CHAPITRE 4

Value at Risk

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Contents

1 Introduction

1. La VaR d'un portefeuille d'actifs financiers est une mesure quantitative de risque attachée à la valeur du portefeuille.

2. Si un investisseur détient un portefeuille (une position longue), il encourt le risque d'une perte de valeur sur un certain horizon.

3. Les accords de Bâle sur la réglementation des banques requièrent que celles-ci constituent des provisions de capital pour couvrir ce type de risques. Les banques doivent elles-mêmes évaluer la VaR de leurs actifs financiers. The Basel Committee on Banking Supervision (1996) at the Bank for International Settlements uses VaR to require financial institutions such as banks and investment firms to meet capital requirements to cover the market risks that they incur as a result of their normal operations.

4. Une banque pourrait dire que la VaR de son portefeuille, à un certain horizon, est de 20 millions d'euros au niveau de confiance de 95% (ou au niveau de risque statistique de 5%). Autrement dit, il y a 5 chances sur 100 qu'une perte supérieure à 20 millions se produise à l'horizon considéré, si les marchés fonctionnent normalement.

5. Comment peut-on estimer ce montant? Est-il de 20, 5, ou 50 millions? La VaR est définie pour un niveau de risque (comme 5%), un horizon de temps (comme 10 jours), et sur la base d'un modèle d'évaluation du risque financier.

6. La VaR est déduite directement d'un quantile de la distribution (conditionnelle) du rendement du portefeuille.

1.1 VaR d'une position longue

Soit y_t le rendement aléatoire, à une période, d'un portefeuille qui vaut p_t:

$$y_t = 100\Delta \log p_t.$$

Soit $q_t(\alpha)$ le quantile de niveau $\alpha\%$ de la distribution de y_t, défini par la relation

$$\Pr[y_t < q_t(\alpha)] = \alpha.$$

La VaR de niveau α de p_t est la perte minimale qui peut se produire avec une probabilité égale α:

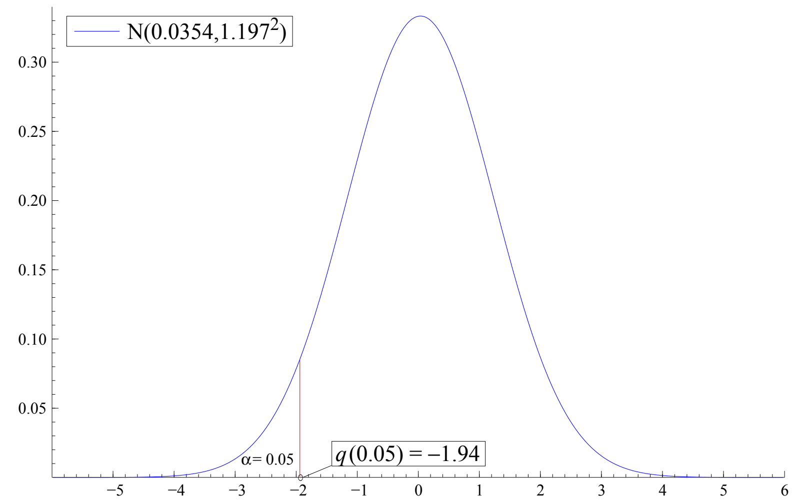
$$VaR_t(\alpha) = -q_t(\alpha) \times p_t.$$

Par exemple, si

$$\begin{aligned} y_t &\sim N(\mu_t, \sigma_t^2), \\ VaR_t(\alpha) &= -(\mu_t + z_\alpha \sigma_t) p_t, \end{aligned}$$

où z_α est le quantile de niveau α de la distribution N(0,1).

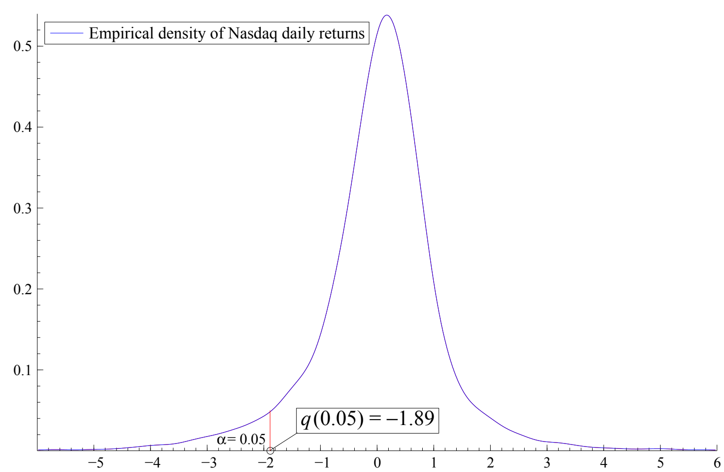
Exemple de VaR (distribution normale)



$$VaR(0.05) = -(\underbrace{0.0354}_{\mu} \underbrace{-1.645}_{z_{0.05}} \times \underbrace{1.197}_{\sigma}) p_t = 1.94\, p_t,$$

Chaire Francqui 0 – p. 43/97

Exemple de VaR (distribution empirique)



On a observé un rendement inférieur à -1.89% dans 5% du nombre total de jours de l'échantillon $\Longrightarrow VaR(0.05) = 1.89\, p_t$.

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Figure 1: VaR du Nasdaq

2 Risk and volatility

On peut mesurer la volatilité comme

$$V_\theta = \mathrm{E}_{t-1}[|R_t - \mu_t|^\theta]$$

Mean absolute deviation or variance. Variance adequate only under quadratic utility or normal distribution of returns.

Risk only exists in the left part of the distribution. We do not want to diversify to reduce the possibility of an unexpected positive return.

VaR

$$VaR_t^\alpha = -sup[r|\mathrm{Pr}_{t-1}[R_t \leq r] \leq \alpha]$$

Basel

1. $\alpha = 0.01$
2. 10 trading days
3. one year for backtesting

3 Estimation methods for VaR

3.1 Historical simulation

The VaR is estimated it as the α th quantile of the empirical distribution of losses:

$$\hat{V}aR_t^\alpha = -R_{\omega:T},$$

where $R_{\omega:T}$ is the αth-order statistic of the data and

$$\omega = [T\alpha] = max\{m|m \leq T\alpha, m \in \mathbb{N}\}.$$

But the empirical quantile is not a good estimator for an extreme quantile.

As seen in the seminar, cannot be revised daily. Because we need a lot of observations to estimate the quantiles.

3.2 Bootstrap

If IID returns, bootstrapping in the empirical distribution of returns. HS estimation using the replications.

3.3 Modeling the dependence of returns

Returns modeled as

$$R_t = \mu_t + \epsilon_t \sigma_t$$

Then

$$VaR_t^\alpha = \mu_t + q_\alpha \sigma_t$$

The mean is modeled using an ARMA

$$\mu_t = \phi_0 + \phi_1 R_{t-1} + \theta_1 a_{t-1} \qquad a_t = R_t - \mu_t$$

The variance is modeled using a GARCH

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta \sigma_{t-1}^2$$

Finally, the quantile can be determined easily if the distribution of ϵ is normal. Other distributions can be used when estimating the GARCH like the Student of asymmetric Students.

4 Engle and Manganelli: CAViaR 2004

VaR is an estimate of how much a certain portfolio can lose within a given time period, for a given confidence level. The great popularity that this instrument has achieved among financial practitioners is essentially due to its conceptual simplicity: VaR reduces the (market) risk associated with any portfolio to just one monetary amount.

Despite its conceptual simplicity, the measurement of VaR is a very challenging statistical problem and none of the methodologies developed so far gives satisfactory solutions. Since VaR is simply a particular quantile of future portfolio values, conditional on current information, and since the distribution of portfolio returns typically changes over time, the challenge is to find a suitable model for time varying conditional quantiles.

1. provide a formula for calculating VaR_t as a function of variables known at time $t-1$ and a set of parameters that need to be estimated

2. provide a procedure (namely, a loss function and a suitable optimisation algorithm) to estimate the set of unknown parameters

3. provide a test to establish the quality of the estimate.

Some first estimate the volatility of the portfolio, perhaps by GARCH or exponential smoothing, and then compute VaR from this, often assuming normality.

The volatility approach assumes that the negative extremes follow the same process as the rest of the returns and that the distribution of the returns divided by standard deviations will be independent and identically distributed, if not normal.

CAViarR: model directly the quantile of the distribution of returns. We want the quantile to evolve smoothly: so autoregressive. We want explanations: the x.

$$VaR_t^\alpha = \beta_0 + \beta_1 VaR_{t-1}^\alpha + l(\beta_2, R_{t-1}, VaR_{t-1}^\alpha)$$

where different forms for the function l were proposed.

1. symmetric slopes in R
2. asymmetric slopes $l = \beta_2 R_{t-1}^+ + \beta_3 R_{t-1}^-$
3. adaptative $l = \beta_0([1 + \exp(G[R_{t-1} + VaR_{t-1}^\alpha])]^{-1} - \alpha) \qquad G > 0$

Estimation using the methods of the quantile regression.

5 Backtesting VaR estimates

Imposed by Basel Committee on Banking Supervision. Backtesting over one year (250 obs). Backtesting is based on a failure process

$$I_t^\alpha = \mathbb{1}(R_t < -VaR_t^\alpha), \qquad t = T+1, ..., T+n$$

One year ahead for testing, $n = 250$. A VaR estimate is accurate iff

$$\mathrm{E}_{t-1}[I_t^\alpha] = \alpha.$$

5.1 Likelihood ratio test

The number of failures x has a binomial distribution with parameters α and n. Kuipec (1995)

$$LR_{uc} = 2\log\left[\left(1 - \frac{x}{n}\right)^{n-x}\left(\frac{x}{n}\right)^x\right] - 2\log[(1-\alpha)^{n-x}\alpha^x] \sim \chi^2(1).$$

$\chi^2(1) = 3.84$ at 95%. Low power in small samples.

We could devise a test for

$$\mathrm{E}_{t-1}[I_t^\alpha | I_{t-1}^\alpha] = \alpha.$$

It is a test of independence. LR_{ind}. The test for conditional coverage is

$$LR_{cc} = LR_{uc} + LR_{ind} \sim \chi^2(2)$$

$\chi^2(2) = 5.99$ at 95%.

5.2 CAViaR

A dynamic quantile test based on a regression of $I_t^\alpha - \alpha$ on its lags and the other values contained in the conditioning set. See if these variables are significant.

5.3 Hurlin 2011

Backtesting Value-at-Risk : A GMM Duration-based Test. The duration between hits is a geometric distribution. Using properties of this distribution, Hurlin finds polynomials which are zero under the null. The first polynomial corresponds to the first moment. Test if these polynomials are jointly zero. Has to determine the number of polynomials: 1, 2 or 3.

6 References

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