JOURNAL OF ECONOMIC THEORY **69**, 71–103 (1996)
ARTICLE NO. 0038

Inequality Reducing Properties of Composite Taxation*

MICHEL LE BRETON

LEQAM-GREQAM, Université d'Aix-Marseille II, Château La Farge, Route des Milles, F-13290 Les Milles, France

PATRICK MOYES

CNRS and LARE, Université Bordeaux IV, Avenue Léon Duguit, F-33608 Pessac, France

AND

ALAIN TRANNOY

THEMA, Université de Cergy-Pontoise, 33 Boulevard du Port, F-95011 Cergy-Pontoise, France

Received June 16, 1993; revised November 27, 1994

On the one hand, we know that, the more progressive a tax schedule, the more equally distributed the after tax incomes (U. Jakobsson, *J. Public Econ.* **5** (1976), 161–168). On the other hand, most tax systems involve a complex procedure where individual incomes are successively subjected to different rounds of taxation. We study in the paper the inequality reducing properties of the composition of different taxation schemes and we identify necessary and sufficient conditions for after tax inequality to decrease when the degree of progressivity of one element (or more) of the tax system is increased. For the sake of generality, we assimilate the set of income distributions with the space of probability measures over an interval of the real line, and any function that associates after tax incomes to before tax incomes is viewed as a taxation scheme. This general approach proves to be particularly powerful when investigating the restrictions the egalitarian principle imposes on the set of admissible taxation schemes. *Journal of Economic Literature* Classification Numbers: D31, D63, H24.

1. INTRODUCTION

Apart from financing the production of public goods, redistributing incomes in order to cut (relative) income differentials is certainly one of the

* This paper forms part of the research program of the network "The Distribution and Redistribution of Income" financed by European Communities Contract ERBCHRXCT940647. We have benefited from extremely valuable comments from the associate editor and two anonymous referees. Needless to say, none of them should be held responsible for any remaining deficiencies.

0022-0531/96 $18.00

main objectives of any income tax system (see, e.g., Brown and Jackson [4, Chap. 2]). Though the reduction of income inequality through the income tax system seems to be on the agenda of most policy makers, there is, however, far less agreement regarding the appropriate way the redistributive impact of taxation should be assessed. Following the seminal paper of Atkinson [3], it is now a well-established tradition to appeal to the (relative) Lorenz criterion in order to measure inequality. We say that a taxation scheme is equalising with respect to a given income distribution if the (relative) Lorenz curve of the after tax incomes is nowhere below that of the before tax incomes. If this holds true whatever the pre-tax income distribution, then we say that the taxation scheme is universally equalising. A second feature mostly associated with income taxation refers to the progressivity of the tax schedule. Though this has long been anticipated, Jakobsson [8] was the first to show that a non-decreasing average tax rate (equivalently a non-increasing retention rate) is both a necessary and sufficient condition for post-tax incomes to be more equally distributed than pre-tax incomes according to the Lorenz criterion.[1] A direct consequence of this result is that substituting a more progressive schedule for a less progressive one would result in more equally distributed after tax incomes.

All the results obtained in the literature up to now focus on a *single tax schedule*, though income taxation in practice is a complex process where different schedules and deductions interact when determining after tax income. Indeed, in most developed countries, the tax units are successively subjected to different rounds of taxation such as national insurance contributions and the income tax. When the national insurance contributions are deductible from the income tax base, then the tax system can be viewed as the composition of different schedules. A formally similar situation arises when some transfer payments, such as family allowances, are included into the income tax base. The abatements on taxable income designed to compensate for professional expenses and family composition constitute another example of such a composite process. Are the results in the literature still valid when the composite nature of the income tax system is explicitly introduced into the analysis? Intuition suggests that an increase in the degree of progressivity of one component of the tax system would imply more equally distributed post-tax incomes. However, progressive taxation may fail to be inequality reducing when incomes are submitted successively to different rounds of taxation. For example, substituting a more progressive system of national insurance contributions for a less progressive one, even in the case of a strictly progressive income tax

[1] Independently, Fellman [6] and Kakwani [9] proved that a non-decreasing average tax rate implies a decrease in inequality as measured by the (relative) Lorenz criterion. Related work includes Eichhorn *et al.* [5], Latham [13], and Thon [27] among others.

schedule, may well result in an increase in inequality after tax. The identification of the circumstances under which such an apparently paradoxical situation may occur is of prime interest for the design of tax reforms. Indeed, it is certainly a basic consistency requirement for a tax policy that an equality-improving modification of one element of the tax system be not thwarted after tax. In this paper, we derive a set of conditions that will guarantee that the inequality reducing effect of a modification of a particular component of the tax system be preserved after tax. For instance, the paper indicates the way the income tax schedule should be adjusted for inequality-reducing increases (decreased) in the national insurance contributions to imply an overall reduction in post-tax inequality.

For the sake of generality, we consider the largest possible class of income distributions, i.e., the set of probability measures defined over an arbitrary interval of the real line. This general class includes in particular the set of discrete distributions for populations of fixed or variable sizes as well as the set of purely continuous income distributions the literature has mainly focused on. Similarly, we place no restriction on the set of taxation schemes so that any function that associates a post-tax income to a pre-tax income does constitute an admissible candidate. Introducing as few restrictions as possible from the outset proves to be particularly suitable for investigating the relationship between the equalising properties of taxation and the regularity conditions to be satisfied by the taxation scheme. All along the paper, we adopt the relative inequality point of view which is largely predominant in the literature. If one focuses rather on absolute income differentials as is suggested by Kolm [10], then the absolute Lorenz ordering proves to be the right criterion in order to measure inequality (Moyes [19]). It is a straightforward exercise to derive the absolute analogues to our results (for an introduction, see Moyes [21]) and we will not pursue in this direction.[2]

The organization of the paper is as follows. We introduce in Section 2 the general notation and definitions we employ throughout the paper. Section 3 is concerned with the inequality reducing properties of elementary taxation schemes. On the one hand, we identify the necessary and sufficient conditions for a taxation scheme to be inequality reducing and we further prove that the distinction often made in the literature between alternative classes of income distributions plays actually no role. Regarding the comparison of different taxation schemes, a number of alternative

[2] It is interesting to note that the absolute Lorenz ordering allows for negative incomes, what the relative Lorenz criterion does not. It is possible to adapt the relative Lorenz criterion in order to compare distributions with negative incomes. On the one hand, the normative interpretation of such an inequality ordering is far from being clear. On the other hand, the relationship between the progressivity of the tax schedule and its inequality reducing properties no longer holds.

definitions of the equalising power of a taxation scheme have been provided in the literature. We show that actually all definitions are equivalent under rather mild restrictions on the class of taxation schemes. We exploit all the preceeding results in Section 4 where we investigate the inequality reducing properties of composite taxation schemes. Without loss of generality, we focus on tax systems consisting of two distinct elementary tax schedules, for instance the national insurance contribution scheme and the income tax schedule. We first check that the composition of inequality reducing tax schedules is overall inequality reducing in a static framework. Considering next potential tax reforms, we consider successively two cases: (i) the tax reform affects one component of the tax system, either one being fixed, and (ii) both elements of the tax system are affected by the tax reform. In either case, we point at a set of conditions which guarantee that all equality-improving modification of *either* component of the tax system or *both* implies an overall inequality reduction. In a number of circumstances, the policy maker is interested in the comparison of tax reforms that do not modify the aggregate revenue to be collected. We extend in Section 5 the seminal result of Hemming and Keen [7] and we further indicate necessary and sufficient conditions for the composition of revenue neutral taxation schemes to be inequality reducing. Finally, Section 6 concludes the paper, summarizing our main findings.

2. Preliminary Definitions and Notation

We assume that incomes before tax belong to an interval $V := [\underline{u}, \bar{u}] \subseteq \mathbb{R}$ with $0 \leqslant \underline{u} \leqslant \bar{u} < +\infty$. We will sometimes need to exclude the origin and we let $V^* := V \setminus \{0\}$. A pre-tax income distribution is a compactly supported *probability measure* P over V. The general set of pre-tax income distributions over V is denoted as $\mathscr{P}(V) := \{\text{Probability measures } P \text{ over } V \mid \mu_P > 0\}$, where μ_P represents the mean of P. As the preceeding definition indicates, we restrict attention to distributions with positive means. Indeed, as it will become clear later, considering distributions with non-positive means makes no sense in the relative Lorenz framework. This general definition of a distribution allows for a simultaneous analysis of discrete and continuous distributions as well as distributions over a finite support. The following subsets of $\mathscr{P}(V)$ will play a crucial role in our analysis. The set of *discrete distributions for a population of size* n is defined by

$$\mathscr{D}_n(V) := \left\{ P \in \mathscr{P}(V) \;\middle|\; P = \sum_{k=1}^{n} \frac{1}{n} \delta_{u_k} \text{ with } u_k \leqslant u_{k+1}, \forall k \right\}, \tag{2.1}$$

where δ_{u_k} is the Dirac mass at point u_k. The set of *discrete distributions for populations of variable sizes* is defined by $\mathscr{D}(V) := \bigcup_{n=2}^{\infty} \mathscr{D}_n(V)$. Given

$P \in \mathscr{P}(V)$, we denote as F_P its distribution function and as F_P^{-1} its *right inverse distribution function* defined by

$$F_P^{-1}(t) := \mathrm{Inf}\{u \in V \mid F_P(u) \geqslant t\}, \qquad \forall t \in [0, 1]. \tag{2.2}$$

Following Atkinson [3], it is now a well-established tradition to appeal to the Lorenz criterion in order to measure inequality. We first introduce the Lorenz curve of distribution $P \in \mathscr{P}(V)$ given by

$$L_P(t) := \frac{\int_0^t F_P^{-1}(s)\, ds}{\int_0^1 F_P^{-1}(s)\, ds}, \qquad \forall t \in [0, 1]. \tag{2.3}$$

Actually, $L_P(t)$ represents the proportion of total income possessed by the $t \times 100\ \%$ poorest income units in situation P. The Lorenz ordering of distributions involves the comparison of the income shares accruing to the different fractions of the population. Precisely:

Definition 2.1. Given $P, Q \in \mathscr{P}(V)$, we say that P *weakly dominates* Q *in the (relative) Lorenz sense*, which we write $P \geq_{\mathrm{L}} Q$, if $L_P(t) \geqslant L_Q(t)$, for all $t \in [0, 1]$.

We denote as $>_{\mathrm{L}}$ the asymetric component of $\geq_{\mathrm{L}}$ defined by $P >_{\mathrm{L}} Q$, if and only if, $P \geq_{\mathrm{L}} Q$ and *not* $[Q \geq_{\mathrm{L}} P]$. Since we are ultimately interested in the comparison of pre-tax and post-tax incomes, we will focus throughout the paper on the functions mapping before tax into after tax incomes; hence, the following definition:

Definition 2.2. A *taxation scheme* is a mapping $G: V \to \mathbb{R}_+$ that associates post-tax income $G(u)$ to pre-tax income u.

The associated *tax schedule* T is defined by $G(u) = u - T(u)$, where $T(u)$ is the tax liability of a taxpayer with pre-tax income u. We note in passing that $T(u)$ is not restricted to be non-negative. We emphasize that our definition of a taxation scheme is very general. On the one hand, we do not assume any regularity conditions such as continuity or monotonicity. On the other hand, our definition of a taxation scheme takes into account the situation where incomes are simultaneously subjected to different taxes as it is the case in the United Kingdom with the payment of the national insurance contributions and the income tax or in France with the recent introduction of the so-called generalised social contribution.[3] We denote as

[3] Typically, we have $G(u) := u - g(u) - h(u)$, where $g(u)$ represents the national insurance contribution or the generalised social contribution paid by a tax unit with pre-tax income u while $h(u)$ represents its income tax. This is different from what we have called a composite taxation scheme, where the former tax is deductible from the income that is subjected to the later tax (see Section 4).

$\mathscr{F}(V)$ the general set of taxation schemes defined over V. Though the taxation schemes in practice are not differentiable, most of them however are piecewise differentiable. We let $\mathscr{F}^*(V)$ represent the set of piecewise $C1$ taxation schemes over V. Given a taxation scheme $G \in \mathscr{F}(V)$ and a pre-tax distribution $P \in \mathscr{P}(V)$, we denote as $P^G := G(P(V))$ the resulting after tax distribution. It is certainly a desirable property for a tax schedule that after tax incomes be more equally distributed than before tax incomes. The next definition is central to the paper:

DEFINITION 2.3. Let $P \in \mathscr{L}(V) \subseteq \mathscr{P}(V)$ and $G \in \mathscr{F}(V)$. We will say that G is equalising (or inequality reducing) over $\mathscr{L}(V)$ if $P^G \geq_{\mathrm{L}} P$, for all $P \in \mathscr{L}(V)$.

The inequality reducing property we consider is really a very strong one since it requires that after tax incomes are more equally distributed than before tax ones, for *all* pre-tax distributions that belong to a given subset of distributions. This should be contrasted with *conditional equalisation*, where the tax schedule is required to be inequality reducing for a *given* pre-tax distribution (see, e.g., Latham [13]). One may invoke two reasons for justifying the approach followed in the paper. The positive reason is that there is some degree of uncertainty regarding the actual pre-tax distribution, due in part to statistical measurement errors and to the possible incentive effects of taxation, and that this uncertainty must be taken into account by the policy maker. The normative reason is that we want equalisation to hold for all possible subgroups of tax units. Requiring equalisation over a large set of pre-tax distributions precisely allows one to attain these two objectives. We note in passing that $\mathscr{S}^*(V) \subset \mathscr{S}^\circ(V)$ implies that $E(\mathscr{S}^\circ(V)) \subseteq E(\mathscr{S}^*(V))$, where

$$E(\mathscr{L}(V)) := \{G \in \mathscr{F}(V) \mid P^G \geq_{\mathrm{L}} P, \forall P \in \mathscr{L}(V)\} \tag{2.4}$$

is the set of equalising taxation schemes over $\mathscr{L}(V) \subseteq \mathscr{P}(V)$: the larger the set of distributions, the smaller the class of equalising taxation schemes.[4] The next two properties are closely related to equalisation as we will show in a while. The former condition requires that the positions of the tax units on the income scale are not modified by taxation.

DEFINITION 2.4. Let $G \in \mathscr{F}(V)$. We will say that G is *rank preserving* if G is non-decreasing over V, i.e., $G(u) \leqslant G(v)$, for all $u, v \in V$ such that $u < v$.

[4] This is no longer true when one restricts the set of distributions in such a way that taxation does not modify aggregate income as in the Hemming and Keen [7] approach (see Section 5 below).

Rank preservation is best seen as an *incentive compatibility* property (see, for example the concept of *monotonicity* in L'Ollivier and Rochet [16]). Indeed, even in a model without explicit behaviour, there is always free disposal of the endowment so that the tax units have the possibility (to some extent) of manipulating the result of the tax policy. Rank preservation guarantees that no individual has an incentive to withdraw part of his or her endowment since he or she cannot be made better-off after tax as a result. Progressive taxation usually requires that higher incomes be taxed more heavily than lower incomes. A number of alternative definitions of a progressive schedule have been proposed in the literature. Here we follow the current practice in public finance and require a non-increasing average retention rate. Precisely:

DEFINITION 2.5. Let $G \in \mathscr{F}(V)$. We will say that G is progressive if $G(u)/u \geqslant G(v)/v$, for all $u, v \in V^*$ such that $u < v$.

We note that the definition above is equivalent to $T(u)/u \leqslant T(v)/v$, for all $u < v$, i.e., a non-decreasing average tax rate (see, e.g., Lambert [11, Chap. 6]).

3. INEQUALITY REDUCTION AND ELEMENTARY TAXATION SCHEMES

Which conditions imposed on the tax system will guarantee that income differentials are reduced after tax? We examine successively the static framework, where the aim of the analysis is to identify the properties a taxation scheme need satisfy in order that after tax incomes be more equally distributed than before tax incomes, and a comparative framework, where one is interested in the evaluation of the equalising power of alternative tax schedules.

Given the different classes of distributions we introduced above, we may reasonably expect that, the larger the class of admissible distributions, the stronger the conditions to be imposed on the taxation schemes. We proceed in two successive stages and consider first the case of discrete distributions of variable size. The result below, the proof of which is relegated to the Appendix, constitutes a crucial step towards the subsequent results.

PROPOSITION 3.1. *Let $G \in \mathscr{F}(V)$. Then G is equalising over $\mathscr{D}(V)$, if, and only if*:

(a) *$G(u)/u$ is non-increasing with u over V^*.*

(b) *G is non-decreasing with u over V.*

Therefore, necessary and sufficient conditions for after tax incomes to be more equally distributed than before tax incomes, whatever the pre-tax distribution, is that the taxation scheme be rank-preserving and progressive. This result has been anticipated for a long time but it is only recently that it has been formally established. The sufficiency part was proved by Marshall *et al.* [18, Proposition 2.4] for the case where $u \neq 0$. Jakobsson [8] was the first to recognize that the result could be stated as an equivalence. His proof however contained a number of flaws that were corrected by Eichhorn *et al.* [5] and Thon [27]. Furthermore, the fact that he restricted his attention to populations of fixed sizes as well as to non-decreasing taxation schemes actually made him unable to point at a number of interesting features of an equalising scheme.

Remark 3.1. Condition (a) in Proposition 3.1 is also necessary when one restricts attention to the case of a fixed population size. However, a non-increasing average retention rate does not guarantee more equally distributed after tax incomes unless further restrictions are placed on the taxation schemes.

The class of taxation schemes with a non-increasing average retention rate has received some attention under the name (up to a reversal of sign) of star-shaped functions (see Thon [27]). The progressivity of the tax schedule is actually a rather weak condition: concave taxation schemes are obviously progressive but convexity is no impediment to a taxation scheme being progressive. For instance $G(u) := \ln(1+\exp(u/2))$ is convex and progressive over $V := [0, \bar{u}]$.[5] For differentiable taxation schemes, condition (a) in Proposition 3.1 is equivalent to $G'(u)\, u/G(u) \leqslant 1$, all $u \in (\underline{u}, \bar{u})$. In the public finance literature, the elasticity of the taxation scheme is actually known as the *residual income progression* (see Lambert [11, Chap. 6]). Though many taxation schemes in practice are not differentiable everywhere, most of them have left and right derivatives at every point. Denoting respectively as $G^-(u)$ and $G^+(u)$ the left and the right derivatives of G at u, condition (a) in Proposition 3.1 in the case of a piecewise differentiable taxation scheme may be written

$$\operatorname{Sup}\left\{\frac{G^+(u)\,u}{G(u)}, \frac{G^-(u)\,u}{G(u)}\right\} \leqslant 1, \qquad \forall u \in [\underline{u}, \bar{u}]. \tag{3.1}$$

Remark 3.2. The fact that equalisation does also require that the positions of the income units not be modified by taxation is rather counter-intuitive since the Lorenz ordering of distributions is insensitive to a permutation of incomes.

[5] One easily checks that a necessary condition for a convex function $G\colon [0, \bar{u}] \to \mathbb{R}_+$ to be progressive is that $G(0) \geqslant 0$.

It is interesting to note that rank preservation is no longer necessary for equalisation over $\mathscr{D}_n(V)$ as it can be readily shown. Letting $V := [0, \frac{1}{2}]$ and taking $G(u) := 1 - u$, we check that G is equalising over $\mathscr{D}_2(V)$. Indeed, choose $P := \frac{1}{2}(\delta_u + \delta_v) \in \mathscr{D}_2(V)$ with $u < v$. For $P^G \geq_{\mathrm{L}} P$, it is sufficient that $(1-v)/(2-u-v) \geqslant u/(u+v)$ or equivalently that $v - v^2 \geqslant u - u^2$. But the later inequality is clearly satisfied since $g(t) := t - t^2$ is increasing over $(-\infty, \frac{1}{2}]$. We note in passing that the identification of the regularity conditions implied by equalisation over discrete distributions is a difficult problem that has received no definite answer at present.[6]

The next point we would like to stress is that an equalising taxation scheme cannot be discontinuous over the range of pre-tax incomes with the exception of the origin $(u = 0)$. Precisely:

PROPOSITION 3.2. *Let $G \in \mathscr{F}(V)$. If G is equalising over $\mathscr{D}(V)$, then G is continuous over V^*.*

Proof. From Proposition 3.1, we know that, if G is equalising over $\mathscr{D}(V)$, then $G(u)/u$ is necessarily non-increasing with u over V^*. Thus $G(u)/u$ has a left-hand side and a right-hand side limit in each point $u \in V^*$. Suppose for a contradiction that $G(u)/u$ is discontinuous in u° and denote respectively as ξ_1 and ξ_2 the left-hand side limit and the right-hand side limit of $G(u)/u$ at u°; clearly, $\xi_1 > \xi_2$. Furthermore, let $(u_k)_{k=1,2,...,\infty}$ (resp. $(v_k)_{k=1,2,...,\infty}$) be a sequence converging on the left (on the right) to u° and consider $P_k \in \mathscr{D}_n(V)$ defined by $P_k := (1/n)(\delta_{u_k} + (n-1)\,\delta_{v_k})$. Then, for all $k = 1, 2, ..., \infty$, we have

$$\frac{G(u_k)}{u_k} \geqslant \xi_1 > \xi_2 \geqslant \frac{G(v_k)}{v_k} \tag{3.2}$$

and thus

$$\frac{G(u_k)}{G(v_k)} > \frac{\xi_1}{\xi_2} \frac{u_k}{v_k}. \tag{3.3}$$

Now choose $\varepsilon > 0$ such that $(\xi_1/\xi_2)(1-\varepsilon) > 1 + \varepsilon$. For k sufficiently large, we have

$$\frac{G(u_k)}{G(v_k)} > \frac{\xi_1}{\xi_2}(1-\varepsilon) \qquad \text{and} \qquad \frac{u_k}{v_k} < 1 + \varepsilon. \tag{3.4}$$

We conclude that $P_k >_{\mathrm{L}} P_k^G$, hence, a contradiction. The cases $u = \underline{u}$ and $u = \bar{u}$ may be treated in a similar way. ∎

[6] Moyes and Nizard [23] have provided a full characterization of the taxation schemes that are inequality reducing according to the absolute Lorenz criterion.

The idea underlying the result above is quite simple: if G is discontinuous at some income u°, then we can find $\varepsilon > 0$ such that the absolute value of the difference $G(u^\circ + \varepsilon) - G(u^\circ - \varepsilon)$ can be made arbitrarily large compared to ε, so that Lorenz domination fails. Eichhorn *et al.* [5] have shown that continuity of G follows from the fact that G is non-decreasing and progressive. Our proof is slightly more general since it makes use only of the progressivity condition and the fact that before and after tax incomes are positive. Moyes [20] has even shown that equalisation imposes the taxation schemes to verify some kind of Lipschitzian properties.

The preceeding results concern the particular case where distributions are discrete. Quite interestingly, there is no loss of generality concentrating on the analysis of discrete distributions as the following result indicates.

PROPOSITION 3.3. *Let $G \in \mathscr{F}(V)$. Then G is equalising over $\mathscr{P}(V)$ if, and only if, it is equalising over $\mathscr{D}(V)$.*

Proof. It makes use of an approximation argument introduced in Le Breton [14]. Let $P \in \mathscr{P}(V)$ and denote as $[0, \bar{u}]$ its support. Then, P is the limit for the weak topology of a sequence $(P_n)_{n=1, 2, ..., \infty}$ with $P_n \in \mathscr{D}_n(V)$ and $\mathrm{Supp}(P_n) \subset [0, \bar{u}]$ (see Parthasarathy [24]). Let $P_n := (1/n) \sum_{k=1}^{n} \delta_{u_k}$ with $u_k \leqslant u_{k+1}$, for all $k = 1, 2, ..., \infty$. Since G is equalising over $\mathscr{D}(V)$, we have by definition $L_{P_n^G}(t) \geqslant L_{P_n}(t)$, for all $t \in [0, 1]$. Furthermore, because G is rank preserving (Proposition 3.1), we have

$$L_{P_n^G}(t) := \frac{\int_0^t G(F_{P_n}^{-1}(s))\, ds}{\int_0^1 G(F_{P_n}^{-1}(s))\, ds}, \qquad \forall t \in [0, 1]. \tag{3.5}$$

Since $(P_n)_{n=1, 2, ..., \infty}$ converges weakly to P, we have $\lim_{n \to \infty} F_{P_n}^{-1}(t) = F_P^{-1}(t)$, for λ almost every $t \in [0, 1]$ (where λ is the Lebesgue measure over $[0, 1]$) (see Le Breton [14]). Since G is continuous (Proposition 3.2) and since everything is uniformly (with respect to n) bounded, we deduce from the Lebesgue dominated convergence theorem that $L_{P^G}(t) \geqslant L_P(t)$, for all $t \in [0, 1]$. ∎

From now on, we will assume that the taxation schemes under consideration be non-decreasing. On the one hand, we have seen that rank preservation follows from equalisation for populations of variable sizes. On the other hand, dropping this mild restriction will unecessarily complicate the proofs.

Since the class of universally equalising schemes has been characterized, the next question is to know when it is the case that a taxation scheme is more inequality reducing than another taxation scheme. Regarding the comparison of alternative taxation schemes, a number of alternative definitions of the equalising power of a taxation scheme have been proposed in the literature. An obvious question is then to know to which extent and under which conditions these alternative definitions are equivalent. We start with the following definition which is certainly the less controversial one:

DEFINITION 3.1. Let $G, H \in \mathscr{F}(V)$ be non-decreasing and let $\mathscr{L}(V) \subseteq \mathscr{D}(V)$. Then we will say that G is more equalising than H, which we write $G \geq_{\mathrm{EQ}} H$, over $\mathscr{L}(V)$, if $P^G \geq_{\mathrm{L}} P^H$, for all $P \in \mathscr{L}(V)$.

A taxation scheme has more equalising power than another taxation scheme, whatever the pre-tax distribution, if the post-tax incomes resulting from the application of the former scheme are more equally distributed than the post-tax incomes resulting from the application of the later scheme. The following technical result will be needed in a subsequent proof:

LEMMA 3.1. *Let $G, H \in \mathscr{F}(V)$ be non-decreasing and assume that G is more equalising than H over $\mathscr{D}(V)$. Then, if H is constant over a subinterval $[\underline{v}, \bar{v}] \subseteq V$, so is G.*

Proof. Assume that $H(s) = \xi$, for all $s \in [\underline{v}, \bar{v}]$, and suppose for a contradiction that, for some $u, v \in [\underline{v}, \bar{v}]$ with $u < v$, it is the case that $G(u) < G(v)$. Choosing $P := (1/n)((n-1)\,\delta_u + \delta_v)$, we obtain $P^H = \delta_\xi$ and $P^G = (1/n)((n-1)\,\delta_{G(u)} + \delta_{G(v)})$. But this implies that $P^H >_{\mathrm{L}} P^G$; hence, a contradiction. ▮

The following proposition shows that our definition of a more equalising taxation scheme is actually equivalent to the alternative definitions proposed in the literature.

PROPOSITION 3.4. *Let $G, H \in \mathscr{F}(V)$ be non-decreasing and $\mathscr{L}(V) \subseteq \mathscr{D}(V)$. Then the three following statements are equivalent*:

(a) *$P^G \geq_{\mathrm{L}} P^H$, for all $P \in \mathscr{L}(V)$.*

(b) *There exists $\phi\colon H(V) \to \mathbb{R}_+$ non-decreasing and progressive over $H(V)$ such that $G = \phi \circ H$.*

(c) *$G(u)/H(u)$ is non-decreasing with u over V.*

Proof. We first consider the equivalence of statements (a) and (b). Because condition (b) is clearly sufficient for (a) to hold, we only have to prove it is also necessary. Since G is non-decreasing, we can split the interval V into a finite number of subintervals $[u_k, u_{k+1}]$ over which G is alternatively increasing and constant. From Lemma 3.1, we know that, if H is constant over $[u_k, u_{k+1}]$, so is G. Therefore, there exists a function $\phi\colon H(V) \to \mathbb{R}_+$ such that $G = \phi \circ H$. Furthermore, ϕ is non-decreasing. It remains to show that ϕ is equalising. Appealing to Proposition 3.1, this amounts to checking that $\phi(s)/s$ be non-increasing with s over $H(V)$. Choose $\zeta, \xi \in H(V)$ such that $\zeta < \xi$. Since H is non-decreasing, we have $\zeta = H(u)$ and $\xi = H(v)$, for some $u, v \in V$ with $u < v$. We claim that

$$\frac{\phi(H(u))}{H(u)} = \frac{G(u)}{H(u)} \geqslant \frac{G(v)}{H(v)} = \frac{\phi(H(v))}{H(v)}. \tag{3.6}$$

Indeed, suppose this is false; there exists $u < v \in V$ such that $G(u)/H(u) < G(v)/H(v)$. Taking $P := (1/n)((n-1)\,\delta_u + \delta_v)$ and making use of an argument similar to that employed in the Proof of Proposition 3.1, we obtain $P^H >_{\mathrm{L}} P^G$; hence, a contradiction. We consider next the equivalence of statements (a) and (c). In the case where $0 \notin V$ or $G(0)$, $H(0) \neq 0$, sufficiency follows from Marshall *et al.* [18, Proposition 2.4]. In the either case, sufficiency results from Proposition 3.1. Regarding the converse implication, consider the factorization $G = \phi \circ H$ whose existence has been proven above and choose $u, v \in V$ such that $u < v$. We want to show that $G(u)/H(u) \geqslant G(v)/H(v)$, equivalently $\phi(H(u))/H(u) \geqslant \phi(H(v))/H(v)$. Since H is non-decreasing, $H(u) \leqslant H(v)$ and the result follows from the fact that ϕ is equalising and Proposition 3.1. ▮

The monotonicity condition (c) in Proposition 3.4 is very useful in order to evaluate the capacity of different taxation schemes for reducing inequality in practice (see Trannoy *et al.* [28] for an application). Therefore, the alternative definitions of a more progressive scheme, captured by conditions (b) and (c) in Proposition 3.4, are equivalent, so we can state:

DEFINITION 3.2. Let $G, H \in \mathscr{F}(V)$ be non-decreasing. We will say that G is *more progressive than H over V*, which we write $G \geq_{\mathrm{P}} H$, if either condition (b) or condition (c) of Proposition 3.4 holds.

In the case where G and H are differentiable, we obtain the well-known condition in public finance that the elasticity of G be less than the elasticity of H, i.e.,

$$\frac{G'(u)\,u}{G(u)} \leqslant \frac{H'(u)\,u}{H(u)}, \qquad \forall u \in (\underline{u}, \bar{u}). \tag{3.7}$$

This can be compared with the differentiable characterization of the *more risk averse than* relation in the risk literature (Pratt [25]). In the case of piecewise differentiable taxation schemes, condition (3.7) would write

$$\frac{G^+(u)\,u}{G(u)} \leqslant \frac{H^+(u)\,u}{H(u)} \quad \text{and} \quad \frac{G^-(u)\,u}{G(u)} \leqslant \frac{H^-(u)\,u}{H(u)}, \qquad \forall u \in [\underline{u}, \bar{u}]. \tag{3.8}$$

Remark 3.3. The factorization property (b) in Proposition 3.4 does not survive if we consider more general definitions of a taxation scheme.

Indeed, define now a taxation scheme as a function $\phi: \mathscr{D}_n(V) \to \mathscr{D}_n(\mathbb{R}_+)$ and consider the subclass of linear taxation schemes identified by $n \times n$ matrices $A := [a_{ij}]$. Assuming pure redistribution, so that aggregate income is not modified, a taxation scheme A will be equalising if, and only if, A is a doubly stochastic matrix (see Marshall and Olkin [17, A.4]). It seems quite natural to adapt our definition of a more equalising taxation scheme in this new context by saying that *the doubly stochastic matrix A is more equalising than the doubly stochastic matrix B* if

$$Ax \geq_{\mathrm{L}} Bx, \qquad \forall x := (x_1, ..., x_n) \in \mathbb{R}^n_+. \tag{3.9}$$

Similarly, the equivalent to condition (b) in Proposition 3.4 would be written

There exists a doubly stochastic matrix C such that $A = CB$. (3.10)

The equivalence between conditions (3.9) and (3.10) was conjectured by Kakutani. Actually, the conjecture is false and holds only in special cases (see Marshall and Olkin [17] for references).

4. Inequality Reduction and Composite Taxation Schemes

In this section we focus on tax systems such that pre-tax incomes are successively submitted to different rounds of taxation. Such tax systems can be represented by q-tuples $\Phi := (\phi_1, \phi_2, ..., \phi_q)$, where ϕ_k is the scheme operating at stage k. Therefore, Φ associates to pre-tax income u post-tax income $\phi_q(\phi_{q-1}(\cdots \phi_1(u) \cdots))$. Without loss of generality we restrict attention to composite tax systems involving two taxation schemes. Henceforth, a tax system will be an ordered pair (H, G) such that $G \circ H(u) \equiv G(H(u))$ is the income a tax unit with pre-tax income u is left with after the tax system has been effective. It is convenient in what follows to think of a tax system where national insurance contributions (NIC) are systematically deducted from the income tax base, so that H may be viewed as the national insurance contribution scheme and G as the income taxation scheme.[7] A direct consequence of Proposition 3.4 is that the set of equalising taxation schemes is closed under composition. In other words, the composition of two rank preserving and progressive taxation schemes is equalising. Precisely:

Remark 4.1. Let $H \in \mathscr{F}(V)$ and $G \in \mathscr{F}(H(V))$ be non-decreasing. Then H and G progressive over V and $H(V)$ respectively imply that $G \circ H$ is equalising over $\mathscr{P}(V)$.

[7] This is just a matter of convenience and our analysis applies equally to other possible interpretations (see the discussion in the Introduction).

However, the converse implication is not true and it is not necessary that both H and G be equalising for $G \circ H$ to be equalising as the following example demonstrates.

EXAMPLE 4.1. Choose $u^\circ, u^* \in V$ such that $\underline{u} < u^\circ < u^* < \bar{u}$ and define $\underline{v} := \beta^\circ \underline{u}$, $v^\circ := \beta^\circ u^\circ$, $v^* := \beta^* u^*$, $\bar{v} := \beta^* \bar{u}$. Let

$$H(u) := \begin{cases} \alpha u, & \text{for} \quad \underline{u} \leqslant u < u^\circ, \\ u - (1 - \alpha u^\circ), & \text{for} \quad u^\circ < u \leqslant \bar{u}, \end{cases} \tag{4.1}$$

and

$$G(v) := \begin{cases} v, & \text{for} \quad \underline{v} \leqslant v < v^*, \\ v^* + \gamma(v - v^*), & \text{for} \quad v^* < v \leqslant \bar{v}, \end{cases} \tag{4.2}$$

with $0 < \alpha < 1$ and $0 < \gamma < 1$. Clearly, H and G are increasing, H is regressive, and G is progressive (actually, H is strictly regressive over $[u^\circ, \bar{u}]$ and G is strictly progressive over $[v^*, \bar{v}]$). Choosing $v^* = \alpha u^\circ$ and $\gamma < \alpha$, one easily checks that $G \circ H$ is progressive over V.

We are primarily interested in signing the effect on after tax inequality of a modification of one component of the tax system (H, G). More precisely, we would like to know if, other things equal, an increase in the degree of progressivity of one element of the tax system reduces after tax inequality as measured by the Lorenz criterion. Given two tax systems (H°, G°) and (H^*, G^*), we are looking for conditions that guarantee that $G^* \circ H^*$ be more equalising than $G^\circ \circ H^\circ$ over $\mathscr{P}(V)$. There are three possible cases we will consider successively: (i) $H^* = H^\circ = H$ and $G^* \neq G^\circ$, (ii) $H^* \neq H^\circ$ and $G^* = G^\circ = G$, and (iii) $H^* \neq H^\circ$ and $G^* \neq G^\circ$.

When it is the later taxation scheme that is subject to a modification, we obtain the following obvious result:

PROPOSITION 4.1. *Let $H \in \mathscr{F}(V)$ and $G^\circ, G^* \in \mathscr{F}(H(V))$ be non-decreasing. Then, $G^* \circ H \geq_{\mathrm{EQ}} G^\circ \circ H$ over $\mathscr{P}(V)$ if, and only if, $G^* \geq_{\mathrm{EQ}} G^\circ$ over $\mathscr{P}(H(V))$.*

According to Proposition 4.1, the only way to improve the distribution of after tax incomes according to the Lorenz criterion, when the first stage tax schedule is fixed, is to substitute a more progressive taxation scheme for a less progressive one in the later stage.

In the case where it is only the first component of the tax system that is subject to a change, the result is less obvious. In particular, substituting a more progressive taxation scheme for a less progressive one in the first stage of taxation no longer guarantees that post-tax incomes are more equally distributed as the next example indicates.

EXAMPLE 4.2. Given $V := [\underline{u}, \bar{u}]$, choose $u^\circ, u^* \in V$ such that $\underline{u} < u^\circ < u^* < \bar{u}$. Let $H^*(u) := \beta^* u$ and

$$H^\circ(u) := \begin{cases} \beta_1^\circ u, & \text{for} \quad \underline{u} \leqslant u < u^\circ, \\ \alpha^\circ + \beta_2^\circ u, & \text{for} \quad u^\circ < u \leqslant \bar{u}, \end{cases} \tag{4.3}$$

with $0 < \beta_1^\circ < \beta^* < 1$, $\beta_2^\circ := \beta_1^\circ + \varepsilon/(u^* - u^\circ)$, $\alpha^\circ := -\varepsilon u^\circ/(u^* - u^\circ)$, and $\varepsilon > 0$. Clearly, H° and H^* are increasing, and H^* is strictly more progressive than H°. Letting $\underline{v} := \beta^\circ \underline{u}$, $v^\circ := \beta^\circ u^\circ$, $v^* := \beta^* u^*$, $\bar{v} := \beta^* \bar{u}$, we next define

$$G(v) := \begin{cases} v, & \text{for} \quad \underline{v} \leqslant v < v^\circ, \\ v^\circ + \gamma(v - v^\circ), & \text{for} \quad v^\circ < v \leqslant \bar{v}, \end{cases} \tag{4.4}$$

with $0 < \gamma < 1$ and note that G is increasing and strictly progressive over $[\underline{v}, \bar{v}]$. Choose next $0 < \varepsilon < (1 - \gamma)\, \beta_1^\circ(\beta^* - \beta_1^\circ)(u^* - u^\circ)/((1 - \gamma)\, \beta_1^\circ + \gamma\beta^*)$. Then, for all $P := (1/n) \sum_{i=1}^n \delta_{u_i}$, where $u_i \in [u^\circ, u^*]$, for all $i = 1, 2, ..., n$ $(n \geqslant 2)$ and $u_i \leqslant u_{i+1}$, for all $i = 1, 2, ..., n-1$ with a strict inequality for at least one i, we have $P^{G \circ H^\circ} >_{\mathrm{L}} P^{G \circ H^*}$.

Contrary to what may be conjectured, the progressivity of the income tax does not reinforce the redistributive incidence of the substitution of a more progressive NIC schedule for a less progressive one.

Arnold [2] and Moyes [22] have shown that the only taxation schemes that preserve after tax the Lorenz ordering of distributions involve proportional taxes. It follows that G linear is a sufficient condition for H^* more equalising than H° to imply that $G \circ H^*$ be more inequality reducing than $G \circ H^\circ$. On the one hand, this condition is too restrictive to be of any practical relevance. On the other hand, the fact that H^* is more equalising than H° tells actually a bit more than what Arnold [2] or Moyes [22] did assume. Indeed, it tells us that H^* is more progressive than H° [Proposition 3.4] and we will make extensive use of this information in what follows. We first introduce an obvious technical result that will be useful in later proofs:

LEMMA 4.1. *Let $H^\circ, H^* \in \mathscr{F}(V)$. Then $H^*(u)/H^\circ(u)$ non-increasing with u over V is equivalent to*

$$\frac{H^*(\lambda u)}{H^*(u)} \leqslant \frac{H^\circ(\lambda u)}{H^\circ(u)}, \qquad \forall \lambda \geqslant 1 \textit{ and } \forall u, \lambda u \in V. \tag{4.5}$$

In order to shorten notation, we will let $V_H := H^\circ(V) \cup H^*(V)$. Before stating our results, we need an additional definition.

DEFINITION 4.1. Let $H^\circ, H^* \in \mathscr{F}(V)$. We will say that H^* is individually welfare superior to H° over V, which we write $H^* \geq_{\mathrm{IW}} H^\circ$, if $H^*(u) \geqslant H^\circ(u)$, for all $u \in V$.

Since H° and H^* are non-decreasing, individual welfare superiority of H^* over H° is actually equivalent to the fact that P^{H^*} stochastically dominates to the first-order P^{H°, for all $P \in \mathscr{P}(V)$. Typically, this means that, whatever the pre-tax distribution, the tax every income unit is liable to is no greater under scheme H^* than under scheme H°, which is a very restrictive condition. Considering the case where a reform of the former component of the tax system implies a decrease in the tax liability of every taxpayer, we obtain:

PROPOSITION 4.2. *Let $H^\circ, H^* \in \mathscr{F}(V)$ and $G \in \mathscr{F}(V_H)$ be non-decreasing. Then the two following statements are equivalent*:

(a) *$G(\vartheta s)/G(s)$ is non-increasing with s, for all $\vartheta \geqslant 1$ and all s, $\vartheta s \in V_H$.*

(b) *$H^* \geq_{\mathrm{IW}} H^\circ$ and $H^* \geq_{\mathrm{EQ}} H^\circ$ over $\mathscr{P}(V) \Rightarrow G \circ H^* \geq_{\mathrm{EQ}} G \circ H^\circ$ over $\mathscr{P}(V_H)$.*

Proof. (a) $\Rightarrow$ (b): Since H° is non-decreasing, we deduce from condition (a) that

$$\frac{G(\vartheta H^\circ(\lambda u))}{G(H^\circ(\lambda u))} \leqslant \frac{G(\vartheta H^\circ(u))}{G(H^\circ(u))}, \tag{4.6}$$

for all $\vartheta, \lambda \geqslant 1$ and all $u, \lambda u \in V$. In particular, taking $\vartheta = H^*(\lambda u)/H^\circ(\lambda u)$, the preceeding inequality may be written

$$\frac{G(H^*(\lambda u))}{G((H^*(\lambda u)/H^\circ(\lambda u))\, H^\circ(u))} \leqslant \frac{G(H^\circ(\lambda u))}{G(H^\circ(u))}. \tag{4.7}$$

But, since H^* is more equalising than H°, appealing to Propositions 3.1 and 3.2, and Lemma 4.1, we deduce that

$$\frac{H^*(\lambda u)}{H^*(u)} \leqslant \frac{H^\circ(\lambda u)}{H^\circ(u)}. \tag{4.8}$$

Furthermore, because G is non-decreasing, using (4.7) and (4.8), we obtain

$$\frac{G(H^*(\lambda u))}{G(H^*(u))} \leqslant \frac{G(H^*(\lambda u))}{G((H^*(\lambda u)/H^\circ(\lambda u))\, H^\circ(\lambda u))} \leqslant \frac{G(H^\circ(\lambda u))}{G(H^\circ(u))}. \tag{4.9}$$

Clearly, $G \circ H^\circ$ and $G \circ H^*$ are non-decreasing as the composition of two non-decreasing schemes. Appealing to Propositions 3.1 and 3.2, we conclude that $G \circ H^* \geq_{\mathrm{EQ}} G \circ H^\circ$ over $\mathscr{P}(V)$.

(b) $\Rightarrow$ (a): We show that, if condition (b) holds, then $G(\vartheta s)/G(s)$ is non-increasing with s, for all $\vartheta \geqslant 1$ and all s, $\vartheta s \in V_H$. Let $H^\circ(t) := t$ and $H^*(t) := \vartheta t$, for all $t \in V$, with $\vartheta \geqslant 1$ and choose $P := (1/n)((n-1)\,\delta_u + \delta_v)$, with $u < v$ ($u, v \in V$). Clearly, $H^* \geq_{\mathrm{IW}} H^\circ$ and $H^* \geq_{\mathrm{EQ}} H^\circ$ over $\mathscr{P}(V)$, and therefore $G \circ H^* \geq_{\mathrm{EQ}} G \circ H^\circ$ over $P(V_H)$. But this is equivalent to

$$L_{P G \circ H^*}\left(\frac{k}{n}\right) = \frac{kG(\vartheta u)}{(n-1)\,G(\vartheta u) + G(\vartheta u)}$$

$$\geqslant \frac{kG(u)}{(n-1)\,G(u) + G(v)} = L_{P G \circ H^\circ}\left(\frac{k}{n}\right), \tag{4.10}$$

for all $k = 1, 2, ..., n-1$. Upon simplifying, (4.10) reduces to $G(\vartheta u)/G(u) \geqslant G(\vartheta v)/G(v)$, which is true, for all $\vartheta \geqslant 1$ and all u, ϑu, v, $\vartheta v \in V_H$. ∎

It is interesting to note that G need not be progressive for a progressivity improving modification of the former component of the tax system to imply an overall inequality reduction. For differentiable taxation schemes, condition (a) of Proposition 4.2 demands that the elasticity of G be non-increasing,[8] which is compatible with an elasticity everywhere greater than 1. When the tax liability unambiguously increases for every taxpayer, we obtain along a similar reasoning:

PROPOSITION 4.3. *Let $H^\circ, H^* \in \mathscr{F}(V)$ and $G \in \mathscr{F}(V_H)$ be non-decreasing. Then the two following statements are equivalent*:

(a) *$G(\vartheta s)/G(s)$ is non-decreasing with s, for all $\vartheta \geqslant 1$ and all s, $\vartheta s \in V_H$.*

(b) *$H^* \leq_{\mathrm{IW}} H^\circ$ and $H^* \geq_{\mathrm{EQ}} H^\circ$ over $\mathscr{P}(V) \Rightarrow G \circ H^* \geq_{\mathrm{EQ}} G \circ H^\circ$ over $\mathscr{P}(V_H)$.*

We find it convenient to state a intermediate technical result that will be needed in subsequent proofs:

[8] In the terminology of public finance, this is equivalent to a non-decreasing residual progression (see, e.g., Lambert [11, Chap. 6]).

LEMMA 4.2 (Aczel [1, pp. 144–145]). *Let $G \in \mathscr{F}(S)$ be non-decreasing, where $S \subseteq \mathbb{R}_+$. Then $G(s) = \beta s^\eta$, for all $s \in S$, is the only solution to the functional equation*

$$\frac{G(\vartheta s)}{G(s)} = \frac{G(\vartheta r)}{G(r)}, \qquad \forall \vartheta \geqslant 1 \text{ and } \forall r, s \in S. \tag{4.11}$$

Relaxing in a second stage the constraints imposed on the pair $\{H^\circ, H^*\}$ results in stronger conditions to be fulfilled by G for inequality to decrease after tax. Actually, G must exhibit constant residual progression everywhere as the following result indicates:

PROPOSITION 4.4. *Let $H^\circ, H^* \in \mathscr{F}(V)$ and $G \in \mathscr{F}(V_H)$ be non-decreasing. Then the two following statements are equivalent*:

(a) $G(s) = \beta s^\eta$ ($\beta > 0, \eta \geqslant 0$), *for all* $s \in V_H$.

(b) $H^* \geq_{\mathrm{EQ}} H^\circ$ *over* $\mathscr{P}(V) \Rightarrow G \circ H^* \geq_{\mathrm{EQ}} G \circ H^\circ$ *over* $\mathscr{P}(V_H)$.

Proof. (a) $\Rightarrow$ (b): Assuming that condition (a) holds, we must show that

$$\frac{H^*(\lambda u)}{H^*(u)} \leqslant \frac{H^\circ(\lambda u)}{H^\circ(u)}, \tag{4.12}$$

for all $u, \lambda u \in V$, implies that

$$\frac{G(H^*(\lambda u))}{G(H^*(u))} \leqslant \frac{G(H^\circ(\lambda u))}{G(H^\circ(u))}. \tag{4.13}$$

The proof parallels the proof of sufficiency in Proposition 4.2. Given Lemma 4.2, condition (a) implies that

$$\frac{G(\vartheta H^\circ(\lambda u))}{G(H^\circ(\lambda u))} = \frac{G(\vartheta H^\circ(u))}{G(H^\circ(u))}, \tag{4.14}$$

for all $\vartheta \geqslant 1$. In particular, taking $\vartheta = H^*(\lambda u)/H^\circ(\lambda u)$, the preceeding equality may be rewritten

$$\frac{G(H^*(\lambda u))}{G((H^*(\lambda u)/H^\circ(\lambda u))\, H^\circ(\lambda u))} = \frac{G(H^\circ(\lambda u))}{H^\circ(u)}. \tag{4.15}$$

But since G is non-decreasing, using (4.12), we obtain

$$\frac{G(H^*(\lambda u))}{H^*(u)} \leqslant \frac{G(H^*(\lambda u))}{G((H^*(\lambda u)/H^\circ(\lambda u))\, H^\circ(\lambda u))} = \frac{G(H^\circ(\lambda u))}{H^\circ(u)}. \tag{4.16}$$

Appealing once again to Propositions 3.1 and 3.2, we conclude that $G \circ H^* \geq_{\mathrm{EQ}} G \circ H^\circ$ over $\mathscr{P}(V)$.

(b) $\Rightarrow$ (a): We show that, if condition (b) holds, then $G(\vartheta s)/G(s) = G(\vartheta r)/G(r)$, for all $\vartheta > 1$ and all $s, \vartheta s, r, \vartheta r \in V_H$. Let $H^\circ(t) := t$ and $H^*(t) := \vartheta t$, for all $t \in V$, with $\vartheta \geqslant 1$. By definition, $H^* \geq_{\text{EQ}} H^\circ$ over $\mathscr{P}(V)$, and it follows from condition (b) that $G \circ H^* \geq_{\text{EQ}} G \circ H^\circ$ over $\mathscr{P}(V_H)$. In particular, choosing $P := (1/n)((n-1)\,\delta_u + \delta_v)$, with $u < v$ $(u, v \in V)$, this implies that

$$L_{P^{G \circ H^*}}\left(\frac{k}{n}\right) = \frac{kG(\vartheta u)}{(n-1)\,G(\vartheta u) + G(\vartheta v)} \geqslant \frac{kG(u)}{(n-1)\,G(u) + G(v)} = L_{P^{G \circ H^\circ}}\left(\frac{k}{n}\right), \tag{4.17}$$

for all $k = 1, 2, ..., n-1$. Letting now $H^\circ(t) := \vartheta t$ and $H^*(t) := t$, for all $t \in V$, with $\vartheta \geqslant 1$, and choosing P as above, we obtain by similar reasoning

$$L_{P^{G \circ H^*}}\left(\frac{k}{n}\right) = \frac{kG(\vartheta u)}{(n-1)\,G(\vartheta u) + G(\vartheta v)} \leqslant \frac{kG(u)}{(n-1)\,G(u) + G(v)} = L_{P^{G \circ H^\circ}}\left(\frac{k}{n}\right), \tag{4.18}$$

for all $k = 1, 2, ..., n-1$. Combining (4.17) and (4.18), we finally get $G(\vartheta u)/G(u) = G(\vartheta v)/G(v)$, which is true, for all $\vartheta \geqslant 1$ and all $u, \vartheta u, v, \vartheta v \in V_H$. Using Lemma 4.2 we conclude that $G(s) = \beta s^\eta$ $(\beta > 0,\ \eta \geqslant 0)$, for all $s \in V_H$. ▮

We insist once again on the fact that a progressive G is neither necessary nor sufficient for overall inequality to decrease when a more progressive H^* is substituted for a less progressive H°. In addition, Propositions 4.2 to 4.4 indicate the nature of the trade-off between the restrictions we impose on the pair $\{H^\circ, H^*\}$ and the restrictions on G we obtain as a result.

We finally consider the more complex situation where $H^* \neq H^\circ$ and $G^* \neq G^\circ$. Contrary to what intuition would suggest, an increase in the degree of progressivity of the income tax does not guarantee that a modification of the NIC schedule which is progressivity neutral will result in more equally distributed post-tax incomes, as the following example makes clear.

EXAMPLE 4.3. Let $H^\circ(u) := \beta^\circ u, H^*(u) := \beta^* u$, with $0 < \beta^\circ < \beta^* = (1+\varepsilon)\,\beta^\circ < 1$ and $0 < \varepsilon < 1$. Clearly, H° and H^* are increasing, and H^* is weakly more progressive than H°. Choose $u^\circ, u^* \in V$ such that $\underline{u} < u^\circ < u^* < \bar{u}$ and let $\underline{v} := \beta^\circ \underline{u}$, $v^\circ := \beta^\circ u^\circ$, $v^* := \beta^* u^*$, $\bar{v} := \beta^* \bar{u}$. Consider next

$$G^\circ(v) := \begin{cases} v, & \text{for } \ \underline{v} \leqslant v < v^\circ, \\ v^\circ + \gamma^\circ(v - v^\circ), & \text{for } \ v^\circ < v \leqslant \bar{v}, \end{cases} \tag{4.19}$$

and

$$G^*(v) := \begin{cases} v, & \text{for} \quad \underline{v} \leqslant v < w^\circ, \\ w^\circ + \gamma^*(v - w^\circ), & \text{for} \quad w^\circ < v \leqslant \bar{v}, \end{cases} \tag{4.20}$$

with $0 < \gamma^* = (1-\varepsilon)\,\gamma^\circ < \gamma^\circ < 1$ and $w^\circ < v^\circ$. Then G° and G^* are increasing, and G^* is more progressive than G° (actually, G^* is strictly more progressive than G° over $[v^\circ, \bar{v}]$). However, for all $P := (1/n) \sum_{i=1}^n \delta_{u_i}$, where $u_i \in [u^\circ, u^*]$, for all $i = 1, 2, ..., n$ $(n \geqslant 2)$ and $u_i \leqslant u_{i+1}$, for all $i = 1, 2, ..., n-1$ with a strict inequality for at least one i, we have $P^{G^\circ \circ H^\circ} >_{\mathrm{L}} P^{G^* \circ H^*}$.

Considering the case where a reform of the former component of the tax system implies a decrease in the tax liability of every taxpayer, we check that a non-increasing elasticity of after tax income is both necessary and sufficient for an overall inequality reduction after tax. Precisely:

PROPOSITION 4.5. *Let $H^\circ, H^* \in \mathscr{F}(V)$ and $G^\circ, G^* \in \mathscr{F}^*(V_H)$ be non-decreasing. Then the two following statements are equivalent*:

(a) *There exists a non-decreasing ϕ: $V_H \to \mathbb{R}_+$, with $\phi^-(s)\, s/\phi(s)$ non-increasing with s, for all $s \in V_H$, such that $G^* \geq_{\mathrm{P}} \phi \geq_{\mathrm{P}} G^\circ$.*

(b) *$H^* \geq_{\mathrm{IW}} H^\circ$ and $H^* \geq_{\mathrm{EQ}} H^\circ$ over $\mathscr{P}(V) \Rightarrow G^* \circ H^* \geq_{\mathrm{EQ}} G^\circ \circ H^\circ$ over $\mathscr{P}(V)$.*

Proof. (a) $\Rightarrow$ (b): Assuming that (a) holds, we deduce from Proposition 4.2 that $\phi \circ H^* \geq_{\mathrm{EQ}} \phi \circ H^\circ$ over $\mathscr{P}(V)$. Similarly, it follows from Propositions 3.4 and 4.1 that $\phi \circ H^\circ \geq_{\mathrm{EQ}} G^\circ \circ H^\circ$ and $G^* \circ H^* \geq_{\mathrm{EQ}} \phi \circ H^*$, over $\mathscr{P}(V)$. Upon combining, we obtain $G^* \circ H^* \geq_{\mathrm{EQ}} G^\circ \circ H^\circ$ over $\mathscr{P}(V)$.

(b) $\Rightarrow$ (a): We show that, if condition (b) holds, then we can find a non-decreasing ϕ with $\phi^-(r)\, r/\phi(r) \geqslant \phi^-(s)\, s/\phi(s)$, for all $r, s \in V_H$ $(r < s)$, such that $G^* \geq_{\mathrm{P}} \phi \geq_{\mathrm{P}} G^\circ$. Let $H^\circ(t) := \lambda^\circ t$ and $H^*(t) := \lambda^* t$, for all $t \in V$, with $0 < \lambda^\circ \leqslant \lambda^*$ (Note that $V_H = H^*(V^*)$). Clearly, $H^* \geq_{\mathrm{IW}} H^\circ$ and $H^* \geq_{\mathrm{EQ}} H^\circ$ over $\mathscr{P}(V)$. Choose next $P := (1/n)((n-1)\,\delta_u + \delta_v)$ with $u, v \in V^*$ $(u < v)$. Assuming that condition (b) holds, we have $G^* \circ H^* \geq_{\mathrm{EQ}} G^\circ \circ H^\circ$ over $\mathscr{P}(V)$, hence $P^{G^* \circ H^*} \geq_{\mathrm{L}} P^{G^\circ \circ H^\circ}$. Since G° and G^* are non-decreasing, this implies in turn that

$$L^{P^{G^* \circ H^*}}\left(\frac{k}{n}\right) = \frac{kG^*(\lambda^* u)}{(n-1)\, G^*(\lambda^* u) + G^*(\lambda^* v)}$$

$$\geqslant \frac{kG^\circ(\lambda^\circ u)}{(n-1)\, G^\circ(\lambda^\circ u) + G^\circ(\lambda^\circ v)} = L^{P^{G^\circ \circ H^\circ}}\left(\frac{k}{n}\right), \tag{4.21}$$

for all $k = 1, 2, ..., n-1$, from which we deduce that

$$\frac{G^\circ(\lambda^\circ v)}{G^\circ(\lambda^\circ u)} \geqslant \frac{G^*(\lambda^* v)}{G^*(\lambda^* u)} \quad (\geqslant 1). \tag{4.22}$$

Taking the logarithms of (4.22) and using the fact that $u < v$, we obtain

$$\frac{\ln G^\circ(\lambda^\circ v) - \ln G^\circ(\lambda^\circ u)}{\ln \lambda^\circ v - \ln \lambda^\circ u} \geqslant \frac{\ln G^*(\lambda^* v) - \ln G^*(\lambda^* u)}{\ln \lambda^* v - \ln \lambda^* u} \quad (\geqslant 0), \tag{4.23}$$

which is true for all $0 < \lambda^\circ \leqslant \lambda^*$ and all $u < v$ $(u, v \in V)$. Letting $\vartheta = (v/u) > 1$, this may be written equivalently

$$\frac{\ln G^\circ(\vartheta\lambda^\circ u) - \ln G^\circ(\lambda^\circ u)}{\ln \vartheta} \geqslant \frac{\ln G^*(\vartheta\lambda^* u) - \ln G^*(\lambda^* u)}{\ln \vartheta} \quad (\geqslant 0), \tag{4.24}$$

which is true for all $\vartheta \geqslant 1$ and all $\lambda^\circ u \leqslant \lambda^* u$ $(\lambda^\circ u, \lambda^* u \in V_H)$. Taking the limit of both sides of (4.24) when $\vartheta \to 1$, we deduce that

$$\min\left\{\frac{G^{\circ -}(r)\, r}{G^\circ(r)}, \frac{G^{\circ +}(r)\, r}{G^\circ(r)}\right\} \geqslant f(s) := \max\left\{\frac{G^{* -}(s)\, s}{G^*(s)}, \frac{G^{* +}(s)\, s}{G^*(s)}\right\}, \tag{4.25}$$

for all $r \leqslant s$ $(r, s \in V_H)$. Define next

$$\tilde{f}(r) := \max_{r \leqslant s} \{f(s) \mid s \in V_H\}, \qquad \forall r \in V_H, \tag{4.26}$$

and note that, by construction, $\tilde{f}$ is non-increasing and non-negative over V_H. Finally, we let

$$\phi(s) := \exp \int_{\underline{s}}^{s} \frac{\tilde{f}(r)}{r}\, dr, \qquad \forall s \in V_H, \tag{4.27}$$

where $\underline{s} = \inf\{V_H\}$. Therefore, we have identified a non-decreasing function $\phi \in \mathscr{F}^*(V_H)$ with non-increasing elasticity such that $G^* \geq_{\mathrm{P}} \phi \geq_{\mathrm{P}} G^\circ$, which makes the proof complete. ∎

We note that, though an increase in the degree of progressivity of the income tax is not sufficient for the inequality reducing effect of a modification of the NIC to be preserved, it is however necessary. When the tax liability unambiguously increases for every taxpayer, we show along a similar reasoning that a non-decreasing elasticity of after tax income is necessary and sufficient for more equally distributed after tax incomes.

PROPOSITION 4.6. *Let $H^\circ, H^* \in \mathscr{F}(V)$ and $G^\circ, G^* \in \mathscr{F}^*(V_H)$ be non-decreasing. Then the two following statements are equivalent*:

(a) *There exists a non-decreasing* $\phi\colon V_H \to \mathbb{R}_+$, *with* $\phi^-(s)\, s/\phi(s)$ *non-decreasing with s, for all* $s \in V_H$, *such that* $G^* \geq_{\mathrm{P}} \phi \geq_{\mathrm{P}} G^\circ$.

(b) $H^* \leq_{\mathrm{IW}} H^\circ$ *and* $H^* \geq_{\mathrm{EQ}} H^\circ$ *over* $\mathscr{P}(V) \Rightarrow G^* \circ H^* \geq_{\mathrm{EQ}} G^\circ \circ H^\circ$ *over* $\mathscr{P}(V)$.

Without a suitable amendment of the income tax schedule, a progressivity-enhancing modification of the NIC is unlikely to reduce after tax income differentials. Propositions 4.5 and 4.6 tell precisely how the income tax schedule should be adjusted for some general transformations of the NIC to be overall inequality reducing. Relaxing in a second stage the constraints imposed on the pair $\{H^\circ, H^*\}$ results in stronger conditions to be fulfilled by G for inequality to decrease after tax. As the following result indicates, a necessary and sufficient condition for an overall inequality reduction after tax is that there exists an isoelastic taxation scheme ϕ such that G^* is more progressive than ϕ and ϕ in turn is more progressive than G°. Precisely:

PROPOSITION 4.7. *Let* $H^\circ, H^* \in \mathscr{F}(V)$ *and* $G^\circ, G^* \in \mathscr{F}(V_H)$ *be non-decreasing. Then the two following statements are equivalent*:

(a) *There exists* $\phi(s) := \beta s^\eta$ $(\beta > 0,\ \eta \geqslant 0)$, *for all* $s \in V_H$, *such that* $G^* \geq_{\mathrm{P}} \phi \geq_{\mathrm{P}} G^\circ$.

(b) $H^* \geq_{\mathrm{EQ}} H^\circ$ *over* $\mathscr{P}(V) \Rightarrow G^* \circ H^* \geq_{\mathrm{EQ}} G^\circ \circ H^\circ$ *over* $\mathscr{P}(V)$.

Proof. (a) $\Rightarrow$ (b): Assuming that (a) holds, we deduce from Proposition 4.4 that $\phi \circ H^* \geq_{\mathrm{EQ}} \phi \circ H^\circ$ over $\mathscr{P}(V)$. Similarly, it follows from Propositions 3.4 and 4.1 that $\phi \circ H^\circ \geq_{\mathrm{EQ}} G^\circ \circ H^\circ$ and $G^* \circ H^* \geq_{\mathrm{EQ}} \phi \circ H^*$, over $\mathscr{P}(V)$. Upon combining, we obtain $G^* \circ H^* \geq_{\mathrm{EQ}} G^\circ \circ H^\circ$ over $\mathscr{P}(V)$.

(b) $\Rightarrow$ (a): We show that, if condition (b) holds, then we can find a non-decreasing ϕ with $\phi(\vartheta r)/\phi(r) = \phi(\vartheta s)/\phi(s)$, for all $r, s \in V_H$ and all $\vartheta \geqslant 1$, such that $G^* \geq_{\mathrm{P}} \phi \geq_{\mathrm{P}} G^\circ$. Let $H^\circ(t) := \lambda^\circ t$ and $H^*(t) := \lambda^* t$, for all $t \in V$, with $\lambda^\circ, \lambda^* > 0$. Clearly, $H^* \geq_{\mathrm{EQ}} H^\circ$ over $\mathscr{P}(V)$. Choose next $P := (1/n)((n-1)\,\delta_u + \delta_v)$ with $u, v \in V$ $(u < v)$. Assuming that condition (b) holds, we have $G^* \circ H^* \geq_{\mathrm{EQ}} G^\circ \circ H^\circ$ over $\mathscr{P}(V)$, hence $P^{G^* \circ H^*} \geq_{\mathrm{L}} P^{G^\circ \circ H^\circ}$. Since G° and G^* are non-decreasing, this implies in turn that

$$L_{P^{G^* \circ H^*}}\left(\frac{k}{n}\right) = \frac{kG^*(\lambda^* u)}{(n-1)\, G^*(\lambda^* u) + G^*(\vartheta \lambda^* u)}$$

$$\geqslant \frac{kG^\circ(\lambda^\circ u)}{(n-1)\, G^\circ(\lambda^\circ u) + G^\circ(\vartheta \lambda^\circ u)} = L_{P^{G^\circ \circ H^\circ}}\left(\frac{k}{n}\right), \tag{4.28}$$

for all $k=1, 2, ..., n-1$, where $\vartheta = v/u > 1$. This is equivalent to

$$\frac{G^\circ(\vartheta\lambda^\circ u)}{G^\circ(\lambda^\circ u)} \geqslant \frac{G^*(\vartheta\lambda^* u)}{G^*(\lambda^* u)} \quad (\geqslant 1). \tag{4.29}$$

Setting $r := \lambda^\circ u$ and $s := \lambda^* u$ and taking the logarithms of (4.29), we deduce that there exists $\eta \geqslant 0$ such that

$$\frac{\ln G^\circ(\vartheta r) - \ln G^\circ(r)}{\ln \vartheta} \geqslant \eta \geqslant \frac{\ln G^*(\vartheta s) - \ln G^*(s)}{\ln \vartheta}, \tag{4.30}$$

for all $r, s \in V_H$ and all $\vartheta \geqslant 1$. Letting $\phi(z) := \beta z^\eta$ ($\beta > 0$ arbitrary), (4.30) can be rewritten equivalently as

$$\frac{G^\circ(\vartheta r)}{G^\circ(r)} \geqslant \frac{\phi(\vartheta r)}{\phi(r)} = \vartheta^\eta = \frac{\phi(\vartheta s)}{\phi(s)} \geqslant \frac{G^*(\vartheta s)}{G^*(s)}, \tag{4.31}$$

for all $r, s \in V_H$ and all $\vartheta \geqslant 1$. Therefore, we have identified a non-decreasing isoelastic function ϕ, such that $G^* \geq_{\mathrm{P}} \phi \geq_{\mathrm{P}} G^\circ$, which makes the proof complete. ▌

The conditions we obtained for a composite tax system to be overall inequality reducing involve roughly speaking the elasticity of residual income and are extremely restrictive. We do not think, however, one need be too pessimistic regarding our results. In particular, we believe they may provide useful guidelines for the policy maker. For instance, they indicate the way the income tax schedule should be adjusted for an unambiguously inequality reducing increase in national insurance contributions not to be thwarted after tax.

5. Inequality Reduction and Composite Taxation Schemes under Revenue Neutrality

In many situations, the policy maker is involved in the comparison of alternative tax proposals that do not modify the tax revenue to be raised. How does this *revenue neutrality* constraint affect the results we derived in the previous sections?

For the sake of exposition, it is convenient to examine first the situation where the objective of the tax authorities is only to redistribute income accross the tax units so that the tax revenue is zero. Given a taxation scheme $G \in \mathscr{F}(V)$ and a subset $\mathscr{L}(V) \subseteq \mathscr{P}(V)$, we denote as

$$\mathscr{L}_G(V) := \left\{ P \in \mathscr{L}(V) \,\middle|\, \int_V (G(u) - u)\, dP(u) = 0 \right\} \tag{5.1}$$

the set of income distributions with the property that aggregate income is not modified by taxation.[9] Within this particular framework, it is then natural to require equalisation not over the whole set of distributions but on the subset of distributions with the property that aggregate income is not modified after tax. Precisely, we have:

DEFINITION 5.1. Let $\mathscr{L}(V) \subseteq \mathscr{P}(V)$ and $G \in \mathscr{F}(V)$. We will say that G is *equalising* (*or inequality reducing*) over $\mathscr{L}_G(V)$ if $P^G \geq_L P$, for all $P \in \mathscr{L}_G(V)$.

When no restriction is placed on aggregate after tax income, it has been shown that progressivity and rank preservation are necessary and sufficient for post-tax incomes to be more equally distributed than before tax incomes (Propositions 3.1 and 3.3). As we will see in a while, equalisation in the narrower sense of the definition above implies much weaker conditions to be satisfied by a taxation scheme. The following property will play a crucial role in the context of purely redistributive taxation schemes.

DEFINITION 5.2. Let $G \in \mathscr{F}(V)$. We will say that G satisfies the *single-crossing condition* if there exists $u^\circ \in V$ such that $G(u) - u \geqslant 0$, for all $u \leqslant u^\circ$ and $G(u) - u \leqslant 0$, for all $u^\circ \leqslant u$.

Restricting attention to *non-decreasing taxation schemes* and *continuous distributions*, Hemming and Keen [7] (see also Thistle [26]) have shown that, in a purely redistributive context, the single-crossing condition is necessary and sufficient for after tax incomes to be more equally distributed than before tax incomes.[10] One may question whether this interesting result does still hold for more general classes of pre-tax distributions and/or taxation schemes. On the one hand, continuous distributions may be considered reasonable approximations of real world income distributions only in the case of arbitrary large populations. On the other hand, many tax systems in the real world involve a reranking of tax units so that post-tax incomes are decreasing with pre-tax incomes, at least locally.

DEFINITION 5.3. Let $G \in \mathscr{F}(V)$ and assume that G satisfies the single-crossing condition. Then we will say that G is *class monotone* if $G(u) \leqslant u^*$,

[9] Given any arbitrary $G \in \mathscr{F}(V)$, it may perfectly be the case that $\mathscr{L}_G(V)$ be empty. For $\mathscr{L}_G(V)$ not to be empty, G must cross at least once the diagonal in the $V \times \mathbb{R}_+$ space.

[10] It is fair to point out the fact that the idea of single crossing goes back at least to Marshall *et al.* [18, Proposition 2.4]. Actually, the single-crossing condition is used as an intermediate step in the proof of Jakobsson [8, Proposition 1].

for all $u \leqslant u^\circ$ and $G(u) \geqslant u^\circ$, for all $u^* \leqslant u$, where $u^\circ := \text{Sup}\{u \mid G(u) > u\}$ and $u^* := \text{Inf}\{u \mid G(u) < u\}$.

We will say that a tax unit with pre-tax income u is contributing [benefiting] under $G \in \mathscr{F}(V)$ if $G(u) - u < 0$ [$G(u) - u > 0$], which allows one to distinguish two classes of tax units. Then class monotonicity requires that the after tax income of any benefiting individual cannot be larger than the before tax income of any contributing individual. Conversely, the after tax income of any contributing individual cannot be smaller than the before tax income of any benefiting individual. Therefore, a reranking of the income units is allowed for under class monotonicity, but its extent is limited. We insist that single crossing and class monotonicity are not independent, since by definition class monotonicity makes sense only for those taxation schemes that fulfill the single-crossing condition. The single-crossing and the class monotonicity conditions are much weaker conditions than progressivity and rank preservation respectively in the context of purely redistributive taxation schemes as the following example makes clear.

EXAMPLE 5.1. Given $V \subseteq \mathbb{R}_+$, choose $u^\circ, u^* \in V$ such that $u^\circ - \underline{u} = u^* - u^\circ = \bar{u} - u^* = \zeta > 0$ and define

$$G(u) := \begin{cases} u + \zeta, & \text{for} \quad \underline{u} \leqslant u < u^\circ, \\ u, & \text{for} \quad u^\circ \leqslant u \leqslant u^*, \\ u - \zeta, & \text{for} \quad u^* < u \leqslant \bar{u}. \end{cases} \tag{5.2}$$

Clearly, G verifies single crossing and class monotonicity, but G is not progressive, monotone, nor even continuous.

The next result indicates that single crossing and class monotonicity imply and are implied by equalisation under the zero revenue constraint.[11]

PROPOSITION 5.1. *Let $G \in \mathscr{F}(V)$. Then the two following statements are equivalent*:

(a) *G is equalising over $\mathscr{P}_G(V)$.*

(b) *G verifies the single-crossing and the class monotonicity conditions.*

[11] The proof of this result, which is rather lengthy, is omitted; we refer the interested reader to Le Breton *et al.* [15].

It should be stressed that the fact that a taxation scheme is equalising over $\mathscr{D}_G(V)$ does not guarantee that it is also equalising over $\mathscr{P}_G(V)$ as the following example demonstrates.

EXAMPLE 5.2. Given $V \subseteq \mathbb{R}_+$ and $u^\circ, u^* \in V$ $(u^\circ < u^*)$, define

$$G(u) := \begin{cases} u, & \text{for} \quad u \neq u^\circ, u^*, \\ u - \zeta, & \text{for} \quad u = u^\circ, \\ u + \eta, & \text{for} \quad u = u^*, \end{cases} \tag{5.3}$$

where $\zeta, \eta > 0$ and ζ/η is not a rational number. It follows that

$$\mathscr{D}_G(V) := \left\{ P = \frac{1}{n} \sum_{k=1}^{n} \delta_{u_k} \,\middle|\, \text{Supp } P \cap \{u^\circ, u^*\} = \varnothing \right\}. \tag{5.4}$$

Clearly, G is equalising over $\mathscr{D}_G(V)$ but, since G does not verify the single-crossing condition, it is not equalising over $\mathscr{P}_G(V)$.

This arises because, in the purely redistributive case, continuity is not a necessary condition for equalisation over $\mathscr{P}_G(V)$. If $G \in \mathscr{F}(V)$ is continuous, then single crossing and class monotonicity are necessary and sufficient conditions for G to be inequality reducing over $\mathscr{D}_G(V)$.

We consider next the case of a revenue neutral tax reform so that aggregate post-tax income is not modified when substituting taxation scheme G for taxation scheme H. Given $G, H \in \mathscr{F}(V)$ and a subset $\mathscr{L}(V) \subseteq \mathscr{P}(V)$, we denote as

$$\mathscr{L}_{G,H}(V) := \left\{ P \in \mathscr{L}(V) \,\middle|\, \int_V (G(u) - H(u))\, dP(u) = 0 \right\} \tag{5.5}$$

the set of income distributions such that aggregate income is not modified when taxation scheme G is substituted for taxation scheme H.

DEFINITION 5.4. Let $G, H \in \mathscr{F}(V)$ and let $\mathscr{L}(V) \subseteq \mathscr{D}(V)$. Then we will say that G *is more equalising than* H, which we write $G \geq_{\text{EQ}} H$, over $\mathscr{L}_{G,H}(V)$, if $P^G \geq_{\text{L}} P^H$, for all $P \in \mathscr{L}_{G,H}(V)$.

Our purpose here is to identify the set of conditions to be imposed on the pair $\{G, H\}$ that guarantee that G be more equalising than H over $\mathscr{L}_{G,H}(V)$. The two following conditions generalise in a natural way single crossing and class monotonicity when two alternative taxation schemes are to be compared.

DEFINITION 5.5. Let $G, H \in \mathscr{F}(V)$. We will say that G *single crosses* H if there exists $u^\circ \in V$ such that $G(u) - H(u) \geqslant 0$, for all $u \leqslant u^\circ$ and $G(u) - H(u) \leqslant 0$, for all $u^\circ \leqslant u$.

DEFINITION 5.6. Let $G, H \in \mathscr{F}(V)$ and assume that G single crosses H. Then we will say that G *class dominates* H if $G(u) \leqslant h^*$, for all $u \leqslant h^\circ$ and $G(u) \geqslant h^\circ$, for all $h^* \leqslant u$, with $h^\circ := \mathrm{Sup}\{H(u) \mid G(u) > H(u)\}$ and $h^* := \mathrm{Inf}\{H(u) \mid G(u) < H(u)\}$.

The following result, whose proof parallels the proof of Proposition 5.1, indicates that single crossing and class domination are necessary and sufficient conditions for a taxation scheme to be more inequality reducing than another one under revenue neutrality.

PROPOSITION 5.2. *Let $G, H \in \mathscr{F}(V)$. Then the two following statements are equivalent*:

(a) $G \geq_{\mathrm{EQ}} H$ *over* $\mathscr{P}_{G,H}(V)$.

(b) *G single crosses and class dominates H.*

Turning finally to the case of composite taxation schemes, we would like to identify the conditions that guarantee that an equality-improving modification of an element of the tax system will imply an overall inequality reduction, imposing tax revenue neutrality at each stage of the taxation process. Restricting attention to tax systems of the form (H°, G) and (H^*, G), we obtain:

PROPOSITION 5.3. *Let $H^\circ, H^* \in \mathscr{F}(V)$ and $G \in \mathscr{F}(V_H)$ be non-decreasing. Then the two following statements are equivalent*:

(a) $G(u) = \alpha + \beta u$ $(\beta > 0, \alpha \geqslant -\beta \underline{u})$, *for all* $u \in V_H$.

(b) $H^* \geq_{\mathrm{EQ}} H^\circ$ *over* $\mathscr{P}_{H^\circ, H^*}(V) \Rightarrow G \circ H^* \geq_{\mathrm{EQ}} G \circ H^\circ$ *over* $\mathscr{P}_{G \circ H^\circ, G \circ H^*}(V)$.

Proof. Since the implication (a) $\Rightarrow$ (b) is obvious, we concentrate on the proof that (b) $\Rightarrow$ (a). We argue directly and suppose that condition (b) holds. Choose first $u, v, w \in V$ with $u < v \leqslant w$ and let $P := (1/n)(\delta_u + \delta_v + (n-2)\,\delta_w)$. Define then $H^\circ(s) := s$, for all $s \in V$ and

$$H^*(s) := \begin{cases} (u+v)/2, & \text{for } s \in [u, v] \\ s, & \text{for } s \in [\underline{u}, u) \cup (v, \bar{u}]. \end{cases} \tag{5.6}$$

By construction, $P \in \mathscr{P}_{H^\circ, H^*}(V)$ and H^* single crosses and class dominates H°. Appealing to Proposition 5.2, we conclude that $P^{H^*} \geq_{\mathrm{L}} P^{H^\circ}$ over $\mathscr{P}_{H^\circ, H^*}(V)$. Given our assumption, this implies in turn that

$P^{G \circ H^*} \geq_{\mathrm{L}} P^{G \circ H^\circ}$ over $\mathscr{P}_{H^\circ, H^*}(V)$. In particular, for $t = 2/n$, we must have

$$L_{P^{G \circ H^*}}(t) = \frac{2G((u+v)/2)}{2G((u+v)/2) + (n-2)\, G(w)}$$

$$\geqslant \frac{G(u)+G(v)}{G(u)+G(v)+(n-2)\, G(w)} = L_{P^{G \circ H^\circ}}(t), \tag{5.7}$$

which reduces to $2G((u+v)/2) \geqslant G(u)+G(v)$. Choose next $u, v, w \in V$ with $w \leqslant u < v$ and let $P := (1/n)((n-2)\, \delta_w + \delta_u + \delta_v)$. Defining H° and H^* as above, we observe that by construction $P \in \mathscr{P}_{H^\circ, H^*}(V)$ and H^* single crosses and class dominates H°. Therefore, we conclude that $P^{G \circ H^*} \geq_{\mathrm{L}} P^{G \circ H^\circ}$ over $\mathscr{P}_{H^\circ, H^*}(V)$. In particular, for $t = (n-2)/n$, we must have

$$L_{P^{G \circ H^*}}(t) = \frac{(n-2)\, G(w)}{2G((u+v)/2) + (n-2)\, G(w)}$$

$$\geqslant \frac{(n-2)\, G(w)}{G(u)+G(v)+(n-2)\, G(w)} = L_{P^{G \circ H^\circ}}(t), \tag{5.8}$$

which reduces to $2G((u+v)/2) \leqslant G(u)+G(v)$. Therefore, G must verify the functional equation

$$G\left(\frac{u+v}{2}\right) = \frac{G(u)+G(v)}{2}, \qquad \forall u, v \in V_H. \tag{5.9}$$

Since G is non-decreasing by assumption, we deduce that there is at most a countable set of discontinuities and $G(u) = \alpha + \beta u$ ($\beta > 0$, $\alpha \geqslant -\beta \underline{u}$), for all $u \in V_H$, is the only solution to (5.9) (see Aczel [1]). ∎

Revenue neutrality at each stage of the taxation process is a natural condition to impose when the different tax schedules are under the control of independent administrative bodies. One may think of a decentralized system where the government fixes the tax revenues to be collected by the different administrations but leaves them to choose the most appropriate way of raising these funds. The result above indicates the restrictions that are to be placed on the later stage tax schedule for after tax inequality to decrease when the former stage tax schedule is made more progressive. From a more technical point of view, requiring revenue neutrality only at the later stage of the taxation process results in rather uninteresting situations. Indeed, suppose we want that

$$\int_V (G \circ H^*(u))\, dP(u) = \int_V (G \circ H^\circ(u))\, dP(u), \qquad \forall P \in \mathscr{P}(V). \tag{5.10}$$

Choosing $P := (1/n) \sum_{i=1}^{n} \delta_{u_i}$, with $u_1 = \cdots = u_n = u$ ($u \in V$), we deduce from (5.10) that $G \circ H^* = G \circ H^\circ$. It follows that either G is a constant function, when $H^* \neq H^\circ$, or $H^* = H^\circ$, when G is increasing.

6. Summary and Concluding Remarks

The aim of the paper was to investigate the overall inequality reducing properties of the tax systems involving different stages of taxation. For the sake of generality, we focused on the set of probability measures which comprises, as particular cases, discrete and continuous distributions.

We derive in Section 3 a number of results concerning elementary taxation schemes that generalise to some extent the results in the literature. In particular, we prove that there is no loss of generality in focusing on purely discrete distributions when one is interested in the inequality reducing properties of taxation. We also prove the equivalence of the different definitions of a more progressive taxation scheme proposed in the literature. We exploit these results in Section 4 where we consider the equalising properties of composite tax systems. Precisely, we indicate necessary and sufficient conditions for a tax system (H^*, G^*) to be more inequality reducing than a tax system (H°, G°). Contrary to what intuition suggests, increasing the degree of progressivity of one element of the tax system may well imply an increase in inequality after tax. Our results stress the key role played by the elasticity of the later stage taxation scheme within the inequality reducing process. Precisely, when the later tax schedule is fixed, we showed that:

(i) A constant elasticity is both necessary and sufficient for an increase in the degree of progressivity of the former stage taxation scheme to generate after tax a reduction in inequality.

(ii) This condition can be substantially weakened if one requires in addition that the new schedule is nowhere below (resp. above) the old schedule in the former stage of taxation. In this case, a non-increasing (resp. non-decreasing) elasticity is both necessary and sufficient for overall inequality to be reduced.

When the later tax schedule is allowed to vary, our results indicate the way the adjustment should be made in order that overall inequality decreases when one substitutes a more progressive schedule for a less progressive one in the former stage of the taxation process.

(iii) The later taxation scheme must be adjusted in such a way that there exists a non-decreasing and isoelastic function ϕ which separates G° and G^* according to the *more progressive than* ordering.

(iv) If one requires in addition that the new schedule is nowhere below (resp. above) the old schedule in the former stage of taxation, then the separating function ϕ needs only exhibit a non-increasing (resp. non-decreasing elasticity).

Our results may at first sight appear limited compared to other results in the literature (in particular the public finance literature). We have deliberately decided to search for uncontroversial results which do not depend to some degree on particular features such as the pre-tax distribution. This is to be contrasted with the *conditional approach* (see, e.g., Lambert and Pfähler [12]) where the conditions obtained typically involve the pre-tax distribution and the tax schedule. We believe that our results, though limited as they are, may add to the existing knowledge, shedding some light on the way the different elements of the current tax systems do interact when determining the overall redistributive effect of taxation.

Throughout the paper, we have restricted attention to non-negative incomes with the further requirement that aggregate income be positive. Distributions allowing for negative incomes are generally excluded from most studies on income inequality because the relative inequality indices, and therefore the Lorenz ordering, are not defined for distributions with zero means. Considering only positive incomes rules out such a possibility, but nothing prevents us from considering distributions with some negative incomes as long as the means of these distribution are positive. Indeed, one obtains a new Lorenz curve which possesses most of the properties of the traditional Lorenz curve: it is convex and scale invariant, and any progressive transfer moves the curve upwards. However, this modified version of the Lorenz curve is no longer increasing with the population share; actually, it crosses once from below the abscissa at some point $0 < t^* < 1$. Though the interpretation of this new Lorenz curve is not straightforward, there is however no technical objection to the use of the Lorenz criterion for comparing distributions with negative incomes as long as means are positive. The problem is that the definition of progressivity, which originates in the concept of star-shaped functions with respect to the origin (see Marshall and Olkin [17]), cannot be extended in such a way that the equivalence between progressive taxation and inequality-reducing taxation survives when negative incomes are allowed for. However, we insist on the fact that the results obtained under the revenue neutrality constraint remain valid in the case of distributions with negative incomes. We finally note that the approach to inequality measurement which focuses on absolute differentials rather than on relative differentials does not raise this kind of problem. The equivalence between more equally distributed after tax incomes, according to the absolute Lorenz criterion, and a non-decreasing tax liability (Moyes [21]) still holds for distributions with negative incomes, and possibly negative means.

Appendix

Proposition 3.1. *Let $G \in \mathscr{F}(V)$. Then G is equalising over $\mathscr{D}(V)$ if, and only if*:

(a) *$G(u)/u$ is non-increasing with u over V^*.*

(b) *G is non-decreasing with u over V.*

Proof of Proposition 3.1.

Sufficiency Part. There are two cases to be considered.

Case 1. $\underline{u} \neq 0$. The proof is given in Marshall *et al.* [18, Proposition 2.4].

Case 2. $\underline{u} = 0$. Consider the distribution $P := (1/n)((q-1)\,\delta_0 + \sum_{j=q}^{n} \delta_{u_j})$ with $0 < u_q \leqslant u_{q+1} \leqslant \cdots \leqslant u_n$. We must show that, for all $k = q, q+1, ..., n$,

$$\frac{(q-1)\,G(0) + \sum_{j=q}^{k} G(u_j)}{(q-1)\,G(0) + \sum_{j=q}^{n} G(u_j)} \geqslant \frac{\sum_{j=q}^{k} u_j}{\sum_{j=q}^{n} u_j}, \tag{A.1}$$

which, upon computation, reduces to

$$\left((q-1)\,G(0) + \sum_{j=q}^{k} G(u_j)\right)\left(\sum_{i=k+1}^{n} u_i\right) \geqslant \left(\sum_{i=k+1}^{n} G(u_i)\right)\left(\sum_{j=q}^{k} u_j\right). \tag{A.2}$$

Consider now the distribution $Q := 1/(n-q) \sum_{i=q}^{n} \delta_{u_i}$. Appealing to Marshall *et al.* [18, Proposition 2.4], we obtain $Q^G \geq_{\mathrm{L}} Q$, i.e.,

$$\frac{\sum_{j=q}^{k} G(u_j)}{\sum_{i=q}^{n} G(u_i)} \geqslant \frac{\sum_{j=q}^{k} u_j}{\sum_{i=q}^{n} u_i} \tag{A.3}$$

or equivalently

$$\left(\sum_{j=q}^{k} G(u_j)\right)\left(\sum_{i=k+1}^{n} u_i\right) \geqslant \left(\sum_{i=k+1}^{n} G(u_i)\right)\left(\sum_{j=q}^{k} u_j\right). \tag{A.4}$$

Combining (A.4) and the fact that by definition $G(s) \geqslant 0$ gives precisely (A.1).

Necessity Part. We first prove that G equalising over $\mathscr{D}_n(V)$ implies that $G(u)/u$ is non-decreasing with u over V^*. Consider any $u, v \in V^*$ with $u < v$. If $G(u) \geqslant G(v)$, then obviously $G(u)/u \geqslant G(v)/v$. Therefore, we restrict ourselves to the case where $G(u) < G(v)$. Take $P \in \mathscr{D}_n(V)$ defined by $P := (1/n)(\delta_u + (n-1)\,\delta_v)$. Since $P^G \geq_{\mathrm{L}} P$, the income share of the $(1/n) \times 100\ \%$ poorest individuals is not smaller after tax than their income share before tax, i.e.,

$$L_P\left(\frac{1}{n}\right) = \frac{u}{u + (n-1)\,v} \leqslant \frac{G(u)}{G(u) + (n-1)\,G(v)} = L_{P^G}\left(\frac{1}{n}\right). \tag{A.5}$$

Similarly, the income share of the $(1/n) \times 100\ \%$ richest individuals is not greater after tax than their income share before tax, i.e.,

$$\begin{aligned} 1 - L_P\left(\frac{n-1}{n}\right) &= \frac{v}{u + (n-1)\,v} \\ &\geqslant \frac{G(v)}{G(u) + (n-1)\,G(v)} = 1 - L_{P^G}\left(\frac{n-1}{n}\right). \end{aligned} \tag{A.6}$$

Combining these two inequalities, we finally deduce that $G(u)/u \geqslant G(v)/v$. We next prove that G equalising over $\mathscr{D}_n(V)$ implies that $G(u)$ is non-decreasing with u over V. There are two possible cases.

Case 1. $\underline{u} \neq 0$. The proof is given in Eichhorn *et al.* [5].

Case 2. $\underline{u} = 0$. Suppose that $G(0) > G(v)$ for some $v \in V^*$. From Case 1, we deduce that, for all $u \leqslant v$, $G(u) < G(0)$. Let $\eta = \lim_{u \to 0^+} G(u)$; we have $G(0) > \eta$. Consider a sequence $(v_n)_{n=1, 2, ..., \infty}$ converging to 0 and define $P_n := (1/n)(\delta_0 + (n-1)\,\delta_{v_n}) \in \mathscr{D}(V)$, for all $n = 1, 2, ..., \infty$. We claim that, for n sufficiently large, $L_{P^G}((n-1)/n) < L_P((n-1)/n)$ or equivalently

$$\frac{n-2}{n-1} > \frac{(n-1)\,G(v_n)}{G(0) + (n-1)\,G(v_n)}. \tag{A.7}$$

Upon simplifying, (A.7) reduces to $G(0) > [(n-1)/(n-2)]\,G(v_n)$, which is clearly satisfied when $n \to \infty$. Therefore, for n sufficiently large, we have not $[P_n^G \geq_{\mathrm{L}} P_n]$, hence a contradiction. ∎

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