JOURNAL OF ECONOMIC THEORY **41**, 248–269 (1987)

Measures of Inequality As an Aggregation of Individual Preferences about Income Distribution: The Arrowian Case*

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June 4, 1984; revised June 3, 1986

An inequality preorder is a preorder on a simplex which satisfies symmetry and strict Schur-convexity (the mathematical equivalent of the principle of transfers of Pigou and Dalton). It is shown that we cannot aggregate individual inequality preorders to a collective one if we are interested in Arrow's aggregation rules. The proof uses an interesting result of Kalai, Muller and Satterthwaite (*Public Choice* **34** (1979), 87–97). Moreover, we prove further results for the aggregation of individual inequality indices when we allow cardinality and interpersonal comparibility of utility. *Journal of Economic Literature* Classification Numbers: 024, 025, 914.
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1. INTRODUCTION

There is wide agreement (see, for example, Atkinson [2], Sen [15], Kolm [8], Fei and Fields [5]) on the minimal properties that must be required of an inequality index, which are symmetry and the principle of transfers of Dalton. Symmetry means that the measure is invariant up to a permutation between the ith and the jth components of the vector representing the amount received by each individual in the society, and the principle of tranfers of Dalton is that a finite sequence of transformations transferring income from the rich to the poor has to decrease the value of the inequality index. A theorem of Hardy, Littlewood, and Polya, spelled out in Dasgupta, Sen and Starrett [4], shows that the requirement of the principle of transfers of Dalton is equivalent to the mathematical property of Schur-convexity. (See Section 2 for an exact definition.) We shall say that a preorder on a simplex is an inequality preorder if it satisfies sym-

* This paper resulted from discussions held at a seminar on inequality and social choice theory with J. Lainé, C. Mouton and J. R. Uriarte as regular co-participants. We are grateful to them, to S. Barbera, and to an anonymous referee for helpful suggestions.

0022-0531/87 $3.00
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metry and Schur-convexity. Here, we deal with the question, can we interpret an inequality preorder as an aggregation of individual preferences about income distributions? We can expect that we will maximize the chance of a positive answer if we assume that individual preference are also inequality preorders.

In doing so, we can interpret the preferences of individuals as their opinions on what is socially right; Sen [16] points out that the problem of this approach is in arriving at these distributional judgments rather than in aggregating such judgments. We agree with him and we do not claim that we can find a population with such judgments; but our problem is a purely theoretical one: we only investigate the economic meaning of inequality measures. Sen [17] has pointed out that, ordinal and noncomparable informational bases are acceptable when we want aggregate value judgments, and here, we mainly follow this advice.

First, we are interested in aggregating inequality preorders in an Arrow framework: the aggregation rule satisfies Pareto, independence of irrelevant alternatives and non-dictatorship. The question we face seems clear: is the restriction of the domain of the collective choice rule introduced by symmetry and Schur-convexity sufficient to avoid the impossibility of Arrow's theorem?

In Fig. 1, we illustrate the restrictions on the domain of preferences induced by symmetry and Schur-convexity in the case of a simplex of dimension 2. Any indifference curve through x^* must lie within the hatched part (and be symmetric).

Finally, our problem seems related to a problem stated by Hamada [6] but he considers a space of inequality opinions smaller than ours; he

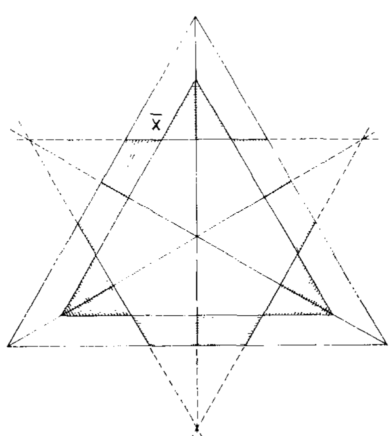

FIG. 1. Restrictions on the indifference curves of an inequality preorder.

studies only the majority rule and, more important, he does not impose the domain restrictions on the range of the collective choice rule.

The result of this paper is the determination of the impossibility of an Arrowian way of aggregating the opinions of individuals on inequality. The proof uses an interesting result of Kalai, Muller and Satterthwaite [7].

The paper is organized as follows: in Section 2, we introduce definitions and some basic results that are needed. Then in Section 3, we state and prove the impossibility result when there is a finite number of individuals concerned by the distribution. When this number becomes large, it is easier to rank income distributions and we prove in Section 4 that the impossibility result remains valid with an infinite number of individuals. In Section 5, we give an extension of our result when we enlarge our choice space to $\mathbb{R}^l_+$.[1] In Section 6, we state the general problem of aggregating individual inequality indices (and no more just inequality preorders) under less narrow informational framework than the ordinal non-comparable one.[1] In particular, we show that Roberts' results [13] apply to the aggregation of individual inequality indices.

2. Notations, Definitions and Basic Results

We are concerned with the distribution of a single, divisible object among l individuals ($l \geqslant 3$). The available amount of the object to be distributed will be normalized to one, so that $S_l = \{(x_1, ..., x_l) \in \mathbb{R}^l_+, \sum_{i=1}^l x_i = 1\}$ denotes the set of feasible distributions. For a given distribution vector x in S_l, x_i denotes the share of the ith individual. Let $\precsim$ be a complete preorder over S. As usual, $\prec$ and $\sim$ are the asymmetric and symmetric parts of $\precsim$, respectively.

Definition 2.1. $\precsim$ is said to be continuous if $\forall x \in S_l$, the sets $\{y \in S_l\colon y \precsim x\}$ and $\{y \in S_l\colon x \precsim y\}$ are closed in S_l.

Definition 2.2. A square matrix of order l, $B = (b_{ij})$, $1 \leqslant i, j \leqslant l$ is bistochastic if $b_{ij} \geqslant 0$, $\forall i, j$, $\sum_{i=1}^l b_{ij} = 1$, $\forall j$; $\sum_{j=1}^l b_{ij} = 1$, $\forall i$.

A permutation matrix is a bistochastic matrix which has exactly one positive entry in each row and each column.

Let B_l and P_l be the set of bistochastic and permutation matrices of order l, respectively.

[1] We thank an anonymous referee for these suggestions.

DEFINITION 2.3. $\precsim$ is said to be symmetric if $\forall x \in S_l$, $\forall P \in P_l$, we have

$$Px \sim x.$$

DEFINITION 2.4. $\precsim$ is said to be strictly Schur-convex if $\forall x \in S_l$, $\forall B \in B_l$, we have

$$Bx \prec x$$

when $Bx \neq Px \; \forall P \in P_l$.

Let $\mathfrak{P}_l$ denote the set of complete preorders over S_l which are continuous and strictly Schur-convex. From now on, an element, $\precsim$, of $\mathfrak{P}_l$ will be called an inequality preorder and for any $x, y \in S_l$, $x \precsim y$ will mean that "distribution x is at least as equal as distribution y." For a more thorough analysis of the motivations behind these formal definitions, see [15, 19].

Remark 2.1. As is shown in Le Breton, Trannoy and Uriarte [11], any inequality preorder is symmetric.

We shall need the following version of a very-well-known theorem of Hardy, Littlewood and Polya.

THEOREM 2.1. *If x and y are two vectors in S_l ordered so that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_l$ and $y_1 \leqslant y_2 \leqslant \cdots \leqslant y_l$ the first three of the following conditions are equivalent, and the last three are equivalent as well.*

(i) *There exists a bistochastic matrix B (which is not a permutation matrix), such that $y = Bx$.*

(ii) $y_1 + \cdots + y_k \geqslant x_1 + \cdots + x_k$, *all $k \leqslant l-1$ (with strict inequality for at least one k).*

(iii) *For any strictly convex function $\mathfrak{f}$ defined on $\mathbb{R}$, we have $\mathfrak{f}(y_1) + \cdots + \mathfrak{f}(y_l) < \mathfrak{f}(x_l) + \cdots + \mathfrak{f}(x_l)$.*

(i′) *There exists a bistochastic matrix B such that $y = Bx$.*

(ii′) $y_1 + \cdots + y_k \geqslant x_1 + \cdots + x_k$ *for all $k \leqslant l-1$.*

(iii′) *For any convex function $\mathfrak{f}$ defined on $\mathbb{R}$, we have $\mathfrak{f}(y_1) + \cdots + \mathfrak{f}(y_l) \leqslant \mathfrak{f}(x_1) + \cdots + \mathfrak{f}(x_l)$.*

Proof. See Berge [3] and Dasgupta, Sen and Starrett [4]. ∎

To any distribution $x \in S_l$, we associate the distribution $x^* = (x_{\sigma(1)}, ..., x_{\sigma(l)}) \in S_l$, where σ is a permutation on the set $\{1, 2, ..., l\}$ such that

$$x_{\sigma(1)} \leqslant x_{\sigma(2)} \leqslant, ..., \leqslant x_{\sigma(l)}.$$

By Remark 2.1 $x \sim x^*$ for any $\precsim \in \mathfrak{P}_l$. Then we have defined the equivalence class of x^* and, in the sequel, $S_{l/\sim}$ is the associated quotient space.

For any $x \in S_l$, we define now the Lorenz curve as the function $\mathscr{L}_x(\cdot)$ defined on $[0, 1]$ by

$$\mathscr{L}_x(0) = 0$$

$$\mathscr{L}_x\left(\frac{k}{l}\right) = \sum_{i=1}^{k} x_i^* \qquad \forall k = 1,..., l$$

and

$$\mathscr{L}_x\left(t\frac{(k-1)}{l} + (1-t)\frac{k}{l}\right) = t\mathscr{L}_x\left(\frac{k-1}{l}\right) + (1-t)\,\mathscr{L}_x\left(\frac{k}{l}\right) \qquad \forall t \in]0, 1[.$$

Remark 2.2. The equivalence between conditions (i) and (ii) in Theorem 2.1 implies that any distribution y which Lorenz-dominates a distribution x has to be declared strictly less unequal ($x \prec y$). Then we have the following consequence of Theorem 2.1.

COROLLARY 2.1. *If x and y are two vectors in S_l ordered so that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_l$ and $y_1 \leqslant y_2 \leqslant \cdots \leqslant y_l$ and if $\exists k$ such that $y_1 + \cdots + y_k > x_1 + \cdots + x_k$ and $\exists k'$ such that $x_1 + \cdots + x_{k'} > y_1 + \cdots + y_{k'}$ then there exists $\precsim_i \in \mathfrak{P}_l$, $i = 1, 2, 3$, with*

$$x \prec_1 y, \qquad y \prec_2 x, \qquad x \sim_3 y.$$

Proof. The existence of k such that $y_1 + \cdots + y_k > x_1 + \cdots + x_k$ implies the existence of at least one strictly convex function $\mathfrak{f}_1\ \mathbb{R} \to \mathbb{R}$ such that

$$\mathfrak{f}_1(y_1) + \cdots + \mathfrak{f}_1(y_l) \geqslant \mathfrak{f}_1(x_1) + \cdots + \mathfrak{f}_1(x_l).$$

For at least one, the inequality is strict. Suppose on the contrary that $\mathfrak{f}(x_1) + \cdots + \mathfrak{f}(x_l) \geqslant \mathfrak{f}(y_1) + \cdots + \mathfrak{f}(y_l)$ for any $\mathfrak{f}$ strictly convex $\mathbb{R} \to \mathbb{R}$. Let $\tilde{\mathfrak{f}}: \mathbb{R} \to \mathbb{R}$ a convex function and K a closed and bounded interval of $\mathbb{R}$, containing the points $x_1,..., x_l$, $y_1,..., y_l$. We have $\tilde{\mathfrak{f}}|_K = \lim_{n \to \infty} \mathfrak{f}_n|_K$, for the uniform convergence topology, where the $\mathfrak{f}_n$, $n \geqslant 1$ are strictly convex functions: $\mathbb{R} \to \mathbb{R}$. We then deduce that for any convex function $\tilde{\mathfrak{f}}: \mathbb{R} \to \mathbb{R}$ we have $\tilde{\mathfrak{f}}(x_1) + \cdots + \tilde{\mathfrak{f}}(x_l) \geqslant \tilde{\mathfrak{f}}(y_1) + \cdots + \tilde{\mathfrak{f}}(y_l)$ and thus, by using the equivalence between (ii)′ and (iii)′, we obtain a contradiction.

Further, we denote $\oint_1 : \mathbb{R} \to \mathbb{R}$, a strictly convex function such that

$$\oint_1 (y_1) + \cdots + \oint_1 (y_l) > \oint_1 (x_1) + \cdots + \oint_1 (x_l).$$

We define $\precsim_1$ as the complete preorder induced by the function

$$S_l \to \mathbb{R}$$

$$(z_1, \ldots, z_l) \to \oint_1 (z_1) + \cdots + \oint_1 (z_l).$$

It is easy to show that $\precsim_1$ enjoys the desired properties. In the same way we prove the existence of a $\precsim_2 \in \mathfrak{P}_l$ such that $y \prec_2 x$ (using a strictly convex $\oint_2 : \mathbb{R} \to \mathbb{R}$).

Last, we define $\oint_3 : \mathbb{R} \to \mathbb{R}$ by

$$\oint_3 (x) = t \oint_1 (x) + (1-t) \oint_2 (x)$$

with

$$t \left[\oint_1 (x_1) + \cdots + \oint_1 (x_l) \right] + (1-t) \left[\oint_2 (x_1) + \cdots + \oint_2 (x_l) \right]$$
$$= t \left[\oint_1 (y_1) + \cdots + \oint_1 (y_l) \right] + (1-t) \left[\oint_2 (y_1) + \cdots + \oint_2 (y_l) \right].$$

We see at once such a t exists on $]0, 1[$ and that the preorder induced by $\oint_3$ has the desired properties. ▮

$M = \{1, 2, \ldots, j, \ldots, m\}$ denotes the set of voters. Each of them has a preference about income distribution which belongs to $\mathfrak{P}_l$. An aggregation rule on $\mathfrak{P}_l^m$ is a function f:

$$f: \underbrace{\mathfrak{P}_l \times \mathfrak{P}_l \times \cdots \times \mathfrak{P}_l}_{m\text{-times}} \to \Sigma$$

where Σ is the set of all complete preorders on S_l. We denote $\precsim_M$ the n-tuple $(\precsim_1, \precsim_2, \ldots, \precsim_j, \ldots, \precsim_m)$ of individual inequality preorders. Two profiles $\precsim_M$ and $\precsim'_M$ agree on a subset A of S_l and we denote $\precsim_M |_A = \precsim'_M |_A$ if $\forall x, y \in A$, $x \precsim_j y$ if and only $x \precsim'_j y$. In the sequel, $f(\precsim_M) = \precsim$.

Definition 2.5. An "Arrow aggregation rule" is an aggregation rule which satisfies the conditions of independence of irrelevant alternatives, weak Pareto and non-dictatorship.

Independence of Irrelevant Alternatives (IIA). $\forall \precsim_M \in \mathfrak{P}_l^m$, $\precsim'_M \in \mathfrak{P}_l^m$ $\forall A \subset S_l$, $\precsim_M|_A = \precsim'_M|_A$ implies $\precsim|_A = \precsim'|_A$.

Weak Pareto (WP): $\forall \precsim_M \in \mathfrak{P}_l^m$, $\forall x, y \in S_l$ $x \prec_j y$ for all $j \in M$ implies $x \prec y$.

Non-dictatorship (ND): $\nexists j \in M / \forall \precsim_M \in \mathfrak{P}_l^m$, $\forall x, y \in S_l$ $x \prec_j y$ implies $x \prec y$.

In order to present the result of Kalai, Muller and Satterthwaite [7] we give some preliminary definitions.

Ⓗ represents a fixed, nonempty subset of Σ.

DEFINITION 2.6.

- A pair of distinct alternatives $x, y \in S_l$ is called trivial (relative to Ⓗ) if all the relations in Ⓗ agree on the set $\{x, y\}$.
- A set of three distinct alternatives $\{x, y, z\}$ $x, y, z \in S_l$ is called a free triple if for every $\precsim \in \Sigma$ there exists $\precsim' \in$ Ⓗ, such that

$$\precsim'|_{\{x,y,z\}} = \precsim|_{\{x,y,z\}}.$$

- Two non-trivial pairs $A = \{x, y\}$ and $B = \{w, z\}$ are said to be strongly connected if $|A \cup B| = 3$ and $A \cup B$ is a free triple.
- Two pairs A and B are said to be connected if a finite sequence of pairs

$$A = B_1, B_2, ..., B_{n-1}, B_n = B$$

exists such that B_i and B_{i+1} are to be strongly connected for each $i = 1, 2, ..., n-1$.

- Ⓗ is called saturating if

(i) the set S_l contains at least two non-trivial pairs.

(ii) every non-trivial pair is connected to every other non-trivial pair.

THEOREM 2.2 (Kalai, Muller and Satterthwaite [7]). *Let f be an aggregation rule on* Ⓗ. *Then, if* Ⓗ *is saturating, f is not an Arrow aggregation rule.*

Proof. Cf. [7, pp. 91–92].[2] ∎

In the next section we prove that the family $\mathfrak{P}_l$ is saturating for $l \geqslant 3$.

[2] The theorem is true for any set of alternatives.

3. THE MAIN RESULT

THEOREM 3.1. *The family $\mathfrak{P}_l$, of inequality preorders on S_l, is saturating for all $l \geqslant 3$.*

Remark 3.1. If $l = 2$, it is easy to show that $\mathfrak{P}_l$ contains one and only one element. It is a trivial consequence of Corollary 2.1.

Proof. The strategy of our proof is the following. First we prove the theorem for $l = 3$. Second we deduce from the study of this case, the general result.

Case 1. $l = 3$

It is easy to see that we merely have to prove the second part of definition (from the corollary, two non-trivial pairs of distributions always exist). Moreover, we shall remark that if $\{x, y, z\}$ is a triple in S_3 such that $\{x, y\}$, $\{y, z\}$ and $\{x, z\}$ are non-trivial, then it is free (it suffices to use continuity arguments with the fact that Schur-convexity is preserved by the operations sup and inf).

Step 1. Each element of $S_{3/\sim}$ may be described by a pair (a, u) with $a \in [0, \frac{1}{3}]$ and $2a \leqslant u \leqslant (1 + a)/2$ (the Lorenz curve associated to (a, u) takes the value a for $\frac{1}{3}$ and the value u for $\frac{2}{3}$ (cf. Fig. 2)). $2a$ and $(1 + a)/2$ are respectively the smallest and the largest values of u (when a is fixed) ensuring thus the convexity of the Lorenz curve.

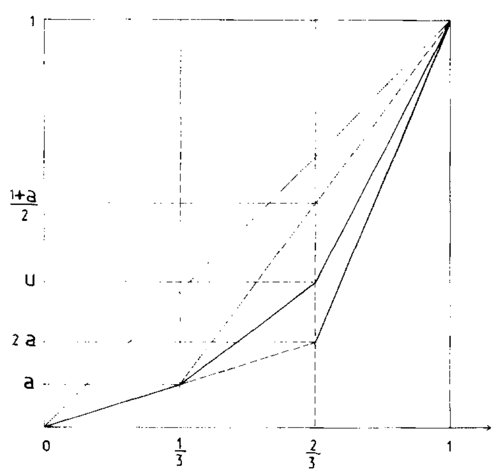

FIG. 2. Identification of a Lorenz curve for three individuals.

Step 2. We consider the two sequences of numbers

$$u_0 = 0, \qquad u_k = \frac{u_{k-1} + 2}{4} \qquad \text{for} \quad k \geqslant 1$$

and

$$a_k = \frac{u_k}{2} \qquad \text{for} \quad k \geqslant 0.$$

We define the following sets:

$$C_{0,0} = \{(a, u) \in S_{3/\sim} : \ a = a_0; u = u_0\}$$
$$C_{0,1} = \{(a, u) \in S_{3/\sim} : \ a = a_0; u_0 < u < u_1\}$$
$$C_{1,1} = \{(a, u) \in S_{3/\sim} : \ a_0 < a < a_1; u_0 < u < u_1\}$$

and for $n \geqslant 2$

$$C_{n-1,n} = \{(a, u) \in S_{3/\sim} : a_{n-2} \leqslant a < a_{n-1}; u_{n-1} \leqslant u < u_n\}$$
$$C_{n,n} = \{(a, u) \in S_{3/\sim} : \ a_{n-1} \leqslant a < a_n; u_{n-1} \leqslant u < u_n\}.$$

It is easy to prove that the sequences $(u_n)_{n \geqslant 0}$ and $(a_n)_{n \geqslant 0}$ converge respectively to $\frac{2}{3}$ and $\frac{1}{3}$. Then the family $(\bigcup_{n \geqslant 0} C_{n,n}) \cup (\bigcup_{n \geqslant 0} C_{n,n+1})$ is a covering of $S_{3/\sim}$. (The construction is represented in Fig. 3.)

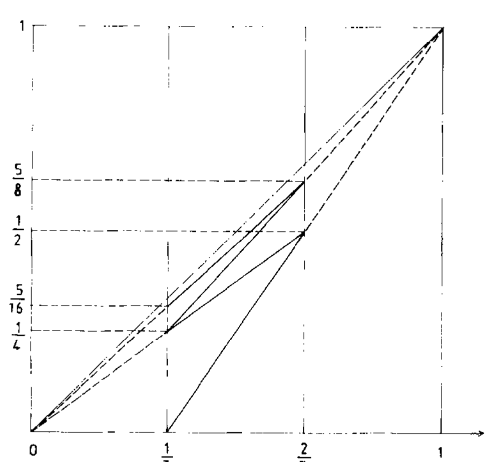

FIG. 3. Covering of $S_{3/\sim}$.

Step 3. Let $\{(a'_1, u'_1), (a'_2, u'_2)\}$ be a non-trivial pair of distributions. Then there exists $n \in \mathbb{N}$, such that

$$(a'_i, u'_i) \in C_{n,n} \cup C_{n,n+1}, \qquad i = 1, 2$$

or

$$(a'_i, u'_i) \in C_{n,n} \cup C_{n-1,n}, \qquad i = 1, 2$$

with $n > 0$.

This is an immediate consequence of our definitions of the sets $C_{n,n}$ and $C_{n,n+1}$.

Step 4. Let $\{(a'_1, u'_1), (a'_2, u'_2)\}$ be a non-trivial pair. Without loss of generality, we assume (Step 3):

$$(a'_i, u'_i) \in C_{n,n} \cup C_{n,n+1}, \qquad n \geqslant 1.$$

Then, there exists a distribution (a'_3, u'_3) such that:

(i) $\{(a'_i, u'_i), i = 1, 2, 3\}$ is a free triple and

(ii) $i \in \{1, 2\}$ such that

$$\{(a'_i, u'_i\}; (a'_3, u'_3)\} \subset C_{n,n}$$

or

$$\{(a'_i, u'_i); (a'_3, u'_3)\} \subset C_{n,n+1}.$$

Take indeed λ in $[0, 1]$ and consider the distribution (a'_3, u'_3) defined by

$$a'_3 = \lambda a'_1 + (1 - \lambda)\, a'_2$$
$$u'_3 = \lambda u'_1 + (1 - \lambda)\, u'_2.$$

The result is a consequence of Step 2.

Step 5. Let us consider the following sets for $n \geqslant 2$.
$\forall \varepsilon > 0$,

$$A^{\varepsilon}_{n,n} = \{(a, u) \in S_{3/\sim} : a_{n-1} - \varepsilon < a < a_{n-1}; u_n - \varepsilon < u < u_n\}$$

and

$$A^{\varepsilon}_{n-1,n} = \{(a, u) \in S_{3/\sim} : a_{n-1} - \varepsilon < a < a_{n-1}; u_{n-1} - \varepsilon < u < u_{n-1}\}.$$

For any ε small enough, these sets contain at least two distributions which constitute a non-trivial pair (use a continuity argument).

The interest in $A^{\varepsilon}_{n,n}$ and $A^{\varepsilon}_{n-1,n}$ rests on the following property: Let

$$\{(a'_1, u'_1)(a'_2, u'_2)\}$$

be a non-trivial pair with

$$(a'_1, u'_1) \in C_{n,n}$$

$$(a'_2, u'_2) \in C_{n,n}.$$

Then there exists $\bar{\varepsilon}$ such that $\forall \varepsilon < \bar{\varepsilon}$, all non-trivial pairs $\{(a'_3, u'_3), (a'_4, u'_4)\} \subset A^{\varepsilon}_{n,n}$ are connected with $\{a'_1, u'_1), (a'_2, u'_2)\}$ (we only have to take $\bar{\varepsilon} = u_n - \mathrm{Sup}[u'_1, u'_2]$).

We have a similar result when $\{(a'_1, u'_1), (a'_2, u'_2)\} \subset C_{n-1,n}$; that is, there exists $\bar{\varepsilon}'$ such that $\forall \varepsilon < \bar{\varepsilon}'$, all non-trivial pairs $\{(a'_2, u'_3), (a'_4, u'_4)\} \subset A^{\varepsilon}_{n-1,n}$ are connected with $\{(a'_1, u'_1), (a'_2, u'_2)\}$ (we only have to take $\bar{\varepsilon}' = a_{n-1} - \mathrm{Sup}[a'_1, a'_2]$).

Now, let us take two non-trivial pairs $\{(a'_1, u'_2), (a'_2, u'_2)\}$, $\{(a''_1, u''_1), (a''_2, u''_2)\}$, belonging respectively to $C_{n,n}$ and $C_{n-1,n}$ with $n \geqslant 2$. From the preceding reasoning, we deduce that they are connected. Take indeed a non-trivial pair $\{(a'_3, u'_3), (a'_4, u'_4)\} \subset A_{n,n}$, $\varepsilon < \bar{\varepsilon}$. We denote $\bar{\varepsilon}'' = a_{n-1} - \mathrm{Sup}(a'_3, a'_4)$ and $\bar{\varepsilon}''' = \mathrm{Inf}(\bar{\varepsilon}', \bar{\varepsilon}'')$. And we only have to take a non-trivial pair $\{(a'_5, u'_5), (a'_6, u'_6)\} \subset A^{\bar{\varepsilon}'''}_{n-1,n}$ to obtain the result.

Obviously, the same lines of arguments allow us to prove that two non-trivial pairs belonging respectively to $C_{n,n}$ and $C_{n,n+1}$ are connected.

Step 6. The binary relation "is connected with" is transitive on the set of all non-trivial pairs.

This, as well as the steps 4 and 5, allows us to prove the theorem for $n \geqslant 2$.

We prove along the same lines that the subsets $C_{1,1}$ and $C_{1,2}$ are also "connected." Thus the proof is complete in the case $l = 3$.

Case 2. $l > 3$

Let x be an element of $S_{l/\sim}$. In the sequel, we shall identify x with its Lorenz curve $\mathscr{L}_x(\cdot)$. Now let x and x' be two elements of $S_{l/\sim}$. From Corollary 2.1, we know that $\{x, x'\}$ is a non-trivial pair if and only if $\exists k_1, k_2 \in \{1,..., l\}$, $k_1 \neq k_2$, such that

$$\mathscr{L}_x\left(\frac{k_1}{l}\right) > \mathscr{L}_{x'}\left(\frac{k_1}{l}\right)$$

and

$$\mathscr{L}_x\left(\frac{k_2}{l}\right) < \mathscr{L}_{x'}\left(\frac{k_2}{l}\right).$$

This implies the existence of at least one point $p \in]0, 1[$, such that $\mathscr{L}_x(p) = \mathscr{L}_{x'}(p)$, $p \in]k/l, (k+1)/l[$ for $k \in \{1,..., l-2\}$, which is either an "intersection" point, or for any $p' \in [k/l, (k+1)/l]$ (or any $p' \in [(k-1)/l, k/l])$ we have $\mathscr{L}_x(p') = \mathscr{L}_{x'}(p')$. Without loss of generality, we consider the first case (indeed the proof remains true for the second case).

Step 7. Let $\{x, x'\}$ be a non-trivial pair in $S_{l/\sim}$ and $p \in]0, 1[$ such that $\mathscr{L}_x(p) = \mathscr{L}_{x'}(p)$. We assume that there exists $k \in \{2,..., l-2\}$ such that $p = k/l$. Then there exists a third distribution, denoted x'', such that $\{x, x', x''\}$ is a free triple and that the Lorenz curves of x and x'' (respectively the Lorenz curves of x' and x'') intersect in one of the two intervals $](k-1)/l, k/l[$, $]k/l, (k+1)/l[$. The proof is left to the reader. (See Fig. 4 for $l = 4$.)

Step 8. Let $\{x, x'\}$, $\{x'', x'''\}$ two non-trivial pairs in $S_{l/\sim}$ and $p_1, p_2 \in]k/l, (k+1)/l[$ for some $k \in \{1,..., l-2\}$, such that $\mathscr{L}_x(p_1) = \mathscr{L}_{x'}(p_1)$ and $\mathscr{L}_{x''}(p_2) = \mathscr{L}_{x'''}(p_2)$.

These two pairs are connected. For this, we use the same reasoning as in case 1, and we consider the distributions in $S_{l/\sim}$, which give the same share to the individuals $1, 2,..., k-1, k$, and also the same share to the individuals $k+2, k+3,..., l$, and whose Lorenz curves take for k/l (resp. $(k+1)/l$) the values $(a_n^{(k)})_{n \geq 0}$ (resp. $(u_n^{(k)})_{n \geq 0}$). The sequence $(a_n^{(k)})_{n \geq 0}$ and $(u_n^{(k)})_{n \geq 0}$ are defined in the following way:

$$u_0^{(k)} = 0, \qquad u_{n+1}^{(k)} = \frac{1}{l-k} + \frac{k(l-k-1)\, u_n}{(k+1)(l-k)} \qquad \text{for} \quad n \geq 1$$

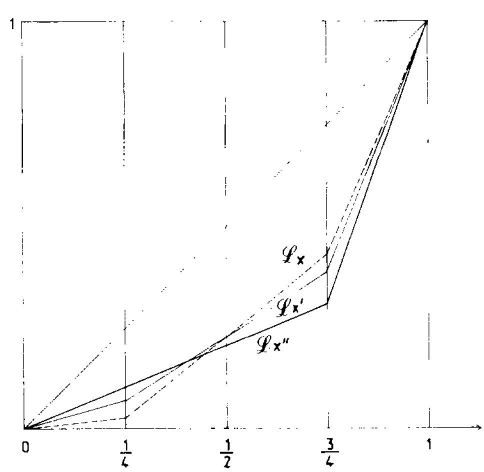

FIG. 4. Illustration of step 7.

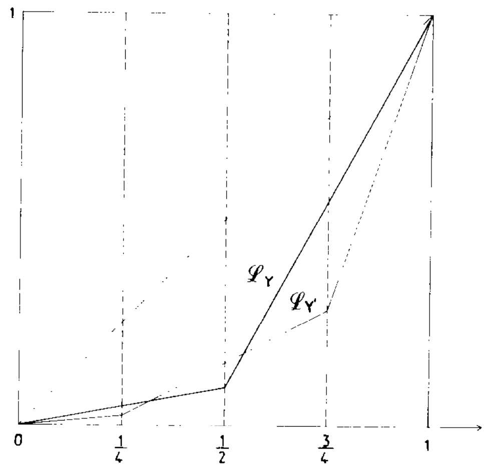

FIG. 5. Illustration of step 9 for four individuals.

and

$$a_n^{(k)} = \frac{k}{k+1} u_n^{(k)} \qquad \text{for} \quad n \geqslant 0.$$

The sequence $(u_n^{(k)})_{n \geqslant 0}$ converges to $(k+1)/l$ and thus the sequence $(a_n^{(k)})_{n \geqslant 0}$ converges to k/l.

Step 9. Let $\{x, x'\}$, $\{x'', x'''\}$ be two non-trivial pairs in $S_{l/\sim}$ and $p_1 \in]k/l, (k+1)/l[$, $p_2 \in](k+1)/l, (k+2)/l[$ for some $k \in \{1, ..., l-3\}$, such that $\mathcal{L}_x(p_1) = \mathcal{L}_{x'}(p_1)$ and $\mathcal{L}_{x''}(p_2) = \mathcal{L}_{x'''}(p_2)$. These two pairs are connected. For this, it suffices to apply step 8 and to build a non-trivial pair $\{y, y'\}$ in $S_{l/\sim}$ such that $\mathcal{L}_y(\cdot)$ and $\mathcal{L}_{y'}(\cdot)$ intersect in $]k/l, (k+1)/l[$ and $](k+1)/l, (k+2)/l[$. (See Fig. 5 for $l=4$.)

With step 7 and an easy transitivity argument, we conclude that in all the cases, two non-trivial pairs are connected. This achieves the proof of the theorem.

THEOREM 3.2. *There are no Arrow aggregation rules on* $\mathfrak{P}_l$.

Proof. Combine Theorems 3.1 and 2.2. ∎

4. THE CASE OF A "LARGE" POPULATION OF INDIVIDUALS INVOLVED IN THE DISTRIBUTION PROCESS

In the previous section we have assumed that the set of individuals has finite, and we may wonder if this assumption plays some role in the analysis. In this section, we show that the previous impossibility result is robust to a continuum assumption. We denote [0, 1] the set of individuals involved in the distribution process. By L^1 we denote the set of integrable functions on [0, 1] (for the Borel σ-algebra and the Lebesgue measure λ). We denote $\tilde{L}^1$ the set of functions f in L^1 such that f is non-negative and $\int_0^1 f(t)\,\lambda(dt) = 1$. An element of $\tilde{L}^1$ is precisely a distribution of a good whose available quantity is normalized to one between a continuum of agents.

Let $f \in \tilde{L}^1$ and $t \in [0, 1]$: we define $f^*(t) = \sup\{r: \lambda(\{s \in [0, 1]: f(s) < r\}) < \tau\}$. It is straightforward to check that f^* is increasing and left-continuous. In some sense f^* is the increasing rearrangement of f. We shall say that the distribution $f \in \tilde{L}^1$ "Lorenz-dominates" the distribution $g \in \tilde{L}^1$ (denoted by $x\ L\ y$) if

$$\int_0^t f^*(r)\,\lambda(dr) \geqslant \int_0^t g^*(r)\,\lambda(dr) \qquad \forall t \in [0, 1].$$

DEFINITION 4.1. We shall say that a linear operator $B: L^1 \to L^1$ is bistochastic if:

(i) $0 \leqslant B \cdot \mathbb{1}_E \leqslant 1$, and

(ii) $\int_0^1 (B \cdot \mathbb{1}_E)(t)\,\lambda(dt) = \lambda(E)$

for any Borel subset E of [0, 1], where $\mathbb{1}_E$ is the indicator function of E.

We shall denote $\mathfrak{B}$ the set of bistochastic operators on L^1.

The following definition is the fundamental one.

DEFINITION 4.2. Let $\precsim$ be a complete preorder on $\tilde{L}^1$. We shall say that $\precsim$ is strictly Schur-convex on $\tilde{L}^1$ if $Bf \prec f$ $\forall B \in \mathfrak{B}$, $\forall f \in \tilde{L}^1$ such that $(Bf)^* \neq f^*$. We shall say that $\precsim$ is symmetric on $\tilde{L}^1$ if $f \sim g$ $\forall f, g \in \tilde{L}^1$ such that $f^* = g^*$.

The following theorem is easily deduced from the continuous version of the theorem of Hardy, Littlewood and Polya stated by Schmeidler [14].

THEOREM 4.1. *Let f and g belong to $\tilde{L}^1$. Then the following statements are equivalent:*

(i) $f\ L\ g \quad f^* \neq g^*$;

(ii) *there exists a bistochastic operator* B *such that* $f = Bg$ *and* $f^* \neq g^*$;

(iii) $\mathfrak{f}(f) < \mathfrak{f}(g)$ *for every real- or* $-\{+\infty\}$*-valued strictly convex symmetric and weakly lower semi-continuous function* $\mathfrak{f}$ *on* $\tilde{L}^1$.

We denote by $\mathfrak{P}_{I,\infty}$ the set of strictly Schur-convex and weakly lower-semi-continuous complete preorders on $\tilde{L}^1$. An element of $\mathfrak{P}_{I,\infty}$ will be called an inequality preorder on $\tilde{L}_1$.[3]

We have the following result:

THEOREM 4.2. *The family* $\mathfrak{P}_{I,\infty}$ *is saturating.*

Proof. Let $\{f, g\}$, $\{u, w\}$, two non-trivial pairs in $L^1_{[0,1]}$.

Step 1. Here, the Lorenz curve associated with the distribution f is the function $\mathscr{L}_f: t \in [0, 1] \to \int_0^t f^*(r)\,dr$. It is not difficult to see that $\mathscr{L}_f$ is a continuous function on $[0, 1]$, with values in $[0, 1]$ such that $\mathscr{L}_f(0) = 0$ and $\mathscr{L}_f(1) = 1$. Moreover, it is convex and increasing.

Conversely, any continuous, convex and increasing function $\mathscr{L}$, defined on $[0, 1]$, with values in $[0, 1]$ satisfying $\mathscr{L}(0) = 0$, $\mathscr{L}(1) = 1$, is the Lorenz curve of a distribution in $L^1_{[0,1]}$. Indeed, since $\mathscr{L}$ is convex on $[0, 1]$, $\mathscr{L}$ is derivable in all but perhaps countably many points of $[0, 1]$. We note $\mathscr{L}'$ its derivative. It is a classical exercise in integration theory, to show that $\mathscr{L}' \in L^1_{[0,1]}$, and that $\forall a, b \in [0, 1]$, $a < b$,

$$\mathscr{L}(b) - \mathscr{L}(a) = \int_a^b L'(t)\,dt.$$

Since $\mathscr{L}(0) = 0$, we have $\mathscr{L}(t) = \int_0^t \mathscr{L}'(r)\,dr$. $\mathscr{L}'$ is a distribution (modulo a.e.) satisfying the conditions required, because in addition to the above properties, $\int_0^1 \mathscr{L}'(t) = \mathscr{L}(1) = 1$ and $\mathscr{L}'$ take non-negative values.

Step 2. From Theorem 4.1 there exist t_1, t_2, t_3 and t_4 belonging to $]0, 1[$, such that

$$\mathscr{L}_f(t_1) > \mathscr{L}_g(t_1)$$
$$\mathscr{L}_f(t_2) < \mathscr{L}_g(t_2)$$
$$\mathscr{L}_u(t_3) > \mathscr{L}_w(t_3)$$
$$\mathscr{L}_u(t_4) < \mathscr{L}_w(t_4).$$

[3] Any inequality preorder on $\tilde{L}^1$ is symmetric. A proof of this property is available from the first author upon request.

From the intermediate value theorem we deduce that there exist $t_5, t_6 \in]0, 1[$ such that

$$\mathscr{L}_f(t_5) = \mathscr{L}_g(t_5)$$

and

$$\mathscr{L}_u(t_6) = \mathscr{L}_w(t_6).$$

Without loss of generality, assume that $t_6 \geqslant t_5$, and denote by M, $\operatorname{Sup}(\mathscr{L}_f(t_5)/t_5, \mathscr{L}_u(t_6)/t_6)$. Then take two linear functions denoted respectively by $\tilde{\mathscr{L}}$ and $\tilde{\mathscr{L}}'$ of respective slopes $M+\varepsilon$ and $M+2\varepsilon$, for $\varepsilon > 0$ sufficiently small. It is not difficult to show that $\tilde{\mathscr{L}}$ and $\tilde{\mathscr{L}}'$ intersect with $\mathscr{L}_u$, $\mathscr{L}_g$, $\mathscr{L}_f$ and $\mathscr{L}_w$ (use step 1 and the intermediate value theorem). At least, we change the slope of $\tilde{\mathscr{L}}$ (resp. $\tilde{\mathscr{L}}'$) in a point t_7 sufficiently close to 1 (resp. $t_7 + \eta$, $\eta > 0$, $t_7 + \eta < 1$), so that the new functions denoted respectively $\tilde{\mathscr{L}}''$ and $\tilde{\mathscr{L}}'''$ are the Lorenz curves of some distributions e and f belonging to $\tilde{L}^1_{[0,1]}$ (see step 1), and so that the pair $\{e, f\}$ is non-trivial (see Fig. 6).

The conclusion is simple. The details are left to the reader. ▌

THEOREM 4.3. *There are no Arrow aggregation rules on* $\mathfrak{P}_{l,\infty}$.

Proof. Combine Theorems 4.2 and 2.2. ▌

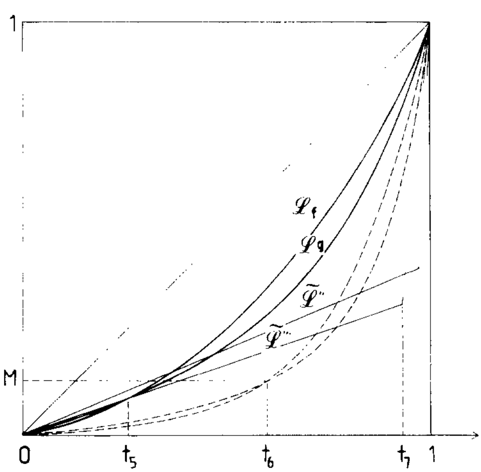

FIG. 6. Construction of the connection for a continuum of individuals.

5. Variable Total Income

In Section 2, the space of distributions under consideration was a simplex; i.e., we assumed that the total endowment was fixed. Classical welfare analysis is often interested in arbitrages between equity and efficiency; precisely we have to compare distributions which do not belong to the same simplex. So the appropriate space of distributions is $\mathbb{R}^l_+$. If we are interested in inequality indices which are decreasing with respect to the components of x (or identically social welfare functions which are increasing with respect to individual incomes) it is shown below that the impossibility result of Section 2 always holds.

Definition 5.1. A complete preorder $\precsim$ over $\mathbb{R}^l_+$ is called an inequality preorder if it is continuous, strictly Schur-convex and decreasing, i.e., $\forall x$, $y \in \mathbb{R}^l_+$ with $x_i \leqslant y_i$ $\forall i = 1, ..., l$ (and strict inequality for at least one i) we have $y \prec x$.

This definition is adapted (in an ordinal perspective) from Shorrocks [18] and must be contrasted with invariance conditions as suggested, for instance, by Fei and Fields [5]. Note that Atkinson [2] also introduces such a monotonicity property before doing its normalization.

The following theorem is the equivalent of the theorem of Hardy, Littlewood and Polya for this context.

Theorem 5.1. *Let $x = (x_1, ..., x_l)$, $y = (y_1, ..., y_l)$ in $\mathbb{R}^l_+$ be such that $x_i \leqslant x_{i+1}$ and $y_i \leqslant y_{i+1}$ $\forall i = 1, ..., l-1$. Then the following conditions are equivalent.*

(i) *there exists a bisuperstochastic matrix $B = (b_{ij})_{1 \leqslant i,j \leqslant l}$ (i.e., a nonnegative matrix which is majorized by a bistochastic one component by component) such that $y = Bx$, and y is not a permutation of x.*

(ii) $y_1 + \cdots + y_k \geqslant x_1 + \cdots + x_k$ $\forall k = 1, ..., l$ *(with strict inequality for at least one k).*

(iii) $\sum_{i=1}^l \mathfrak{f}(y_i) < \sum_{i=1}^l \mathfrak{f}(x_i)$ *for any decreasing and strictly convex function* $\mathfrak{f}: \mathbb{R} \to \mathbb{R}$.

Proof. See Marshall and Olkin [12, pp. 30–31 and 64]. ∎

Remark 5.1. To each distribution, $x = (x_1, ..., x_l)$ belonging to $\mathbb{R}^l_+$ arranged such that $x_i \leqslant x_{i+1}$ $\forall i = 1, ..., l-1$, we associate the function $\mathscr{L}_x(\cdot)$ defined on $[0, 1]$ as follows:

$$\mathscr{L}_x\left(t\cdot\frac{k}{l}+(1-t)\frac{k+1}{l}\right)=t\mathscr{L}_x\left(\frac{k}{l}\right)+(1-t)\,\mathscr{L}_x\left(\frac{k+1}{l}\right)$$
$$\forall k=0,..., l-1, \forall t\in[0, 1]$$

with

$$\mathscr{L}_x(0)=0 \qquad \text{and} \qquad \mathscr{L}_x\left(\frac{k}{l}\right)=\sum_{i=1}^{k} x_i.$$

The function $\mathscr{L}_x(\cdot)$ is called by Shorrocks [18] the generalized Lorenz curve of the distribution x. The condition (ii) in Theorem 5.1 may be written in an equivalent way: $\mathscr{L}_x(t)\leqslant\mathscr{L}_y(t)$ $\forall t\in[0, 1]$ (with strict inequality for at least one t).

Remark 5.2. The partial preorder defined on $\mathbb{R}^l_+$ by one of the equivalent conditions in Theorem 5.1 is called weak majorization by Marshall and Olkin [12].

We shall denote $\mathfrak{P}_l$ the set of inequality preorders on $\mathbb{R}^l_+$.

We have the following theorem.

THEOREM 5.2. *The domain* $\mathfrak{P}_l$ *is saturating*, $\forall l\geqslant 2$.

Proof. The proof parallels exactly that of Theorem 3.1 using now Theorem 5.1 and generalized Lorenz curves rather than Lorenz curves. ∎

We deduce:

THEOREM 5.3. *There are no Arrow aggregation rules on* $\mathfrak{P}_l$, $\forall l\geqslant 2$.

Proof. Combine Theorems 5.2 and 2.2. ∎

6. Aggregation of Inequality Indices

It is tempting to attribute the impossibility results stated in Theorems 3.2 and 5.3 to the poverty of the informational basis with which we were dealing. In particular, we ignored the cardinal content of any inequality index and the possibility of interpersonal comparisons of inequality judgments. It is an easy exercise to prove the existence of "nice" aggregation rules in our restricted domain context, under weaker invariance axioms. So the problem is not an existence one but rather a characterization one. Precisely, we shall show that the fundamental representation theorem proved by Roberts in [13] (and also its corollaries) always holds here. Some notations and definitions are needed.

$\mathfrak{U} \underset{\text{def}}{=}$ set of real-valued functions that may be defined on $S_l \times M$.

$\mathfrak{J} \underset{\text{def}}{=}$ set of real-valued functions on $S_l \times M$ such that $\forall i \in \mathfrak{J}$, $\forall j \in M$ $I(\ ,j)$ is continuous and strictly Schur-convex on S_l.

$\mathfrak{U}|_A \underset{\text{def}}{=}$ set of real-valued functions on $A \times M$, where $A \subset S_l$.

DEFINITION 6.1. A social choice functional on $\mathfrak{J}$ is a mapping f from $\mathfrak{J}$ to Σ, i.e., $f\colon \mathfrak{J} \to \Sigma$.

Let f be a social choice functional on $\mathfrak{J}$. The basic conditions encountered in social choice analysis are the following:

DEFINITION 6.2 (Independence Irrelevant Alternatives: IIA). For any I, $I' \in \mathfrak{J}$, $A \subseteq S_l$, if $I(x,\cdot) = I'(x,\cdot)$ $\forall x \in A$ then $R|_A = R'|_A$, where $R = f(I)$ and $R' = f(I')$.

DEFINITION 6.3. (Weak Pareto Criterion: WP). For any x, $y \in S_l$, for all $I \in \mathfrak{J}$, if $\forall j \in M$, $I(x,j) > I(y,j)$ then xPy, where P denotes the strict preference relation corresponding to $R = f(I)$.

DEFINITION 6.4 (Weak Continuity: WC). f is said to be weakly continuous if $\forall I \in \mathfrak{J}$, $\forall \varepsilon \in \mathbb{R}^M_{++}$ $\exists I' \in \mathfrak{J}$ satisfying $0 \ll I(x,\cdot) - I'(x,\cdot) \ll \varepsilon$ $\forall x \in S_l$ and $f(I) = f(I')$.

Remark 6.1. The regularity property stated in Definition 6.4 is implied by any of the invariance axioms proposed in the literature.

Remark 6.2. $\mathfrak{U}$ (resp. $\mathfrak{J}$) has a product structure. Thus we can use the alternative notation $\prod_{j=1}^{M} \tilde{\mathfrak{U}}$ (resp. $\prod_{j=1}^{M} \tilde{\mathfrak{J}}$), where $\tilde{\mathfrak{U}}$ (resp. $\tilde{\mathfrak{J}}$) is the set of real-valued functions on S_l (resp. the set of continuous and strictly Schur-convex real-valued functions on S_l).

DEFINITION 6.5. (1) A pair $\{x, y\} \subset S_l$ will be said to be trivial if all the functions in $\tilde{\mathfrak{J}}$ agree on $\{x, y\}$ in the following sense: either $I(x) > I(y)$ $\forall I \in \tilde{\mathfrak{J}}$, or $I(x) = I(y)$ $\forall I \in \tilde{\mathfrak{J}}$ or $I(x) < I(y)$ $\forall I \in \tilde{\mathfrak{J}}$.

(2) A triple $\{x, y, z\} \subset S_l$ will be said free if

$$\tilde{\mathfrak{J}}|_{\{x,y,z\}} = \tilde{\mathfrak{U}}|_{\{x,y,z\}}.$$

Two non-trivial pairs in S_l will be said to be

(1) strongly connected if $A \cup B$ is a free triple;

(2) connected if there exists a finite sequence of pairs $A = A_1$, $A_2, \ldots, A_{n-1}$, $A_n = B$ such that A_i and A_{i+1} are strongly connected $\forall i = 1, \ldots, l-1$.

Remark 6.3. It is clear that the previous definitions make sense for an arbitrary choice space X and any $\mathfrak{D} \subset \mathfrak{U}$ such that $\mathfrak{D}$ can be written as a cartesian product $\prod_{i=1}^{M} \tilde{\mathfrak{D}}$. Here we want to avoid new notations.

This remark motivates the following general definition.

DEFINITION 6.6. A domain $\mathfrak{D} \subset \mathfrak{U}$ is said saturating if:

(1) X contains at least two non-trivial pairs;

(2) every non-trivial pair is connected to every other non-trivial pair.

The result which follows is a slight improvement of Roberts' theorem [13].

THEOREM 6.1. *If f is a social choice functional defined on a domain $\mathfrak{D} \subset \mathfrak{U}$ which is saturating, and satisfies* (IIA), (WP) *and* (WC) *then there exists a continuous real-valued function $W: \mathbb{R}^M \to \mathbb{R}$ increasing with an increase in all arguments with the property that for all $u \in \mathfrak{D}$, $x, y \in X$*

$$\text{if} \quad W(u(x, \cdot)) > W(u(y, \cdot)) \text{ then } x \, P \, y.$$

Proof. See Le Breton [10].

Remark 6.4. Le Breton [10] gives an example of non-saturating domain for which the above result does not hold: we can construct social choice functionals on this domain satisfying (IIA), (WP) and (WC) but they are not weakly neutral. In Roberts' terminology with non-saturating domains non-welfare characteristics may play an important role.

From Theorems 3.1 and 6.1 we deduce

THEOREM 6.2. *If f is a social functional on $\mathfrak{J}$, satisfying* (IIA), (WP) *and* (WC) *then there exists a continuous real-valued function $W: \mathbb{R}^M \to \mathbb{R}$ increasing with an increase in all arguments with the property that for all $I \in \mathfrak{D}$, $x, y \in S_I$*

$$\text{if } W(I(x, \cdot)) > W(I(y, \cdot)) \text{ then } x \, P \, y.$$

Theorem 6.2 is fundamental. While the social opinion about inequality is not completely described by the function W, we may interpret it as a social inequality index. Moreover, if we impose conditions stronger that weak continuity, for instance, some invariance axiom, we may have more information about this function. In particular, if we assume non-comparability and ordinality (as in Sections 2 and 3) we deduce the existence of a dictator.

7. Concluding Remarks

The results we have obtained deserve some comments:

1. Here we retain strict Schur-convexity instead of Schur-convexity in order to be true to the literature about the properties of a "nice" measure of inequality. But it will be clear that the proof given in Section 3 remains true in the case of Schur-convexity. So, the impossibility result is not altered by such a type of enlargement of the space of inequality preorders. See Trannoy [19]. Otherwise in [9] Lebreton shows that the impossibility theorems proved in this paper are robust to the introduction of smoothness properties on inequality preorders.

2. We have proved that it is impossible to aggregate individual opinions toward inequality into a social one if we eschew interpersonal comparisons of utility. Sen [17] has stressed that the meaning of such comparisons is not very clear when we aggregate individual value judgments.

3. Does the negative result depend crucially on the conditions imposed by the aggregation rule? A partial answer to this question has been given by the authors in a companion paper [11] (see also [9, 19]). They show that we have a positive result in a topological framework.

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