T. (2006). Adaptive organizations. Journal of Political Economy, 114:956–995.

Dewan, T. and Myatt, D. (2008). Qualities of leadership: Communication, direction and obfuscation. American Political Science Review, 102(3):351–368.

Engelman, D. and Strobel, M. (2008). Inequality aversion, efficiency, and maximin preferences in simple distribution experiments. American Economic Review, 94(4):857–869.

Galeotti, A., Ghiglino, C., and Squintani, F. (2013). Strategic information transmission networks. Journal of Economic Theory, 148(5):1751–1769.

Galeotti, A. and Goyal, S. (2010). The law of the few. American Economic Review, 100(4):1468–1492.

Goyal, S. and Joshi, S. (2003). Networks of collaboration in oligopoly. Games and Economic Behavior, 43(1):57–85.

Goyal, S. and Moraga-González, J. (2006). R&d networks. RAND Journal of Economics, 32(4):686–707.

Hagedoorn, J. (2002). Inter-firm r&d partnerships: an overview of major trends and patterns since 1960 author links open overlay panel. Research Policy, 31(4):477–492.

Hagedoorn, J. (2006). Inter-firm r & d partnering in pharmaceutical biotechnology since 1975: Trends, patterns, and networks. Research Policy, 35(3):431–446.

Hagenbach, J. and Koessler, F. (2010). Strategic communication networks. The Review of Economic Studies, 77(3):1072–1099.

Heidhues, P., Rady, S., and Strack, P. (2015). Strategic experimentation with private payoffs. Journal of Economic Theory, 159:531–551.

Herskovic, B. and Ramos, J. (2021). Aquiring information through peers. American Economic Review, 110(7):2128–2152.

Keller, G., Rady, S., and Cripps, M. (2005). Strategic experimentation with exponential bandits. Econometrica, 73(1):39–68.

Marlats, C. and Ménager, L. (2021). Strategic observation with exponential bandits. Journal of Economic Theory, 193:105232.

Merlino, L., Pin, P., and Tabasso, N. (2023). Debunking rumors in networks. American Economic Journal: Microeconomics, 15(1):467–496.

Powell, W., White, D., Koput, K., and Owen-Smith, J. (2005). Network dynamics and field evolution: The growth of interorganizational collaboration in the life sciences. American Journal of Sociology, 110(4):1132–1205.

Rantakari, H. (2008). Governing adaptation. Review of Economic Studies, 75:1257–1285.

Appendix A: Appendix A: Proofs

We prove the following statement: Let $\mathcal{J} \setminus \mathbf{S}_{-i} = \{j_1, j_2, ...j_L\}$ be such that $e_{j_1,i} \geq ... \geq e_{j_L,i}$. Then $\text{TBR}_i(\mathbf{S}_{-i}) = \{j_1, ...j_l\}$ iff

$$e_{j_l i} > Mean\,\{e_{ji} : j \in \{j_1, ..., j_l\} \cup \mathbf{S}_{-i} \cup \mathcal{I}\} \geq e_{j_{l+1},i}$$

Proof. Note first that uniqueness directly follows from the fact that $\text{TBR}_i(\mathbf{S}_{-i})$ is the intersection of all elements of $Br_i(\mathbf{S}_{-i})$. Then, note that there cannot be $1 \leq l < l' \leq L$ such that $l' \in \text{TBR}_i(\mathbf{S}_{-i})$ while $l \notin \text{TBR}_i(\mathbf{S}_{-i})$, because deviating to informing

l instead of l' would yield an equal or higher payoff, contradicting the fact that $\text{TBR}_i(\mathbf{S}_{-i})$ is unique. Thus there exists $l^* \geq 0$ such that $\text{TBR}_i(\mathbf{S}_{-i}) = \{j_1, ... j_{l^*}\}$. Let now

$$f(l) := Mean\left\{(e_{ki})_{k\in\mathcal{I}\cup\mathbf{S}_{-i}}, e_{j_1 i}, \cdots, e_{j_l i}\right\}$$

for $l = 0, ..., L$. Note that

$$f(l) \geq f(l+1) \Leftrightarrow f(l) \geq e_{j_{l+1}i} \Rightarrow f(l+1) \geq e_{j_{l+2}i} \Leftrightarrow f(l+1) \geq f(l+2).$$

As a consequence the map $f(\cdot)$ is quasi-concave in the sense that

$$f(l) \geq f(l+1) \Rightarrow f(l+1) \geq f(l+2).$$

Hence l^* is the only integer in $\{0, ..., L\}$ such that $f(l^*-1) < f(l^*)$, and $f(l^*) \geq f(l^*+1)$.[23] This proves the second statement.

□

Proof of Lemma 1. Write $\mathcal{J} \setminus \mathbf{S}'_{-i} = \{j_1, \geq, j_L\}$, where $e_{j_1 i} \geq e_{j_2 i} \geq \cdots \geq e_{j_L i}$. Then $\text{TBR}_i(\mathbf{S}_{-i}) \setminus \mathbf{S}'_{-i} = \{j_1, ..., j_l\}$ for some $l \leq L$. By definition of j_l belonging to the tight best-response to $\mathbf{S}_{-i}$ we must have that $e_{j_l i}$ is strictly greater than $Mean(A)$, where

$$\mathbf{A} := \{e_{ji} : \ j \in \mathbf{S}_{-i} \cup \mathcal{I}\} \cup \{e_{j_1 i}, \cdots, e_{j_l i}\} \cup \{e_{ji} : \ j \in \mathbf{S}'_{-i}, e_{ji} > e_{j_l i}\}$$

We want to prove that j_l belongs to $\text{TBR}_i(\mathbf{S}'_{-i})$. Let $\mathbf{A}' := \{e_{ji} : \ j \in \mathbf{S}'_{-i} \cup \mathcal{I}\} \cup \{e_{j_1 i}, \cdots, e_{j_l i}\}$. Then

$$\mathbf{A}' = \mathbf{A} \cup \{e_{ji} : j \in \mathbf{S}'_{-i}, \ e_{ji} \leq e_{j_l i}\}$$

Hence, since $e_{j_l,i} > Mean(\mathbf{A})$, we necessarily also have that $e_{j_l,i} > Mean(\mathbf{A}')$, because every element in $\mathbf{A}' \setminus \mathbf{A}$ is smaller or equal than $e_{j_l i}$. Thus $j_l \in \text{TBR}_i(\mathbf{S}'_{-i})$.

□

A profile $\mathbf{S}$ is *under-informed* if $\mathbf{S}_i \subseteq \text{TBR}_i(\mathbf{S}_{-i})$ for any $i \in \mathcal{I}$. We call $\mathcal{S}_u$ the set of under-informed profiles. Furthermore, for any $i \in \mathcal{I}$, let $\mathbf{B}_i$ be given by

$$\mathbf{B}_i : (\mathbf{S}_i, \mathbf{S}_{-i}) \mapsto (\text{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}), \quad \text{and } B_i(\mathbf{S}_i, \mathbf{S}_{-i}) = \text{TBR}_i(\mathbf{S}_{-i}) \cup \mathbf{S}_{-i}.$$

23. With the convention that $f(-1) < f(0)$ and $f(L+1) \leq f(L)$

DEFINITION A.1. The *sequential best-response map* is constructed as follows. Let $\mathbf{S} = (\mathbf{S}_i)_{i \in \mathcal{I}}$ be an action profile. Then $\mathbf{B} : \mathcal{S} \to \mathcal{S}$ is defined as[24]

$$\mathbf{B}(\mathbf{S}) := \mathbf{B}_I \circ \mathbf{B}_{I-1} \circ ... \circ \mathbf{B}_1(\mathbf{S})$$

We write $B(\mathbf{S}) = \cup_i (\mathbf{B}(\mathbf{S}))_i$.

Proof of Proposition 1. The proof is immediate, remarking that the condition $e_{ji} > \pi_i(\mathbf{S}^*)$ is equivalent to the condition $e_{ji} > \pi_i(\mathbf{S}^* \setminus \{j\})$; meaning that the incentives condition, which says that benefit from communication exceeds the before-communication payoff, can also be expressed in terms of the after-communication payoff.

□

Before proving Theorem 1, we first prove some useful lemmas.

LEMMA A.1. *If* $\mathbf{S}$ *and* $\mathbf{S}'$ *are such that* $\mathbf{S} \subseteq \mathbf{S}'$ *and* $\mathbf{S}'$ *is under-informed then, for any player* i*, we have* $B_i(\mathbf{S}) \subseteq B_i(\mathbf{S}')$*. More importantly,* $\mathbf{B}(\mathbf{S}')$ *is under-informed and* $B(\mathbf{S}) \subseteq B(\mathbf{S}')$*.*

Proof. By assumption, $\mathbf{S}'$ is such that $\mathbf{S}'_i \subseteq \text{TBR}_i(\mathbf{S}'_{-i})$. Hence $\mathbf{S}_{-i} \subseteq \mathbf{S}' \subseteq B_i(\mathbf{S}') = \text{TBR}_i(\mathbf{S}'_{-i}) \cup \mathbf{S}'_{-i}$. Consequently we only need to prove that $\text{TBR}_i(\mathbf{S}_{-i}) \subseteq \text{TBR}_i(\mathbf{S}'_{-i}) \cup \mathbf{S}'_{-i}$. Without loss of generality, we can write $\mathcal{J} \setminus \mathbf{S}_{-i} = \{j_1, \cdots, j_P\} \cup (\mathbf{S}' \setminus \mathbf{S}_{-i})$ where $\{j_1, \cdots, j_P\} = \mathcal{J} \setminus \mathbf{S}'$ and $e_{j_1 i} \geq \cdots \geq e_{j_P i}$.
The set $\text{TBR}_i(\mathbf{S}_{-i})$ can then be written $B \cup \{j_1, \cdots, j_p\}$ (where $B \subseteq \mathbf{S}' \setminus \mathbf{S}_{-i}$), while $\text{TBR}_i(\mathbf{S}'_{-i}) = \mathbf{S}'_i \cup \{j_1, \cdots, j_{p'}\}$. We need to prove that $j_p \in \text{TBR}_i(\mathbf{S}'_{-i})$. Since $j_p \in \text{TBR}_i(\mathbf{S}_{-i})$, we have

$$e_{j_p i} > Mean\, \{e_{ji} :\ j \in \mathcal{I} \cup \mathbf{S}_{-i} \cup B \cup \{j_1, \cdots, j_{p-1}\}\}$$

Thus we have

$$e_{j_p i} > Mean\left(e_{ji} :\ j \in \mathcal{I} \cup \mathbf{S}_{-i} \cup (\mathbf{S}' \setminus \mathbf{S}_{-i}) \cup \{j_1, \cdots, j_{p-1}\}\right),$$

24. Note that map $\mathbf{B}$ depends on the order of players. However, as we will see the important objects do not depend on the order chosen. Note also that $\mathbf{B}_i(\mathbf{S})$ and $(\mathbf{B}(\mathbf{S}))_i$ are different objects; the map $\mathbf{B}$ is not monotonic in the classical sense, as there are simple examples where $\mathbf{S}_i \subseteq \mathbf{S}'_i$ for all i does not imply that $(\mathbf{B}(\mathbf{S}))_i \subseteq (\mathbf{B}(\mathbf{S}'))_i$.

because B consists of the elements of the elements of $\mathbf{S}' \setminus \mathbf{S}_{-i}$ who give the largest share to i. This proves that $j_p \in \text{TBR}_i(\mathbf{S}'_{-i})$, and therefore that $B_i(\mathbf{S}) \subseteq B_i(\mathbf{S}')$.

Let us now prove that $B(\mathbf{S}) \subseteq B(\mathbf{S}')$. By a recursive argument, it is enough to show that $\mathbf{B}_i(\mathbf{S}')$ is under-informed, to be able to repeatedly apply the first point of the lemma. Let $j \neq i$. We must prove that $(\mathbf{B}_i(\mathbf{S}'))_j \subseteq \text{TBR}_j((\mathbf{B}_i(\mathbf{S}'))_{-j})$. Since $(\mathbf{B}_i(\mathbf{S}'))_j = \mathbf{S}'_j$, it amounts to proving that $\mathbf{S}'_j \subseteq \text{TBR}_j((B_i(\mathbf{S}'))_{-j})$. Note that $\mathbf{S}'_j \cap (\mathbf{B}_i(\mathbf{S}'))_{-j} = \emptyset$. Hence

$$\mathbf{S}'_j \subseteq \text{TBR}_j(\mathbf{S}'_{-j}) \setminus (\mathbf{B}_i(\mathbf{S}'))_{-j} \subseteq \text{TBR}_j((\mathbf{B}_i(\mathbf{S}'))_{-j}),$$

because $\mathbf{S}'_{-j} \subseteq (\mathbf{B}_i(\mathbf{S}'))_{-j}$, and applying Lemma 1.

□

LEMMA A.2. *Let $\mathbf{S} \in \mathcal{S}_u$. Then $\mathbf{S}_i \subseteq (\mathbf{B}(\mathbf{S}))_i$ for any $i \in \mathcal{I}$.*

Proof. We have

$$(\mathbf{B}(\mathbf{S}))_i = \text{TBR}_i((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{S}_{i+1}, ..., \mathbf{S}_I), \text{ for } i = 1, ..., I.$$

We show the proposition by induction on i. By definition of $\mathbf{S} \in \mathcal{S}_u$ we have $\mathbf{S}_1 \subseteq \text{TBR}_1(\mathbf{S}_{-1}) = (\mathbf{B}(\mathbf{S}))_1$. Assume that $\mathbf{S}_j \subseteq (\mathbf{B}(\mathbf{S}))_j$ for $j = 1, ..., i-1$. Then

$$\mathbf{S}_{-i} \subseteq ((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{S}_{i+1}, ..., \mathbf{S}_I)$$

and $\mathbf{S}_i \cap (\mathbf{B}(\mathbf{S}))_1 \cup ... \cup (\mathbf{B}(\mathbf{S}))_{i-1} \cup \mathbf{S}_{i+1} \cup ... \cup \mathbf{S}_I)$ by construction. Hence

$$\begin{aligned}
\mathbf{S}_i &\subset Br_i(\mathbf{S}_{-i}) \setminus ((\mathbf{B}(S))_1, ..., \mathbf{B}(S))_{i-1}, \mathbf{S}_{i+1}, ..., \mathbf{S}_I) \\
&\subset Br_i(((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{S}_{i+1}, ..., \mathbf{S}_I)) \\
&= (\mathbf{B}(\mathbf{S}))_i
\end{aligned}$$

by Lemma 1.

□

LEMMA A.3. *If $\mathbf{S}_i \subseteq (\mathbf{B}(\mathbf{S}))_i$ $\forall i$ then $\mathbf{B}^k(\mathbf{S})$ is non-decreasing. In particular if $\mathbf{S}$ is under-informed then $\mathbf{B}^k(\mathbf{S})$ is non-decreasing.*

Proof. Suppose that $\mathbf{S}_i \subseteq (\mathbf{B}(\mathbf{S}))_i$ for any $i \in \mathcal{I}$. We only need to prove that $(\mathbf{B}(\mathbf{S}))_i \circ (\mathbf{B} \circ \mathbf{B}(\mathbf{S}))_i$ and the result follows by induction. We can write the terms of $\mathbf{B}(\mathbf{S})$ recursively:

$$(\mathbf{B}(\mathbf{S}))_i = \text{TBR}_i((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_{i-1}, \mathbf{S}_{i+1}, ..., \mathbf{S}_I), \text{ for } i = 1, ..., I.$$

Also

$$(\mathbf{B}^2(\mathbf{S}))_i = \text{TBR}_i\left((\mathbf{B}^2(\mathbf{S}))_1, ..., (\mathbf{B}^2(\mathbf{S}))_{i-1}, (\mathbf{B}(\mathbf{S}))_{i+1}, ...(\mathbf{B}(\mathbf{S}))_I\right)$$

By assumption we have $\mathbf{S}_{-1} \subseteq (\mathbf{B}(\mathbf{S}))_{-1}$. Moreover $\text{TBR}_1(\mathbf{S}_{-1}) \cap \mathbf{B}(\mathbf{S})_{-1} = \emptyset$. As a consequence

$$\text{TBR}_1(\mathbf{S}_{-1}) \subseteq \text{TBR}_1((\mathbf{B}(\mathbf{S}))_{-1}).$$

Suppose we proved that $(\mathbf{B}(\mathbf{S}))_j \subseteq (\mathbf{B}^2(\mathbf{S}))_j$ for $j = 1, ..., i$ ($i < n$). We now prove that $(\mathbf{B}(\mathbf{S}))_{i+1} \subseteq (\mathbf{B}^2(\mathbf{S}))_{i+1}$, and it will conclude the proof. We have

$$((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_i, \mathbf{S}_{i+2}, ..., \mathbf{S}_I) \subseteq ((\mathbf{B}^2(\mathbf{S}))_1, ..., (\mathbf{B}^2(\mathbf{S}))_i, (\mathbf{B}(\mathbf{S}))_{i+2}, ..., (\mathbf{B}(\mathbf{S}))_I)$$

and $\text{TBR}_{i+1}\left(((\mathbf{B}(\mathbf{S}))_1, ..., (\mathbf{B}(\mathbf{S}))_i, \mathbf{S}_{i+2}, ..., \mathbf{S}_I)\right)$ does not intersect the set $\mathbf{B}^2(\mathbf{S}))_1 \cup ... \cup \mathbf{B}^2(\mathbf{S}))_i \cup (\mathbf{B}(\mathbf{S}))_{i+2} \cup ... \cup (\mathbf{B}(\mathbf{S}))_I$. Consequently it is contained in

$$Br_{i+1}\left((\mathbf{B}^2(\mathbf{S}))_1, ..., (\mathbf{B}^2(\mathbf{S}))_i, (\mathbf{B}(\mathbf{S}))_{i+2}, ..., (\mathbf{B}(\mathbf{S}))_I\right).$$

In other terms $(\mathbf{B}(\mathbf{S}))_{i+1} \subseteq (\mathbf{B}^2(\mathbf{S}))_{i+1}$, and the proof is complete. When $\mathbf{S} \in \mathcal{S}_u$ this follows from Lemma A.2.

□

Proof of Theorem 1. The sequence $(\mathbf{B}^k(\emptyset))_k$ is non-decreasing and bounded above in a finite set. Thus there exist $\underline{\mathbf{S}}^*$ and an integer K such that $\mathbf{B}^K(\emptyset) = \underline{\mathbf{S}}^*$. Let $\mathbf{S}^*$ be a tight Nash equilibrium. We need to show that $\underline{\mathbf{S}}^* \subseteq \mathbf{S}^*$ and the proof will be complete. Both $\emptyset$ and $\mathbf{S}^*$ are under-informed. Thus $\mathbf{B}^k(\emptyset) \subseteq \mathbf{B}^k(\mathbf{S}^*) = \mathbf{S}^*$ for any k by Lemma A.1.

We now prove the existence of a maximum TNE. Let $\{\mathbf{S}^*(k)\}_{k=1,...,K}$ be the set of TNEs, and consider a profile $\mathbf{S}$ such that $\mathbf{S}_i \cap \mathbf{S}_{-i} = \emptyset$, and satisfying the following properties:

$$\mathbf{S}_i^*(k) \subseteq \mathbf{S}_i \subseteq \mathbf{S}_i^*(1) \cup ... \cup \mathbf{S}_i^*(K)\ \forall i,\ \forall k;\ \ \mathbf{S} = \mathbf{S}^*(1) \cup ... \cup \mathbf{S}^*(K).$$

By the monotonicity property, for $k = 1, ..., K$, we have

$$\mathbf{S}_i^*(k) = TBR_i(\mathbf{S}_{-i}^*(k)) \subseteq TBR_i(\mathbf{S}_{-i}) \cup \mathbf{S}_{-i}.$$

Consequently, since $\mathbf{S}_i \cap \mathbf{S}_{-i} = \emptyset$, we have

$$\mathbf{S}_i \subseteq TBR_i(\mathbf{S}_{-i}).$$

In other terms, $\mathbf{S}$ is under-informed. The sequence $\mathbf{B}^k(\mathbf{S})$ is non-decreasing and therefore converges to a TNE $\overline{\mathbf{S}}^*$ such that $\mathbf{S}^*(k) \subseteq \overline{\mathbf{S}}^*$, for all k. This concludes the proof.

□

Proof of Proposition 2. We show that if $\mathbf{S}^* \in TNE$ and $\mathbf{S}^* \subseteq \overline{\mathbf{S}}$ then $\pi_i(\mathbf{S}^*) \geq \pi_i(\overline{\mathbf{S}})$; Therefore, any TNE Pareto-dominates any TNE with a larger set of informed agents. Let $\mathbf{D} = \overline{\mathbf{S}} \setminus \mathbf{S}^*$. We have

$$\pi_i(\overline{\mathbf{S}}) = \frac{m(\mathbf{S}^*)}{m(\overline{\mathbf{S}})}\pi_i(\mathbf{S}^*) + \frac{1}{m(\overline{\mathbf{S}})} \sum_{d \in \mathbf{D}} e_{d,i}.$$

However $e_{d,i} \leq \pi_i(\mathbf{S}^*),\ \forall d \in \mathbf{D}$ because $\mathbf{S}^*$ is tight. Hence $\pi(\overline{\mathbf{S}}) \leq \pi(\mathbf{S}^*)$.

□

Proof of Theorem 2. To prove the theorem, we show that the threshold is lowered for externality matrix $\mathbf{E}'$. Formally, let $j \notin \mathcal{M} = \mathcal{I} \cup \mathbf{S}_{-i}$, and assume that

$$e_{ji} > \frac{1}{m}\left(e_{ii} + \sum_{l \in \mathcal{M}\setminus\{i\}} e_{li}\right)$$

Then

$$e'_{ji} = \varphi_i(e_{ji}) \quad > \quad \varphi_i\left(\frac{1}{m}\left(e_{ii} + \sum_{l \in \mathcal{M}\setminus\{i\}} e_{li}\right)\right)$$

And noting that, for any function φ concave and such that $\varphi(0) \geq 0$, we have $\varphi(\lambda x) > \lambda\varphi(x)$ for $\lambda \in (0, 1)$, we find

$$\begin{aligned} e'_{ji} &\geq \frac{1}{m}\left(\varphi_i(e_{ii}) + \sum_{l \in \mathcal{M}\setminus\{i\}} \varphi_i(e_{li})\right) \\ &\geq \frac{1}{m}\left(e'_{ii} + \sum_{l \in \mathcal{M}\setminus\{i\}} e'_{li}\right) \end{aligned}$$

Hence the communication cannot be reduced under $\mathbf{E}'$: $\text{TBR}_i(\mathbf{S}_{-i} \mid \mathbf{E}) \subseteq \text{TBR}_i(\mathbf{S}_{-i} \mid \mathbf{E}')$. We now prove the last point: first note that, if $\mathbf{T}_{-i} \subseteq \mathbf{S}_{-i}$ then

$$\text{TBR}_i(\mathbf{T}_{-i} \mid \mathbf{E}) \cup \mathbf{T}_{-i} \subseteq \text{TBR}_i(\mathbf{S}_{-i} \mid \mathbf{E}) \cup \mathbf{S}_{-i} \subseteq \text{TBR}_i(\mathbf{S}_{-i} \mid \mathbf{E}') \cup \mathbf{S}_{-i}.$$

Consequently, for any $k \in \mathbb{N}^*$, we have that $\mathbf{B}^k(\emptyset) \subseteq (\mathbf{B}')^k(\emptyset)$, which proves that $\underline{\mathbf{S}}(\mathbf{E}) \subseteq \underline{\mathbf{S}}(\mathbf{E}')$.

□

Proof of Proposition 3. We first show that, for any $\mathbf{S}_{-i}$, we have $\text{TBR}'_i(\mathbf{S}_{-i}) \subseteq \text{TBR}_i(\mathbf{S}_{-i})$. Let π'_i denote the payoff function of player i in the game with sharing matrix $\mathbf{E}'$. Since $\mathbf{E}$ and $\mathbf{E}'$ are symmetric, we have

$$\pi_i(\mathbf{s}_i, \mathbf{S}_{-i}) = \frac{1}{m(\mathbf{S})}\left(1 - \sum_{j \notin \mathcal{M}(\mathbf{S})} e_{ji}\right) \quad \pi'_i(\mathbf{s}_i, \mathbf{S}_{-i}) = \frac{1}{m(\mathbf{S})}\left(1 - \sum_{j \notin \mathcal{M}(\mathbf{S})} e'_{ji}\right).$$

Note that the characterization of TBRs implies that, if $j \in \text{TBR}_i(\mathbf{S}_{-i})$, then

$$e_{ji} > \pi_i(\text{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}), \tag{A.1}$$

meaning that the value of the shares of every informed neighbor strictly exceeds the agent's current payoff. Now, by (A.1), It is sufficient to show that

$$\pi'_i(\text{TBR}'_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}) \geq \pi_i(\text{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}).$$

Let $\mathcal{M} := \mathcal{M}(\text{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i})$ and $m := |\mathcal{M}|$. Then

$$\begin{aligned}
\pi'_i(\text{TBR}'_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}) &\geq \pi'_i(\text{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}) \\
&= \frac{1}{m}\left(1 - \sum_{j \notin \mathcal{M}} e'_{ji}\right) \\
&\geq \frac{1}{m}\left(1 - \sum_{j \notin \mathcal{M}} e_{ji}\right) \\
&= \pi_i\left(\text{TBR}_i(\mathbf{S}_{-i}), \mathbf{S}_{-i}\right)
\end{aligned}$$

which concludes the proof of the first point. We now prove the last point. First note that, if $\mathbf{S}'_{-i} \subseteq \mathbf{S}_{-i}$ then

$$\text{TBR}'_i(\mathbf{S}'_{-i}) \cup \mathbf{S}'_{-i} \subseteq \text{TBR}'_i(\mathbf{S}_{-i}) \cup \mathbf{S}_{-i} \subseteq \text{TBR}_i(\mathbf{S}_{-i}) \cup \mathbf{S}_{-i}.$$

Consequently, for any $k \in \mathbb{N}^*$, we have that $(\mathbf{B}')^k(\emptyset) \subseteq \mathbf{B}^k(\emptyset)$, which proves that $\underline{\mathbf{S}}(\mathbf{E}') \subseteq \underline{\mathbf{S}}(\mathbf{E})$.

□

Proof of Proposition 4. Let $\mathbf{S}^*$ be an TNE. Then there exists $t \in \mathcal{J}$ such that $\mathbf{S}^* = \{j \in \mathcal{J} : j \leq t\}$, because any tight best-response of player i is of the form $\{j \in \mathcal{J} : j \leq t_i\}$. Suppose that there exists another TNE $\hat{\mathbf{S}}^*$, such that $\hat{\mathbf{S}}^* = \{j \in \mathcal{J} : j \leq \hat{t}\}$, with $\hat{t} > t$. Then there exists some $i \in \mathcal{I}$ such that $\hat{t} \in \hat{\mathbf{S}}_i^*$, Thus we may assume without loss of generality that $\hat{\mathbf{S}}_i^* = \hat{\mathbf{S}}^*$ and $\hat{\mathbf{S}}_{-i}^* = \emptyset$ (i informs everyone up to regular agent $\hat{t}$). Since $\pi_i(\hat{\mathbf{S}}_i^*, \emptyset) = \pi_i(\hat{\mathbf{S}}^*) < \pi_i(\mathbf{S}^*) = \pi_i(\mathbf{S}^*, \emptyset)$, player i has a profitable deviation, and it contradicts the fact that $\hat{\mathbf{S}}^*$ is a TNE.

□

Proof of Proposition 5. Let $\mathbf{S}^*$ be a TNE, and suppose that i, j are such that $\mathcal{N}_i \cap (\mathcal{I} \cup \mathbf{S}_{-i}^*) \neq \emptyset$, $\mathbf{S}_i^* = \emptyset$ and $\mathbf{S}_j^* \neq \emptyset$. Then

$$e_i \leq \pi_i^* = \frac{e_{ii} + e_i |\mathcal{N}_i \cap (\mathcal{I} \cup \mathbf{S}_{-i}^*)|}{I + |\mathbf{S}^*|} \text{ and } \; e_j > \pi_j^* = \frac{e_{jj} + e_j |\mathcal{N}_j|}{I + |\mathbf{S}^*|}.$$

Hence

$$\eta_i \geq \frac{e_{ii}}{e_i} + |\mathcal{N}_i \cap (\mathcal{I} \cup \mathbf{S}_{-i}^*)| \geq I + |\mathbf{S}^*| > \frac{e_{jj}}{e_j} + d_j = \eta_j$$

This proves the first point of the proposition.

Now let $i^* \in \{0, ..., I\}$, and consider a profile $\mathbf{S}^*$ such that

$$\mathbf{S}_i^* = \mathcal{N}_i \setminus (\mathcal{I} \cup \mathbf{S}_{-i}^*), \; \forall i \leq i^*, \text{ while } \; \mathbf{S}_i^* = \emptyset \; \forall i > i^*.$$

Note that

$$|M(i^*)| = I + |\mathbf{S}^*|, \text{ and } \; \xi(i^*) = |M(i^*)| \min_{i > i^*} \frac{\pi_i(\mathbf{S}^*)}{e_i}.$$

Observing that $\pi(\mathbf{S}^*) = \frac{e_{ii} + e_i d_i}{|M(i^*)|}$ when $i \leq i^*$, the profile $\mathbf{S}^*$ is a TNE if

$$\min_{i > i^*} \frac{\pi_i(\mathbf{S}^*)}{e_i} \geq 1, \text{ i.e. } \; \xi(i^*) \geq |M(i^*)| \; \text{ and } \; \max_{i \leq i^*} \frac{\pi_i(\mathbf{S}^*)}{e_i} < 1, \text{ i.e. } \; \eta_{i^*} < |M(i^*)|$$

Consequently, if $\eta^* \geq 0$ is such that $\eta^* < m(\eta^*) \leq \xi(\eta^*)$, $\mathbf{S}^*$ is a TNE.

□

Appendix B: Appendix B: Multi-level local externality matrices

In this Appendix, we present two-level-local externality matrices, whose analysis straightforwardly generalizes to multi-level-local externality matrices.

Consider externality matrices such that $e_{ii} = e_0$ for all i, and $e_{ij} \in \{g_{ij} \cdot \underline{e}, g_{ij} \cdot \overline{e}\}$ for all $i, j \neq i$, and with $g_{ij} \in \{0, 1\}$; this class generates a communication game of parameters represented by the quadruplet $(e_0, \underline{e}, \overline{e}, \mathbf{G})$. This class of matrix, although stylized, induces more complexity than binary input externality matrices. Some interesting insights from the analysis of two-level-local externality matrices are that partial communication can emerge, and that degree centrality is not sufficient to describe incentives: distance-two neighbors matter.

It is easily seen that there are two kinds of informers at equilibrium. We associate an index to each player, that is increasing in both the degree and the proportion of high-externality neighbors. Then we can partition adequately those players with low index (i.e., with low degree and preferably linked to low-externality neighbors), and those players with high index (i.e., with higher degree and more linked to high-externality neighbors). Players with a low index inform all uninformed neighbors, players with high index inform high-externality uninformed neighbors only if low-externality regular neighbors are preferably linked to high-index players.

To see this formally, consider an equilibrium with m^* informed agents. A basic observation is that, at equilibrium, if a player i informs a low-externality neighbor, then player i informs all uninformed neighbors; and if a player informs one high-externality neighbor, she informs all uninformed high-externality neighbors. Defining by β_i the proportion of player i's high-externality neighbors, the incentives to inform a low-externality neighbor, for player i, are then given by the condition

$$m^* > \mu_i = \frac{e_0}{\underline{e}} + \Big(\beta_i \frac{\overline{e}}{\underline{e}} + (1 - \beta_i)\Big) d_i$$

Note that index μ_i only depends on primitives of the model. For those players of index lower than m^*, they inform all neighbors.[25] Consider now those players whose index is large enough to violate the above condition. They still find profitable to inform a

25. Then, for two arbitrary equilibria, the respective subsets of players informing all uninformed neighbors in each equilibrium are nested.

high-externality neighbor under the condition

$$m^* > \nu_i(m^*) = \frac{e_0}{\overline{e}} + \beta_i d_i + \frac{\underline{e}}{\overline{e}} \sum_{k \in \mathcal{N}_i \cap \mathbf{S}_{-i}} g_{ik} \cdot I\left[\min_{p \in \mathcal{N}_k \cap \mathcal{I}} \mu_p < m^*\right]$$

Hence, informing a high-externality neighbor is more likely to be valuable when the low-externality regular neighbors of player i are connected to a smaller set of low-index players. Note that, in opposite to the former condition, the RHS depends now on m^*; a simple algorithm allows to find the exact set of informers.

To summarize, to know whether a given configuration with m informed agents is an equilibrium, one has to check that:

(i) All players i of index $\mu_i < m$ inform all uninformed neighbors;

(ii) All players i such that $\mu_i \geq m$ and $\nu_i(m) < m$, inform all high-externality neighbors and only them;

(iii) All players i such that $\mu_i \geq m$ and $\nu_i(m) \geq m$ don't communicate.

Importantly, whereas the index μ does not depend on communication strategies, the index $\nu_i(m)$ does.[26]

This analysis is easily extended to multi-level externality matrices. For externality matrices with $k + 1$ possible externality levels (including the zero level), and by a direct generalization of the above case corresponding to $k = 2$, the set of conditions allowing to identify the equilibrium communication of a given player require to incorporate indexes of players at distance up to $2k$ from that player.

26. Such partial characterization may be complemented by algorithmic computation.