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# GROUP DECISION MAKING UNDER UNCERTAINTY
## *A NOTE ON THE AGGREGATION OF "ORDINAL PROBABILITIES"*

ABSTRACT. This paper is a first attempt to study the problem of aggregation of individual ordinal probabilistic beliefs in an Arrowian framework. We exhibit some properties an aggregation rule must fulfil; in particular we prove the existence of a "quasi-dictator".

## 1. INTRODUCTION

In any decision problem under uncertainty, the decision maker has to determine the prior probability distribution over the possible states of nature. To get the best available knowledge and information a decision maker may summon or consult a panel of experts. When the members of the panel have different prior probabilities, the problem is in aggregating them. A common idea is that there does not exist an aggregation rule which satisfies Arrow's conditions. (Pareto, independance of irrelevant alternatives, non dictatorship). Of course this is true when experts only have a ranking on the elementary events. Here we study the more general case when the experts have a ranking on all the events (i.e. the elements of a given $\sigma$-algebra) which can be supported by a probability in the usual sense. We shall call it an ordinal probability.

We are not interested in the characterization of the set of binary relations over an $\sigma$-algebra which enjoys this property (the so-called problem of "qualitative probability": see for instance Fishburn, 1970; Savage, 1954; Chateauneuf, 1981, 1983; and Chateauneuf and Jaffray, 1983). The reader will notice that the ranking over the elementary events does not admit a natural extension to the whole $\sigma$-algebra i.e. there are several extensions of the ranking to the $\sigma$-algebra which are compatible with the properties of an ordinal probability.

The aim of this note is to study the Arrowian aggregation rules of individual ordinal probabilities. We face two questions: first, is the domain restriction introduced by the properties of an ordinal probability

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sufficient to avoid the impossibility of Arrow's Theorem (Arrow, 1963)? Second, in the case of a positive answer, is the result of the aggregation rule, an ordinal probability?

The problem tackled in this note occurs obviously in group decision-making under uncertainty when we accept the separation principle suggested for instance by A. Hylland and R. Zeckauser, (1979) i.e. we aggregate first the individual preferences on the set of possible acts and then the individual opinions about the likelihood of the different events. The paper is organized as follows. In Section 2 we introduce some notations and definitions. In Section 3 we describe the properties that an Arrowian rule of individual ordinal probabilities must fulfil.

## 2. NOTATIONS AND DEFINITIONS

We denote by $X$ the set of elementary events and $\mathscr{B}_X$ a $\sigma$-algebra of subsets of $X$, so that $(X, \mathscr{B}_X)$ is a measurable space.

DEFINITION 1. A probability on $(X, \mathscr{B}_X)$ is a function P from $\mathscr{B}_X$ to $[0,1]$ which fulfils:

(1) $P(\varnothing)=0$

(2) $P(A^c)=1-P(A),\ \forall A \in \mathscr{B}_X$

(3) $P\left(\bigcup_{n=1}^{\infty} A_n\right)=\sum_{n=1}^{\infty} P(A_n)$ for any sequence $(A_n)_{n\geq 1}$ in $\mathscr{B}_X$ with

$$A_n \cap A_k=\varnothing,\ \forall n,k \geq 1,\ n \neq k.$$

We denote by $\mathscr{P}_X$ the space of probabilities over $(X, \mathscr{B}_X)$.

DEFINITION 2. Let $P_1$, $P_2$ be two elements in $\mathscr{P}_X$. We define the binary relation $\sim$ over $\mathscr{P}_X$ by:

$$P_1 \sim P_2 \text{ iff } P_1(A) \geq P_1(B) \Leftrightarrow P_2(A) \geq P_2(B)$$

for any $A$, $B$ belonging to $\mathscr{B}_X$.

*Remark 1.* It is easy to show that $\sim$ is an equivalence relation over $\mathscr{P}_X$.

DEFINITION 3. An ordinal probability over $(X, \mathscr{B}_X)$ is an element of the quotient space $\mathscr{P}_X/\sim$.[1]

We denote by $M=\{1,2,\ldots,m\}$ the set of voters and by $\sum$ the set of complete, reflexive and transitive relations on $\mathscr{B}_X$. Each voter $i \in M$ has an ordinal probabilistic judgement on $X$, $\lesssim_i$, belonging to $\mathscr{P}_{X/\sim}$. Two elements of $\mathscr{P}_{X/\sim}$, $\lesssim$ and $\lesssim'$ are said to agree on a subset $\mathscr{A}$ of $\mathscr{B}_X$, if for every pair $A$, $B \in \mathscr{A} \lesssim B$ iff $A \lesssim' B$. We denote agreement on $\mathscr{A}$ by $\lesssim|_{\mathscr{A}} = \lesssim'|_{\mathscr{A}}$. Two profiles $\lesssim_M = (\lesssim_{,} \ldots, \lesssim_m)$ and $\lesssim'_M = (\lesssim'_1, \ldots, \lesssim'_m)$ of individual probabilities agree on $\mathscr{A} \subset \mathscr{B}_X$ if $\forall_i \in M$, $\lesssim_{i|\mathscr{A}} = \lesssim'_{i|\mathscr{A}}$.

An aggregation rule (A.R.) on $\mathscr{P}_{X/\sim}$ is a function $F$: $(\mathscr{P}_{X/\sim})^m \rightarrow \sum$. (Notice that the range of an A.R. is not restricted to $\mathscr{P}_{X/\sim}$). An Arrowian A.R. is an A.R. that satisfies unanimity, independence of irrelevent alternatives, and nondictatorship.

*Unanimity* (*U*) Let $\lesssim_M \in (\mathscr{P}_{X/\sim})^m$ be an admissible profile, and let $f(\lesssim_M) = \lesssim$.
The A.R. $f$ satisfies U iff for any $A$, $B \in \mathscr{B}_X$

$$A <_i B \Rightarrow A < B$$
$$\forall_i \in M.$$

*Independence of irrelevant alternatives* (II A) Let $\lesssim_M \in (\mathscr{P}_{X/\sim})^m$ and $\lesssim'_M \in (\mathscr{P}_{X/\sim})^m$ be any two admissible profiles. Let $f(\lesssim_M) = \lesssim$ and $f(\lesssim'_M) = \lesssim'$. The A.R. $f$ satisfies II A iff $\forall \mathscr{A} \subset \mathscr{B}_X$

$$\lesssim_{M|\mathscr{A}} = \lesssim'_{M|\mathscr{A}} \Rightarrow \lesssim_{|\mathscr{A}} = \lesssim'_{|\mathscr{A}}$$

*Nondictatorship* (ND) The A.R. $f$ satisfies ND iff $\exists_i \in M$ such that for any admissible profile $\lesssim_M \in (\mathscr{P}_{X/\sim})^m$ and for any $A$, $B \in \mathscr{B}_X$:

$$A <_i B \text{ implies } A < B$$

where $\lesssim = f(\lesssim_M)$.

A pair of distinct events $A$, $B \in \mathscr{B}_X$ is called trivial (relative to $\mathscr{P}_{X/\sim}$) if all relations in $\mathscr{P}_{X/\sim}$ agree on the set $\{A,B\}$.[2] A set of three distinct events $\{A,B,C\}$ is called a free triple if for every $\lesssim \in \sum$, there exists $\lesssim' \in \mathscr{P}_{X/\sim}$ such that: $<_{|\{A,B,C\}} = \lesssim'_{|\{A,B,C\}}$. Two non-trivial pairs $\{A,B\}$ and $\{C,D\}$ are called strongly connected if $\{A,B,C,D\}$ is a free triple (in particular $\# \{A,B,C,D\} = 3$). Two pairs $\{A,B\}$ and $\{C,D\}$ are called connected if a finite sequence of pairs $\{A,B\} = \{B_1,B_2\}$, $\{B_2,B_3\}, \ldots \ldots \{B_{n-1}, B_n\} = \{C,D\}$

exists, such that $\{B_{i-1}, B_i, B_{i+1}\}$ is a free triple $\forall_i = 2,\ldots,n-1$.

Here, we assume that $X$ is finite and we take $\mathscr{B}_X = 2^X$ (the set of subsets of $X$). We denote by $1,2,\ldots,n$, the elements of $X$ and we assume $n \geq 3$.

## 3. THE RESULT

We state the following result:

PROPOSITION. *Let* $n \geq 4$.

(1) *Let* $f\colon (\mathscr{P}_{X/\sim})^m \to \sum$ *an aggregation rule. If* $f$ *satisfies* (U) *and* (II A), *then there exists* $i \in M$ *such that*:

$$A <_i B \Rightarrow A < B$$

*for any profile* $\lesssim_M = (\lesssim_1, \ldots, \lesssim_m) \in (\mathscr{P}_{X/\sim})^m$ *and for any* $A$, $B \in \mathscr{B}_X$, *unless* $\#\ A = n-1$ *and* $\#\ B = 1$, *where* $\lesssim\, = f(\lesssim_M)$.

(2) *There exist Arrowian aggregation rules on* $(\tilde{\mathscr{P}}_{X/\sim})^m$, *where*

$$\tilde{\mathscr{P}}_X = \{P \in \mathscr{P}_X : P(A) > 0 \qquad \forall \mathrm{A} \in \mathscr{B}_X\ \#\ A = 1\}^3.$$

*Proof of (1).*

*Step 1.* All pairs $\{A, B\}$, $\{C, D\}$ with $A$, $B$, $C$, $D \in \mathscr{B}_X$ and $\#A = \#B = \#C = \#D$ are connected (the proof is left to the reader).

*Step 2.* Any pair $\{A,B\}$, $A$, $B \in \mathscr{B}_X$ with $\#A = \#B = i$, is connected with any else $\{C,D\}$ $C$, $D \in \mathscr{B}_X$ with $\#C = \#D = i+1$ $\quad \forall = 1,\ldots,n-2$. For $i=1$, consider $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3, 4\}$ and $A_4 = \{1,4\}$. For $i \geqslant 2$, consider $A_1 = \{1,2,\ldots,i\}$, $A_2 = \{2,3,\ldots,i+1\}$, $A_3 = \{1,3,4,\ldots,i,\ i+1,\ i+2\}$ and $A_4 = \{1,2,4,\ldots,i,\ i+1,\ i+2\}$ $\{A_1,A_2\}$ (resp. $\{A_3,A_4\}$) is connected with $\{A,B\}$ (resp. $\{C,D\}$) by Step 1. It is obvious that $\{A_1, A_2\}$ is connected with $\{A_3,A_4\}$.

*Step 3.* Let $\{A,B\}$ a non-trivial pair in $\mathscr{B}_X$ with $\#\ A = i$, $\#\ B = j$, and $i < j \leqslant n-1$

$$\exists\, k \in A \quad \text{such that } k \notin B$$

and

$$\exists\, k' \in B \quad \text{such that } k' \notin A$$

*Case 1.* $i \geq 2$. Let $C = (A \cup \{k'\})/\{l\}$ $\quad (\#C = \#A)$ $\quad$ where $l \in A$, $l \neq k$. Then $\{A,B\}$ is strongly connected with $\{A,C\}$.

*Case 2.* $i=1$ with $j \leqq n-2$. Let $l \notin A \cup B$. Let $C=(B \cup \{l\})/\{k'\}$ $(\# C = \# B)$. Then $\{A,B\}$ is strongly connected with $\{B,C\}$.

*Step 4.* We have proven that any non-trivial pair $\{A, B\}$ is connected with any other non-trivial pair $\{C,D\}$, unless $\# A=1$ and $\# B=n-1$ or $\# C=1$ and $\# D=n-1$.

*Step 5.* By using arguments similar to Kalai, Muller and Satterthwaite's ones (1979), we deduce (with Step 4) the existence of a voter (namely $i$) such that:
$A <_i B$ implies $A < B$ for any profile $\lesssim_M = (\lesssim_1, \ldots, \lesssim_m) \in (\mathscr{P}_{X/\sim})^m$ and for any $A$, $B \in \mathscr{B}_x$, unless $\# A=1$ and $\# B=n-1$ or $\# A=n-1$ and $\# B=1$.

Now, let $A$, $B \in \mathscr{B}_X$ with $\# A=1$ and $\# B=n-1$. Let $C \subset B$, $C \in \mathscr{B}_X$. Consider an arbitrary profile $\lesssim_M = (\lesssim_1, \ldots, \lesssim_m) \in (\mathscr{P}_{X/\sim})^m$ such that $A <_i B$. There exists another profile $\lesssim'_M = (\lesssim'_1, \ldots, \lesssim'_m) \in (\mathscr{P}_{X/\sim})^m$ such that $A <'_i C <'_i B$ and $\lesssim_{j|\{A,B\}} = \lesssim'_{j|\{A,B\}}$ $\forall_j \neq i$. We notice that $\lesssim_{M|\{A,B\}} = \lesssim'_{M|\{A,B\}}$. By unanimity $C <' B$, and since $i$ is a dictator on $\{A,C\}$, $A <' C$. Thus, by transitivity, $A <' B$. Then, by II A, we deduce that also: $A < B$.

This concludes the proof of Part (1).

*Proof of* (2). In order to prove the existence of Arrowian aggregation rules it suffices to prove that the voter $i$, whose power is described in Part (1) of the proposition, is not necessarly a dictator when $A <_i B$ for a profile $\lesssim_M = (\lesssim_1, \ldots, \lesssim_m)$ with $\# A=n-1$ and $\# B=1$. Indeed, consider the following rule $f$

$$A \lesssim B \text{ iff } A \lesssim_i B$$
$$\forall A,\ B \in \mathscr{B}_X \text{ unless } \# A=n-1 \text{ and } \# B=1$$

and

$$A \lesssim B \text{ iff } A \lesssim_j B \qquad \forall_j \in M$$
$$\forall A,\ B \in \mathscr{B}_X \text{ with } \# A=n-1 \text{ and } \# B=1$$

where

$$\lesssim\ = f(\lesssim_M).$$

Obviously, $f$ satisfies (U) and (II A). It suffices to prove that $f(\lesssim_M) \in \Sigma$ for any $\lesssim_M \in (\tilde{\mathscr{P}}_{X/\sim})^m$. Without loss of generality, consider a profile $\lesssim_M$ with $A \lesssim_i B$, $\# A=n-1$ and $\# B=1$ and $B <_j A$ for $j \in M/\{i\}$. (It must be noticed that there exists at most such a pair $\{A,B\}$). We have $B<A$. It can be easily shown that there does not exist $C\in\mathscr{B}_X$ such that $A\lesssim_i C\lesssim_i B$ (the reader will notice the role played by the new space $\tilde{\mathscr{P}}_{X/\sim}$). Therefore, only two cases remain to be considered in order to ensure transitivity: $C\lesssim B\lesssim A$ or $B\lesssim A\lesssim C$. The first sequence implies $C\subset A$, and thus $C\lesssim A$. A similar proof is used for the second one. This completes the proof of (2).

QED

Our result deserves comment:

(1) We may remark that the aggregation rule defined in Part (2) of the proposition, takes its values in $\mathscr{P}_{X/\sim}$. i.e. in the set of ordinal probabilities.

(2) Our result can be enforced if we restrict the space of ordinal probabilities $\mathscr{P}_{X/\sim}$ to the antisymmetric ones. A necessary and sufficient condition for an aggregation rule $f$ to be Arrowian is part (1) of the proposition and the existence of at least one pair $\{A,B\}\subset\mathscr{B}_X$ with $\#A=1$ and $\#B=n-1$, such that $i$ does not impose his preference in the case $B<_i A$.

(3) The existence of Arrowian aggregation rules must not hide the great power of decision of voter $i$.

(4) Let us consider, at least, the case $n=3$. It is easily verified that Part 1 of the proposition does not hold unless we assign to the aggregation rule to take its values in $\mathscr{P}_{X/\sim}$. Nevertheless, Part 2 of the proposition remains valid.

## CONCLUDING REMARKS

In this paper, we have given a first description of some properties an Arrowian aggregation rule of individual ordinal probabilities must fulfil. Such a rule appears to be "almost dictatorial".

Our result may be interpreted within a different context: restate $X$ as the set of elementary bundles, and $\mathscr{B}_X$ as the set of composite bundles; let us assign to each individual a preference preordering over $X$; we know that it is impossible to build a collective preference preordering over $X$ using an Arrowian aggregation rule; now consider individual preference preorderings over $\mathscr{B}_X$ satisfying the monotonicity properties of ordinal probabilities. Then, there exists arrowian collective preference preorderings over $\mathscr{B}_X$.

Two problems remain unsolved. First, what happens about Part (2) of the position when we keep the space $\mathscr{P}_{X/\sim}$ instead of $\tilde{\mathscr{P}}_{X/\sim}$? Second, the study of the infinite case (i.e. $\# X = \infty$). We can require for this case some restrictions about the probabilities we are considering (Atomless probabilities, existence of a Radon-Nikodym density.....), in order to take a proper subset of the whole set of probabilities. We can remark that the action of the equivalence relation $\sim$ (defined in Section 2) is not as obvious in the finite case.

## NOTES

[1] It is a binary relation on $\mathscr{B}_X$. Thus two probabilities (in the sense of Definition 1) will be considered as equivalent if they generate the same order over the events (i.e. members of $\mathscr{B}_X$). We don't study the problem of characterizing the binary relations over $\mathscr{B}_X$ which are ordinal probabilities.

[2] We borrow these definitions from Kalai, Muller and Satterthwaite, 1979, p. 91.

[3] Part (1) of the proposition holds again if we substitute $\tilde{\mathscr{P}}_{X|\sim}$ to $\mathscr{P}_{X|\sim}$.

## REFERENCES

Arrow, K. J.: 1963, *Social Choice and Individual Values,* 2nd ed., Wiley, New York.

Chateauneuf, A.: 1981, 'Sur l'Existence d'une Probabilité Compatible avec un Préordre défini sur une Algèbre de Boole', Thèse de 3è cycle, Université Pierre et Marie Curie, Paris 6.

Chateauneuf, A.: 1983, 'Sur l'Existence d'une Probabilité Compatible avec un Préordre Total défini sur une Algèbre de Boole', *Bulletin de Mathématiques Economiques* **18**, 1–16.

Chateauneuf, A. et Jaffray, J. Y.: 1983, 'Archimedian Qualitative Probabilities', Note mimeo. Groupe de Mathématiques Economiques, Université de Paris I.

Fishburn, P. C.: 1970, *Utility Theory for Decision Making,* J. Wiley and Sons, New York.

Hylland, A. and Zeckhauser, R.: 1979, 'The Impossibility of Bayesian Group Decision, Making with Separate Aggregation of Beliefs and Values', *Econometrica* **47** (6), 1321–1336.

Kalai, E., Muller, E. and Satterthwaite, M. A.: 1979, 'Social Welfare Functions when Preferences are Convex, Strictly Monotonic, and Continuous', *Public Choice* **34** (1), 87–97.

Savage, L. J.: 1954, *The Foundations of Statistics*, Wiley, New York

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