UNIT ROOTS TESTS
AND SARIMA MODELS

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Abstract

Differencing at order one takes away the stochastic trend and differencing at seasonal order s, removes the seasonal peaks. We propose a joint unit root test, at order one and at order s, and tabulation for this test.

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JEL classification: C12, C15

1 Introduction

The modeling strategy, popularized by Box and Jenkins (1970), consists in adjusting an ARMA(p,q) model on a univariate stationary time series y_t. This strategy finds its justification in the the Wold representation theorem. In the case of non-stationary time series, the most common case in empiric applications, the series can be decomposed into a trend Z_t and a cycle C_t:

$$y_t = Z_t + C_t \tag{1}$$

Box and Jenkins propose to remove the stochastic trend Z_t by differencing the series. The adjusted model becomes an ARIMA model. The diagnostic method proposed by Box and Jenkins to determine the necessary order of differencing is not very rigorous. The correct method should be a unit root test.

Let us now consider a seasonal time series. It can be decomposed into a trend, a cycle and a seasonal component S_t:

$$y_t = Z_t + C_t + S_t \tag{2}$$

How to model S_t? We have three solutions:

1) a purely deterministic process

2) a stationary process at the order of seasonality

3) an integrated seasonal process

The first solution involves for example the use of dummy seasonal variables in the model. The second and third solutions, which do not exclude the first one, consist in introducing a lag polynomial in L^s, where s is the seasonal periodicity, in the AR part and/or the MA part of the ARMA model, leading to a SARMA or a SARIMA model.

We will investigate in this paper the choice between solution 2 and solution 3. Does the seasonality have a stochastic trend? If so, its modeling involves differencing the series at the order of seasonality. This decision should be based on a seasonal unit root test. Dickey, Hasza and Fuller (1984) have tabulated these tests, the null hypothesis being represented by the filter $(1 - L^s)$. But this test can simply detect a unit root at frequency zero instead of a seasonal root because we have the following equivalence:

$$\Delta_s y_t = \Delta y_t + \Delta y_{t-1} \ldots + \Delta y_{t-s+1} \tag{3}$$

Hylleberg, Engle, Granger and Yoo (1990), have improved the Dickey, Hasza and Fuller (1984) test by developing the filter $(1 - L^s)$ around its roots, $(1, -1, i, -i)$ in the case where $s = 4$, and by proposing a different test for each root. The root 1 goes with the zero frequency or the long term trend, the others with the seasonal movements.

We are dealing with the case of two unit roots, one at the level of the trend $(1 - L)$ and one at the level of seasonality $(1 - L^s)$. We do not decompose the seasonal root, because we want to answer the simple question: what is the necessary order of differentiation for modeling a seasonal time series. As the test accepts the presence of two distinct roots, we don't have the ambiguity existing in the Dickey, Hasza and Fuller test (1984).

Dickey and Pantula (1987) show that sequential tests are more powerful than joint tests in the case of multiple unit roots. To test the presence of two unit roots, they use the following regression:

$$(1 - \rho_1 L)(1 - \rho_2 L)y_t \quad = \quad \epsilon_t \tag{4}$$

After a reparameterization, this is equivalent to the regression:

$$\Delta^2 y_t = \beta_1 \Delta y_{t-1} + \beta_2 y_{t-1} + \epsilon_t \tag{5}$$

where $\beta_1 = (\rho_1\rho_2 - 1)$ and $\beta_2 = -(1 - \rho_1)(1 - \rho_2)$. The test for a second unit root is run by first imposing $\beta_2 = 0$ and testing for $\beta_1 = 0$. If this root is rejected, then a single root is tested for $\beta_2 = 0$ with β_1 free. The distribution of the sequential tests is the same as that of the usual Dickey-Fuller test.

In the case of distinct unit roots, one at order 1 and one at order s, the type of reparameterization done in (4) is no longer possible. Because the sequential tests are not longer possible, we shall build a joint test.

In a first section, we shall explain how to build such a test. In a second section, we shall tabulate this test. Finally, we shall illustrate the method with monthly US data of electric consumption.

2 Joint unit root tests at order 1 and s

2.1 Description of the test

We usually write a SARIMA model as follows:

$$\Delta^d \bar{\varphi}(L) \Delta_s^{ds} \bar{\varphi}_s(L^s)\, y_t = \theta(L)\theta_s(L^s)\, \epsilon_t \tag{6}$$

with $\Delta^d = (1-L)^d$ and $\Delta_s^{ds} = (1-L^s)^{ds}$, with $\bar{\varphi}(L)$, $\theta(L)$, $\bar{\varphi}_s(L)$ and $\theta_s(L)$ being lag polynomials with roots outside the unit circle. d is the level of integration of the series, ds is the level of seasonal integration and s is the order of seasonality. This expression requires d unit roots at order 1 and ds other roots at order s. Usually, d and ds are less than or equal to 1.

To build a test regression, let us suppose that y_t can be represented by an AR model:

$$\varphi(L)\varphi_s(L^s)\,(y_t - TD_t - SD_t) = \epsilon_t \tag{7}$$

where TD_t is the deterministic trend and SD_t the deterministic seasonal component of y_t. We shall have for example $SD_t = \sum_{i=1}^{s} \alpha_i d_i$ or $SD_t = \sum_{i=1}^{s-1} \alpha_i d_i$ if TD_t includes a constant term.

We want to test the existence of a unit root at the order 1 in the polynomial $\varphi(L)$ and at the order s in the polynomial $\varphi_s(L^s)$. If we decompose each polynomial into

$$\begin{aligned} \varphi(L) &= (1-\rho L) - (1-L)\varphi^*(L) && (8)\\ \varphi_s(L^s) &= (1-\phi L^s) - (1-L^s)\varphi_s^*(L^s) && (9) \end{aligned}$$

using a simple polynomial division of $\varphi(L)$ by $(1-L)$ and of $\varphi_s(L^s)$ by $(1-L^s)$. $\varphi^*(L) + \varphi(1) - 1$ and $\varphi_s^*(L^s) + \varphi_s(1) - 1$ are the quotients of these divisions and $\rho = 1 - \varphi(1)$, $\phi = 1 - \varphi_s(1)$ the remainders. By multiplying together these two decompositions we get:

$$\begin{aligned} \varphi(L)\varphi_s(L^s) &= (1-\rho L)(1-\phi L^s) - (1-\rho L)(1-L^s)\varphi_s^*(L^s) - (1-\phi L^s)(1-L)\varphi^*(L) \\ &\quad + (1-L)(1-L^s)\varphi^*(L)\varphi_s^*(L^s) \\ &= (1-\rho L)(1-\phi L^s) + \\ &\quad (1-L)(1-L^s)\left[\varphi^*(L)\varphi_s^*(L^s) - \frac{1-\phi L^s}{1-L^s}\varphi^*(L) - \frac{1-\rho L}{1-L}\varphi_s^*(L^s)\right] \end{aligned}$$

which can be approximated by:

$$\varphi(L)\varphi_s(L^s) \simeq (1-\rho L)(1-\phi L^s) + (1-L)(1-L^s)\psi^*(L) \tag{10}$$

The approximation is exact when $\varphi(L)$ and $\varphi_s(L^s)$ are respectively of degree 1 and s or when we are near $\rho = \phi = 1$. The model becomes:

$$[(1-\rho L)(1-\phi L^s)-(1-L)(1-L^s)\psi^*(L)]\,(y_t-TD_t-SD_t)=\epsilon_t \tag{11}$$

A first regression enables us to extract the deterministic components of the series:

$$\widehat{u}_t = y_t - \widehat{TD}_t - \widehat{SD}_t \tag{12}$$

The estimated residuals represent the stochastic part of the series on which we shall perform the tests using the following regression:

$$\widehat{u}_t = \rho\widehat{u}_{t-1} + \phi\widehat{u}_{t-s} - \rho\phi\widehat{u}_{t-s-1} + (1-L)(1-L^s)\psi^*(L)\widehat{u}_t + \epsilon_t \tag{13}$$

As we notice that $(1-L)(1-L^s)=1-L-L^s+L^{s+1}$ we get the following reparameterization:

$$\begin{aligned}\Delta\Delta_s\widehat{u}_t &= (\rho-1)\widehat{u}_{t-1}+(\phi-1)\widehat{u}_{t-s}-(\rho\phi-1)\widehat{u}_{t-s-1}+\psi^*(L)\Delta\Delta_s\widehat{u}_t+\epsilon_t\\ &= (\rho-1)\Delta_s\widehat{u}_{t-1}+(\phi-1)\Delta\widehat{u}_{t-s}+(\rho+\phi-\rho\phi-1)\widehat{u}_{t-s-1}+\psi^*(L)\Delta\Delta_s\widehat{u}_t+\epsilon_t\end{aligned} \tag{14}$$

Testing the null hypothesis of unit roots at order 1 and s, is the same as testing the joint nullity of the lag parameters of the estimated residuals at order 1 and s.

Remark: The test we have built is similar to the one proposed by Osborn *et al* (1988), but the alternative we consider is wider than theirs. Indeed, their regression is equivalent to:

$$\Delta\Delta_s y_t = \beta_1\Delta_s y_{t-1} + \beta_2\Delta y_{t-s} + \epsilon_t \tag{15}$$

Consequently, the tables we shall give are slightly different from theirs. They cover the cases $s=4$ and $s=12$ for a varieties of deterministic components.

2.2 Tabulation of the tests

Under the null H_0, we have two unit roots:

$$H_0 : \rho = 1 \quad \text{and} \quad \phi = 1 \tag{16}$$

So, we will generate an artificial sample of T observations with:

$$\Delta\Delta_s y_t = \epsilon_t \qquad \text{with } \epsilon_t \sim N(0,1) \tag{17}$$

using $10,000$ replications. We give the tables for $s=4$ and $s=12$. Under the null H_0, $\rho=1$ and $\phi=1$. So implicitly, the regressor $\widehat{u}_{t-s-1}$ disappears under the null. We don't have to impose this restriction in the test regression otherwise the power of the test would decrease. Consequently, the tables cover only the cases $\rho=1$, $\phi=1$ (t test) and $\rho=\phi=1$ (F test). The number of observations T, is a multiple of s and $s+1$. The variance used for the elaboration of the table is the corrected variance, the sum of squared residuals divided by $T-k$, k being the number of regressors. The regressions are made in one step. We generate T observations, and because of the different lags, use $T-s-1$ observations for the regression. Tables 1 ($s = 4$) and 2 ($s = 12$) present the critical values of the F test and tables 3 ($s = 4$) and 4 ($s = 12$) the critical values for the t tests.

The presence of a constant term (as it concerns the low frequencies) changes only the test for $\rho = 1$ and leaves the test for $\phi = 1$ unchanged in large samples. Adding a trend has roughly the same effect, but in small sample the critical value for $\phi = 1$ is a little changed. Finally, adding seasonal dummies increases the power of the test for $\rho = 1$, but decreases that for $\phi = 1$.

The value of s does not seem to have a major effect on the critical values. For the test $\rho = 1$, they are roughly the same. For $\phi = 1$, they differ only when seasonal dummies are added.

Comparing tables 3 and 4 with the usual Dickey-Fuller tables shows that the critical values for $\rho = 1$ are all the time greater. However the critical values for $\phi = 1$ are smaller than the values found in the Dickey-Haza-Fuller tables. So it is difficult to qualify the difference in power than could exist with a sequential test procedure, if it were possible.

3 Empirical application

Let us consider the US electric consumption series. It is a monthly series which covers the period 1953:1-1961:12. As shown in figure 1, it displays a well marked trend and a seasonality which increases over time (partly due to the increase in the number of air-conditioners).

A first regression is used to extract the deterministic components (seasonal dummies and trend):

$$y_t = \underset{[60.93]}{7.97} + \underset{[>4]}{\textstyle\sum_{i=1}^{11}} SD_{it} + \underset{[88.53]}{0.094}\, t$$
$$R^2 = 0.99 \qquad \hat{\sigma} = 0.343 \qquad DW = 0.72$$

The test regression shows that we cannot reject the existence of two unit roots:

$$\Delta\Delta_{12}\hat{u}_t = -\underset{[-2.74]}{0.23}\, \hat{u}_{t-1} - \underset{[-4.46]}{0.45}\, \hat{u}_{t-12} + \underset{[4.82]}{0.56}\, \hat{u}_{t-13} - \underset{[-4.41]}{0.38}\, \Delta\Delta_{12}\hat{u}_{t-2}$$
$$R^2 = 0.48 \qquad \hat{\sigma} = 0.206$$

The t statistics on the $\hat{u}_{t-1}$ and $\hat{u}_{t-12}$ coefficients are respectively -2.74 and -4.46, values which are to be compared with the -3.72 and the -6.33 of the table at 5%. The F test of the joint nullity hypothesis of these two coefficients has a value of 12.07 which is again less than the 22.41 read in the table at 5%. So, we can propose a SARIMA modeling of this series as follows (using the BJTEST option of TSP for estimation):

$$(1 + \underset{[-4.92]}{0.59}\, L^{12})(1 - \underset{[5.31]}{0.50}\, L)\Delta\Delta_{12} y_t = (1 - \underset{[47.59]}{0.97}\, L)\epsilon_t + \underset{[2.76]}{0.0042}$$
$$R^2 = 0.39 \qquad \hat{\sigma} = 0.222 \qquad DW = 1.73$$

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	T	$F: \rho = \phi = 1$ 90%	95%	99%
without constant	60	3.63	4.60	7.03
without seas. dum.	100	3.31	4.20	6.49
without trend	200	2.98	3.81	5.77
with constant	60	6.08	7.39	10.2
without seas. dum.	100	5.83	6.99	9.52
without trend	200	5.36	6.41	8.75
with constant	60	10.51	12.48	16.42
with seas. dum.	100	9.72	11.41	14.56
without trend	200	9.06	10.47	13.34
with constant	60	8.43	9.95	13.32
without seas. dum.	100	8.08	9.41	12.49
with trend	200	7.58	8.86	11.45
with constant	60	13.58	15.83	20.60
with seas. dum.	100	12.39	14.18	17.81
with trend	200	11.30	12.92	16.01

Table 1: Critical values for F seasonal unit root tests with quarterly data
DGP: $\Delta\Delta_4 y_t = \epsilon_t \sim N(0,1)$

	T	F: $\rho = \phi = 1$		
		90%	95%	99%
without constant	120	3.37	4.32	6.56
without seas. dum.	156	3.19	4.15	6.32
without trend	312	2.85	3.68	5.55
with constant	120	5.95	7.10	9.75
without seas. dum.	156	5.82	6.92	9.32
without trend	312	5.38	6.50	8.98
with constant	120	19.72	22.18	27.16
with seas. dum.	156	18.69	20.93	25.72
without trend	312	17.55	19.74	23.89
with constant	120	7.72	9.07	12.17
without seas. dum.	156	7.86	8.97	11.76
with trend	312	7.67	8.90	11.42
with constant	120	22.41	25.11	30.25
with seas. dum.	156	21.37	23.79	28.84
with trend	312	20.02	22.07	26.54

Table 2: Critical values for F: seasonal unit root tests with monthly data
DGP: $\Delta\Delta_{12}y_t = \epsilon_t \sim N(0,1)$

	T	t: $\rho = 1$			t: $\phi = 1$		
		1%	5%	10%	1%	5%	10%
without constant	60	-3.30	-2.56	-2.19	-2.60	-1.94	-1.58
without seas. dum.	100	-3.11	-2.41	-2.00	-2.63	-1.94	-1.58
without trend	200	-2.89	-2.26	-1.89	-2.56	-1.92	-1.56
with constant	60	-4.18	-3.48	-3.15	-2.72	-2.06	-1.69
without seas. dum.	100	-4.06	-3.40	-3.06	-2.71	-2.00	-1.63
without trend	200	-3.88	-3.26	-2.93	-2.56	-1.94	-1.60
with constant	60	-3.61	-2.84	-2.47	-4.73	-3.99	-3.63
with seas. dum.	100	-3.56	-2.81	-2.42	-4.48	-3.86	-3.53
without trend	200	-3.39	-2.67	-2.30	-4.33	-3.77	-3.45
with constant	60	-4.74	4.08	-3.73	-3.30	-2.63	-2.26
without seas. dum.	100	-4.64	-3.99	-3.68	-3.03	-2.33	-1.98
with trend	200	-4.48	-3.89	-3.57	-2.70	-2.08	-1.75
with constant	60	-4.42	-3.67	-3.28	-5.01	-4.25	-3.91
with seas. dum.	100	-4.27	-3.53	-3.15	-4.64	-4.01	-3.69
with trend	200	-4.07	-3.40	-3.02	-4.42	-3.85	-3.53

Table 3: Critical values for t seasonal unit root tests with quarterly data
DGP: $\Delta\Delta_4 y_t = \epsilon_t \sim N(0,1)$

	T	t: $\rho = 1$ 1%	5%	10%	t: $\phi = 1$ 1%	5%	10%
without constant	120	-3.25	-2.53	-2.16	-2.62	-1.83	-1.44
without seas. dum.	156	-3.18	-2.43	-2.06	-2.58	-1.85	-1.47
without trend	312	-3.00	-2.29	-1.91	-2.50	-1.82	-1.47
with constant	120	-4.12	-3.51	-3.18	-2.70	-1.93	-1.55
without seas. dum.	156	-4.06	-3.46	-3.14	-2.62	-1.93	-1.55
without trend	312	-3.99	-3.36	-3.02	-2.52	-1.85	-1.51
with constant	120	-3.64	-2.89	-2.52	-6.84	-6.09	-5.68
with seas. dum.	156	-3.54	-2.86	-2.50	-6.57	-5.93	-5.55
without trend	312	-3.49	-2.77	-2.40	-6.45	-5.77	-5.44
with constant	120	-4.62	-3.99	-3.68	-3.08	-2.30	-1.91
without seas. dum.	156	-4.55	-3.97	-3.68	-3.01	-2.23	-1.88
with trend	312	-4.56	-3.98	-3.67	-2.71	-2.06	-1.71
with constant	120	-4.37	-3.72	-3.35	-7.01	-6.33	-5.92
with seas. dum.	156	-4.29	-3.66	-3.34	-6.78	-6.13	-5.73
with trend	312	-4.18	-3.55	-3.21	-6.54	-5.87	-5.53

Table 4: Critical values for t seasonal unit root tests with monthly data
DGP: $\Delta\Delta_{12}y_t = \epsilon_t \sim N(0,1)$

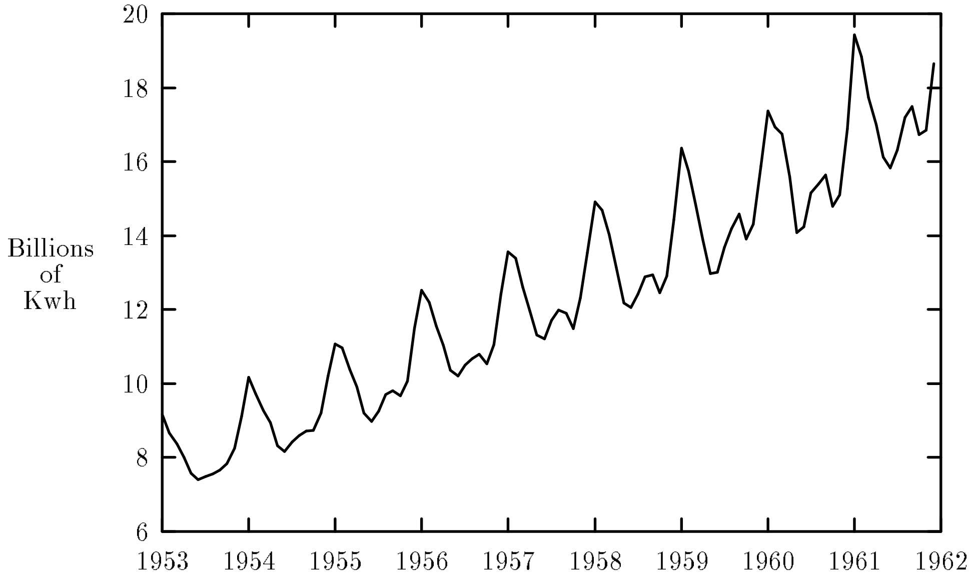

Figure 1: Monthly electric consumption in the USA