Non-linear Sharing Rules

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Abstract

The effect of a change in wealth on its allocation between two attributes is examined when they have both the same utility. We identify three classes of utility function that generate non-linear sharing rules. The divergence between the two shares increases in absolute, average or marginal terms with the total amount of wealth, depending on whether DARA, DRRA and convex risk tolerance are considered. This result allows for a very wide range of applications, from the Arrow-Debreu contingent claims case to the risk-sharing problem, including standard portfolio choice, intertemporal individual consumption, demand for insurance and tax evasion.

(*Keywords: wealth-sharing problem; sharing rules; concavity; convex risk tolerance*)

1 Introduction

Consider an investor who allocates an exogenous wealth over two assets carrying different risk or a consumer who chooses a consumption plan over two periods. Alternatively, look at a couple who share wealth among the two members with unequal weights and whose utility

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function is identical. In spite of the differences in the setting, these three simple decision-making problems all have the structure of a cake-sharing problem with the same features: a decision maker, two ways of allocating the exogenous wealth, the amounts of the two goods expressed in monetary units like the wealth in two states of the world, the same increasing and concave function representing the cardinal utility provided by the two attributes. The correlation in which we are interested is that between the allocation and the amount of wealth. A sharing function maps wealth into the quantity consumed or invested in one good. Under the assumption of identical utility functions, the two sharing functions cannot intersect. If the decision maker prefers to consume more of an attribute for a given level of wealth, this holds for any level of wealth. For convenience, the attribute corresponding to the lower consumption will hereafter be referred to as the *less demanded* attribute. Linear sharing functions occur in many contexts, as when preferences are homothetic in the context of consumption decision or when utility functions have constant absolute or relative risk aversion. Nevertheless, in general, the structure of the problem as such does not impose linear solutions. We show that three forms of non-linear sharing curves (with the linear case as a limit) emerge if and only if the utility function belongs to one of several well-known classes. In this wealth-sharing problem, the marginal propensity to consume the less demanded attribute is decreasing with wealth whenever the utility function has increasing and convex risk tolerance (CT). Suppose now that the utility function only satisfies decreasing relative risk aversion (DRRA). Then the propensity to consume the less demanded attribute decreases as wealth increases. Finally, when the distance between the two sharing curves increases with wealth, the utility function conveys decreasing absolute risk aversion (DARA).[1] Analogous results are already known for

[1] To be specific, in what follows "*increasing*" and "*decreasing*" respectively mean *non-decreasing* and *non-increasing*. "*Absolute*" and "*relative*" risk aversion coefficients are $A(x) = -\frac{v''(x)}{v'(x)}$ and $R(x) = xA(x)$, respec-

some particular frameworks. In an insurance context, Mossin (1968) showed that the amount of coverage is decreasing with wealth when the utility function satisfies DARA. A similar result was found by Arrow (1971) for portofolio choice. Gollier (2001b) establishes the equivalence between the CT class and the concavity of the sharing function in an asset pricing model. Gollier (2001a) and (2007) provides useful connections among many models but it seems fair to say that the true extent of the question's generality has not yet been fully set out nor the numerous applications. Our set up encompasses such models as Arrow-Debreu contingent claim, the standard portfolio model, the intertemporal individual consumption choice problem, the demand for insurance, and the tax evasion decision. The same model may also be adapted to study sharing curves generated by the risk-sharing or the cake-sharing problem between two agents, the latter being highly commended from the prescriptive as well as the descriptive point of view.

The model considered here is particularly simple in that the group utility function is supposed to be additive separable and the utility function attached to each person is the same. Can such a simple model recover any sharing rule belonging to the three classes mentioned above? In the context of collective decision making, the answer is positive. For any feasible sharing rule belonging to one of the classes, a utility function can be found that generates this allocation rule as the solution to the optimization problem. Thus, the model's parsimony is not a "straitjacket" on its ability to explain empirical observed behavior.

However, the positive result does not hold when the model refers to individual decision-making and prices of the two attributes have to be introduced. The more general issue is now to derive any demand function depending on income and prices such that the marginal propensity to consume is decreasing with income. The negative result we found is somewhat unexpected.

tively. "*Risk tolerance*" is defined by $T(x) = \frac{1}{A(x)}$ (see Wilson (1968) or Gollier (2001)).

It means, for instance, that the usual intertemporal consumption model with discounted utility does not generate *any* empirically observable well-shaped demand functions, even if we can choose the utility function from among the whole class of increasing and concave functions.

The outline of the paper is as follows. The next section introduces three non-linear sharing functions on which we focus, the basic model and the characterization result of the non-linear sharing rules. Section 3 provides various interpretations of the result. In Section 4 we explore the constraint imposed by the condition that the utility functions must be the same. Section 5 gives some conclusive observations. All proofs are relegated in the appendix.

2 Non-linear sharing functions

It is convenient to start by defining a sharing function as a reduced form with no specification of any particular structural model. The sharing function $f(y)$ gives the quantity of the *less demanded attribute* consumed (invested) with respect to total wealth y for given prices (p_1, p_2) of the two attributes 1 and 2. There is no ambiguity about the identity of this good since the quantities of the two goods are expressed in the same monetary unit in management and finance applications. When prices are identical, less than one half the wealth is spent on the less demanded attribute. In consumer demand economics, the graph of a sharing function is called the *Engel curve*.

Linear sharing rules arise in individual and group decision making when utility functions are CARA or CRRA (see, among others, Eliashberg and Winkler (1981)). In some configurations, such linear solutions are imposed by the structure of the problem. For instance, Pratt (2000) showed that in a large class of problems of group decision-making under risk Pareto-efficiency of group utility frontiers, combined with the independence property of individual von Neumann-

Morgenstern utilities, results in linear sharing rules.

In general, decision models allow for non-linear solutions. We explore three nested classes of non-linear sharing functions containing the linear class as a particular case and relying on simple properties that capture a growing divergence between the demand for the two attributes when wealth increases: the "*moving away*", "*progressive*" and concave.

First, a sharing function is of the *moving away* M, if the quantity of the less demanded good moves away from the equal split consumption as wealth increases. An equally consumed quantity is defined by $\frac{y}{p_1+p_2}$. Then $f(y)$ belongs to this class when $\frac{y}{p_{1,}+p_2} - f(y)$ is increasing with y. (See Panel (a) Figure 1). It also means that the gap between the demands of the two attributes widens as wealth increases. Equivalently, the moving away phenomenon can be described in terms of expenditure: as wealth increases, the expenditure on the less demanded attribute moves away from the amount corresponding to an equal split.

Second, a sharing function belongs to the *progressive*[2] class P if the average propensity to consume $\frac{f(y)}{y}$ is decreasing with y (see Panel (b), Figure 1). In that case, the ratio between the amounts invested in the two attributes rises with wealth. Equivalently, the difference between the proportions of wealth spent on the two goods increases.

Finally, a sharing function may be classed as concave (see below Panel (c), Figure 1). In this case, the difference between the marginal propensities to consume the two attributes is increasing in wealth. It may be observed further that the difference between the marginal expenditure on the two attributes must also be rising in wealth. Denoting by C the set of concave sharing functions, it is easy to show that $C \subset P \subset M$.

[2] The term originates in the public finance literature, where a progressive tax function means that the ratio of disposable income to gross income is decreasing with income.

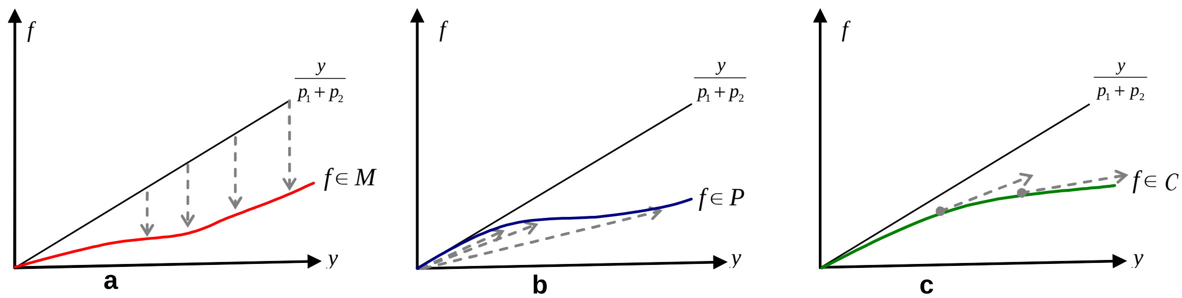

Figure 1. Three types of non-linear sharing functions

The ranking of the two attributes by demand naturally does not translate directly into the same ranking by outlay. In fact, the higher price of the less demanded attribute may mean more expenditure in this attribute for lower levels of income. When the sharing function satisfies one of the above properties, this ranking can be reversed as income increases. It is also easy to see the existence of at most a single crossing among the outlay functions of the two attributes obtains if and only if the sharing function is concave.

2.1 The result

The nested classes of sharing functions presented above are now generated as potential solutions to the following optimization problem, where x_1 and x_2 are the amounts of attributes 1 and 2, the Bernoulli utility function v is assumed to be increasing, strictly concave and differentiable as many times as required, the weight $a \in [0, \frac{1}{2}]$ and the price vector $\mathbf{p}$ has strictly positive components.

$$\max_{x_1,x_2} av(x_1) + (1-a)v(x_2) \tag{\textbf{P}}$$

$$\text{s.t.} \quad p_1x_1 + p_2x_2 = y$$

$$x_1 \geq 0;\ x_2 \geq 0.$$

From the first order conditions of (**P**), it follows that

$$\tfrac{v'(x_1^*)}{v'(x_2^*)} = \tfrac{p_1(1-a)}{p_2 a}. \tag{1}$$

Given the assumptions on the utility function, this condition implies that a higher y induces an increase of the demand for both attributes.[3] We assume

$$\lambda = \tfrac{p_1(1-a)}{p_2 a} > 1 \tag{2}$$

that is $\frac{p_1}{p_2} > \frac{a}{1-a}$. Then the Engel curve for good 1 lies below that for good 2, *i-e.* $x_1^*(y;\cdot) \leq x_2^*(y;\cdot)$. Hence, the sharing function $f(y)$ is identical to $x_1^*(y;\cdot)$. Using the budget constraint, we obtain that the expenditure in attribute 1 is less than half the budget when the two attributes have the same price. In that case, the graph of the outlay function $p_1x_1^*(y;\cdot)$ always lies below the perfectly egalitarian line in terms of expenditure. When the prices are different, the expenditure in attribute 1 is less than $\frac{p_1}{p_1+p_2}y$. This outlay in good 1 ensures that the two attributes are consumed in equal amounts.

The shape of the sharing function generated by the program above depends on that of the utility function. This link will be clarified by using some classes of utility functions that are well-known in the risk literature: $DARA$, decreasing absolute risk aversion; $DRRA$, decreasing relative risk aversion and CT, convex risk tolerance. When the three properties hold for the whole domain, we can state the following general result.

[3] Notice that a sharing function differentiable in the whole domain requires the "Inada condition" $\lim_{x\to 0} v'(x) = \infty$.

Proposition 1 Suppose that $x_1^*(y;\cdot)$ is twice continuously differentiable. Then:

i) $v \in$ DARA $\Longleftrightarrow x_1^*(y;\cdot) \in M$ for all $\lambda \geq 1$

ii) $v \in$ DRRA $\Longleftrightarrow x_1^*(y;\cdot) \in P$ for all $\lambda \geq 1$.

iii) $v \in$ CT $\Longleftrightarrow x_1^*(y;\cdot) \in C$ for all $\lambda \geq 1$.

The *CT* class is perhaps less well known than *DARA* and DRRA, even though it includes some popular utility functions such as the *HARA* class as a limit case and those expressed as a sum of linear and exponential functions (see Bell (1989)). The three statements of Proposition 1 may also be expressed in terms of expenditure rather than consumption. The difference between the expenditure on the less invested attribute and that corresponding to an equal split in consumption decreases iff the utility function is *DARA*. A *DRRA* utility function is necessary and sufficient for the share spent on the less demanded attribute to be decreasing in wealth as well; and a *CT* utility function is required for the marginal share of the less demanded attribute to decrease with wealth.

The relevance of the result to individual and collective decision-making is now discussed.

3 Interpretations

Arrow-Debreu contingency claims

Given two states of the world 1 and 2, with probability a and $1-a$, let x_1 and x_2 be the quantities of the Arrow-Debreu securities demanded in these states and p_1 and p_2 their respective prices. Let y be the initial wealth of the investor, v a state-independent utility function. $x_1^*(y, \mathbf{p}; a)$ gives the demand for the contingent claim with the highest "kernel price" $\frac{p_1}{a}$ namely the price per probability unit. The condition (2) indicates that any risk-adverse decision maker will invest less in the more expensive than in the less expensive asset. In this context, the interpretation of Proposition 1 is the following. The discrepancy between the demand for the cheaper contingent claim and the more expansive one is increasing if and only if the utility function is $DARA$. The average propensity to consume this contingent claim is decreasing with wealth if and only it is $DRRA$. Finally, the marginal propensity to consume the more expensive contingent claim is decreasing with wealth if and only if it is CT.

The standard portfolio model

Consider an agent with initial wealth y that she can invest in a risk-free asset (asset 1) and a risky asset (asset 2). There are two states of the world with probability a and $1-a$, respectively. The excess return of the risky asset is negative in state 1(the gross return is equal to $1+r-r_1$, with $r_1>0$) and positive in state 2 and equal to r_2 (the gross return is equal to $1+r+r_2$). Let z_1 and z_2 be the investment in the two assets, x_1 the final wealth in state 1 and x_2 the final wealth in state 2. They are related by the following constraints: $z_1(1+r)+z_2(1+r-r_1)=x_1$ and $z_1(1+r)+z_2(1+r+r_2)=x_2$. This gives

$$z_2 = \frac{x_2-x_1}{r_1+r_2} \tag{3}$$

and $z_1=\frac{x_1(1+r+r_2)-x_2(1+r-r_1)}{(1+r)(r_1+r_2)}$, which substituted into $z_1+z_2=y$ leads to

$$x_1\frac{r_2}{(1+r)(r_1+r_2)}+x_2\frac{r_1}{(1+r)(r_1+r_2)}=y. \tag{4}$$

The portfolio problem to solve is

$$\max_{x_1,x_2} av(x_1)+(1-a)v(x_2)$$

under constraint (4) as in program (**P**), with $p_1=\frac{r_2}{(1+r)(r_1+r_2)}$ and $p_2=\frac{r_1}{(1+r)(r_1+r_2)}$. The initial condition $\frac{p_1(1-a)}{p_2a}>1$ means $r_2(1-a)>ar_1$, that is the expected return on the risky asset must be greater than on the risk-free asset.

Part (i) of Proposition 1 means that the investment in the risky asset (see 3) is increasing with wealth iff the utility function is $DARA$. Part (ii) states that the proportion of wealth in the good state must be increasing with income iff the utility function is $DRRA$. It translates into an increasing proportion of investment in the risky asset, as it is easy to check by plugging the expression of x_1 from (4) into (3) which results in

$$\frac{z_2}{y} = \frac{x_2}{yr_2} - \frac{1}{r_1 + r_2}. \tag{5}$$

Part (iii) means that wealth in the good state is a convex function of initial wealth, and so wealth in the bad state is a concave function of initial wealth iff the utility function has convex risk-tolerance. Using (3) again this means that convex risk tolerance is necessary and sufficient to ensure that the marginal propensity to consume the risky asset (the risk-free asset) is increasing (decreasing) in initial wealth.

Tax evasion

The similarity of the tax evasion problem with the portfolio problem has long been noted (See Cowell 1990). The taxpayer is confronted with a classic economic problem of choice under risk. Consider a taxpayer who has a fixed gross income y subject to a proportional income tax at rate t. The taxpayer can conceal part of his income, e, while declaring the rest d. There are two states of the world, getting caught (state 1) and not (state 2). The probability of being caught is a and is assumed to be independent of any action by the taxpayer. When caught, the income concealed is subject to surcharge at a rate s. In state 1, the taxpayer pays a tax ty $+se$, whereas in state 2 he pays a tax of td. Declared income and concealed income are thus equivalent respectively to a safe asset with negative return and a risky asset. The return to the safe asset is equal to $1-t$. The excess return to a dollar of evaded with respect to declared income is negative in state 1 and is equal to s (the gross return is $1 - t - s$) and positive in state 2 and is equal to t (the gross return is 1). Let x_1 and x_2 be the final wealth in the two states. They are defined by the following constraints $x_1 = d(1-t) + e(1-t-s)$ and $x_2 = d(1-t) + e$ which give

$$e = \frac{x_2 - x_1}{s+t}$$

and $d = \frac{x_1 - x_2(1-t-s)}{(1-t)(s+t)}$ which if substituted into $d + e = y$ results in

$$x_1 \frac{t}{(1-t)(s+t)} + x_2 \frac{s}{(1-t)(s+t)} = y. \quad (6)$$

The tax evasion reduces to

$$\max_{x_1, x_2} av(x_1) + (1-a)v(x_2)$$

under the same constraint (6) as in program (**P**), with $p_1 = \frac{t}{(1-t)(s+t)}$ and $p_2 = \frac{s}{(1-t)(s+t)}$. The initial condition $\frac{p_1(1-a)}{p_2 a} > 1$ means $t(1-a) > as$, that is, the net expected return of the concealed income must be positive.

Part (i) of Proposition 1 tells us that evaded income is increasing with wealth iff the utility function is *DARA*. Part (ii) states that an increasing proportion of income is concealed iff the utility function is DRRA. Part (iii) tells us that convex risk tolerance is necessary and sufficient to ensure that the marginal propensity to evade is increasing with wealth.

Insurance

Consider an agent with initial wealth Y who faces the risk of a loss of $-X$ (with $X > 0$) in state 1 with probability a. This loss can be covered by an insurance contract where the policyholder can choose the optimal absolute coverage $0 \leq C \leq X$. The premium βC is proportional to the coverage, with $\beta < 1$. The final wealth available in the "bad" state 1 is

$$x_1 = Y - X + (1-\beta)C, \quad (7)$$

and

$$x_2 = Y - \beta C \quad (8)$$

in the "good" state 2. Observe that the uninsured loss $X - C$, denoted z_1, is simply

$$z_1 = x_2 - x_1$$

while $z_2 = Y - X + C$ is the wealth covered by insurance or risk-free. From (8) we get $z_1 = X - \frac{Y - x_2}{\beta}$, and by using (7) we can write $z_2 = \frac{\beta(X-Y) + x_1}{(1-\beta)}$. By substituting into $z_1 + z_2 = Y$, we get :

$$\beta x_1 + (1-\beta)x_2 = Y - \beta X = y \tag{9}$$

The decision problem faced by the policyholder then becomes: $\max\limits_{x_1,x_2} av(x_1) + (1-a)v(x_2)$, under constraint (9). The initial condition $\frac{p_1(1-a)}{p_2 a} > 1$ translates into $\frac{\beta}{a} > \frac{(1-\beta)}{(1-a)}$, that is $\beta > a$, which means a positive loading factor charged by the insurance company over and above the fair insurance premium. In this context, part (i) of Proposition 1 means that the uninsured wealth is increasing with wealth iff the utility function is DARA. Part (ii) states that the proportion of insured wealth decreases with income iff the utility function is DRRA. Part (iii) means that the insured wealth is a concave function of wealth iff the utility function is CT. Observe that due to (9), these results cover the case of a change in expected loss.

The intertemporal consumption setting

Proposition 1 above clarifies the connection between consumption and wealth, in a simple intertemporal setting. An agent lives two periods 1 and 2 and wishes to smooth consumption. His exogenous wealth is y, consumption in the two periods is x_1 and x_2. His saving in the first period is $y - x_1$. The agent has an intertemporal separable utility function where the subjective discount utility factor is $\beta < 1$, which implies $a = \frac{1}{1+\beta}$ and $1 - a = \frac{\beta}{1+\beta}$. There is a risk-free asset bringing an interest r. With $p_1 = 1$ and $p_2 = \frac{1}{1+r}$ the market discount factor, the budget constraint is written as in program **P.** The initial condition $\lambda > 1$, which ensures lower consumption in the first period (when the agent is younger) than in the second period, is obtained when the subjective discount factor β is greater than the market discount factor. Hence the marginal opportunity cost of saving is lower than the intertemporal marginal rate of substitution, meaning that the agent will consume less than half of his wealth in the first period. Statement (i) of Proposition 1 can thus be interpretated as follows: The positive difference between the future and current period consumption is increasing iff the decision maker has a $DARA$ utility function. In other words, this class of utility functions is the largest one that ensures that saving is globally increasing with wealth. Statement (ii) says that the Keynesian concept of average propensity to consume is decreasing for decision makers who have decreasing relative risk aversion. Statement (iii) means that the marginal propensity to consume (in the first period) is decreasing iff the risk tolerance of the decision maker is convex.

Intra-household allocation

This model posits two spouses with the same cardinal utility function. This assumption may reflect a normative point of view, i.e. that one euro of expenditure procures the same marginal utility to any adult person with similar needs. This paves the way to a normative interpretation of the results in this specific model. The spouses have to decide the allocation of the household budget y between them. There are no externalities or public consumption. The private expenditures of the two individuals are x_1 and x_2 and prices are equal to 1. The balance of power among them is captured by the weight a and it is assumed that individual 1 is the "weaker" individual, that is $a < 1/2$. $x_1^*(y, a)$ gives the private expenditure of the weaker individual as a function of the household budget and the weight a. The proposition illuminates the importance of the properties of the cardinal utility function for describing how the consumption of the weaker party relates to the household budget. His marginal part is always less than 1/2 in other words, the difference between the two is increasing with household income iff the utility function satisfies $DARA$. The share of the weaker party decreases iff the utility function belongs to the $DRRA$ type. The marginal portion devoted to the weaker party is decreasing[4] iff the utility function exhibits convex risk tolerance.

Risk-sharing

The second intra-household model is placed in the same framework, but now the income of the household is risky. The spouses earn a random individual incomes z, which are contingent on the realization of θ belonging to a set of states of the world Θ and are not perfectly correlated. So household income becomes a random variable $\gamma : \Theta \to \mathbb{R}$. The two individuals agree to represent risk by a cumulative distribution function $F : \Theta \to [0, 1]$. Let $v(x)$ be the identical vNM utility of the two spouses. Hence, the household solves the following program:

$$\max_{x_1,x_2} \ a \int_{\Theta} v(x_1(\theta))dF(\theta) + (1-a) \int_{\Theta} v(x_2(\theta))dF(\theta), \ \textit{with } a \in (0, \frac{1}{2}] \tag{10}$$

$$\text{s.t.} \quad z_1(\theta) + z_2(\theta) = y(\theta) = x_1(\theta) + x_2(\theta), \ \forall \theta \in \Theta; \quad x_1 \geq 0; \quad x_2 \geq 0.$$

By the Pareto-efficiency condition obtained by Borch (1960, 1962), the consumption in each state of the world should depend only on the total wealth in that state. That is, the function to be maximized can be written

$$a \int_{\Theta} v(x_1(y))dF(\gamma^{-1}(y)) + (1-a) \int_{\Theta} v(x_2(y))dF(\gamma^{-1}(y) \tag{11}$$

under

$$y = x_1(y) + x_2(y), \ \forall y \in \gamma(\Theta).$$

Since wealth is not transferable from one state to another, solving the above program requires solving Program P for any feasible household income y. Then for a given household

[4] The concavity of the intra-household sharing function is shown to be crucial in welfare analysis involving stochastic dominance criteria (See Peluso and Trannoy (2007)).

income, the problem reduces to the simple intra-household allocation model described above. Then, for instance Proposition 1 *(iii)* tells us that a concave risk-sharing function for the weaker individual arises if and only if the utility function is CT. Further results and extensions concerning the concavity of the sharing function of groups with different risk-aversion among members have been explored by Hara et al. (2007).

4 Is a unique utility function restrictive?

An essential feature of the program (P) is the postulation of the same utility function for both attributes. As we have seen, a number of applications can support this assumption. In the case of individual decision-making, this postulate is standard for intertemporal decisions. For group decisions it is more questionable. Might positing identical utility functions impose some restriction on the classes of non-linear sharing functions generated by (**P**)? Our next two propositions give mixed answers.

Proposition 2 proves that the assumption is not restrictive in the group decision-making set-up: we can generate all *moving away* sharing functions through Program (**P**) when prices are fixed.

Proposition 2 *For all $f(y) \in \mathcal{M}$ and $a \in (0, 1/2)$, there exists a continuous differentiable utility function v such that, for all $y \in \mathbb{R}_+, x_1^*(y; a, v)) = f(y)$.*

In this sense, the model with a unique utility function is parsimonious.

It would be important to extend the previous result by introducing prices and requiring the complete recoverability of any "demand" function depending on wealth and prices[5] under the restriction that it remains concave in the wealth dimension. When changes in prices are

[5] Kernel prices in the contingent claims model, or interest rate in the intertemporal consumption setting.

also allowed, the usual *demand function* is $x_1^*(y, \mathbf{p}, a)$ which can be written more compactly as $x_1^*(y, \lambda)$. Unfortunately, the answer is disappointing, as Proposition 3 shows.

Proposition 3 *For some demand functions $f(y, \lambda)$ concave w.r.t. y and decreasing with λ, there does not exist a utility function v such that for all $y \in \mathbb{R}_+$ and for all λ, $x_1^*(y, \lambda; v)) = f(y, \lambda)$*

There are well-defined demand functions concave in wealth that cannot be generated by the program (**P**). This negative result also holds for the two larger classes.since the class of *progressive* and *moving away* sharing functions contains the concave ones, This result is especially telling for the three first interpretations of the model, the Arrow-Debreu contingent claims and the portfolio models, as for the intertemporal consumption model.

5 Concluding remarks

One aim of decision theory is to find regularities that explain the behavior of the decision-maker independently of the context. This article finds one common feature in a cake-sharing problem when the utility maximizing decision-maker is:

1) an *individual* who allocates an exogenous quantity of wealth among two attributes providing utility through the same cardinal function.

2) a *group* with a exogenous wealth to share between two agents with different weights, whose utility is identical.

We have examined the impact of a change in wealth on the optimal allocation among the two attributes. This relation is encapsulated by the sharing function, which can be viewed as a reduced form of the decision process. A very simple model is able to generate very neat non-linear sharing rules whereby the divergence between the two demands increases either in absolute, average or marginal terms as the size of the cake increases. The model is shown to be very parsimonious in the context of collective decision-making theory. In the framework of individual decision-making, the assumption of identical utility functions prevents some sharing functions from being reproduced.

A natural question that arises, in conclusion, is whether the characterization result still obtains when some heterogeneity of preference is allowed. An easy extension is given in Peluso and Trannoy (2005), when the weights and the utility functions depend on the same parameter in the context of risk-sharing. This is a matter for further research, as the extension to a model with more than two attributes.

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Appendix

Proof of Proposition 1

The two first statements have been known since Mossin (1968) and Arrow (1970) in risk bearing and portfolio choice theory, respectively, and are easily proved in the general case. As to the third statement, by choosing unitary prices and differentiating the f.o.c. of (**P**) it follows that: $\frac{\partial x_1^*}{\partial y}(y; \cdot\ ) = \frac{-(1-a)v''(x_2^*)}{-av''(x_1^*)-(1-a)v''(x_2^*)}$. After simple manipulations and using (1), the previous condition may be expressed as in Wilson (1968) $\frac{\partial x_1^*}{\partial y}(y; \cdot\ ) = \frac{T(x_1^*)}{T(x_1^*)+T(x_2^*)}$. Then $\frac{\partial x_1^*}{\partial y}(y; \cdot\ )$ is non-increasing iff $\frac{T(x_2^*)}{T(x_1^*)}$ is non-decreasing with y. Convex risk tolerance ensures this property. To require $\frac{T(x_2^*)}{T(x_1^*)}$ non-decreasing with y for any $x_1^* \leq x_2^*$ leads to the necessary part of the proof, as is easily proved by contradiction.

Proof of Proposition 2.

A constructive proof is now provided of the existence of solutions for the f.o.c. of the program (P) for any $f(y)$.

$$u'(f(y); a) = \tfrac{1-a}{a} u'(y - f(y); a), \text{ for all } y \in \mathbb{R}_{++}. \tag{12}$$

Given the assumption of unitary prices, $\lambda = \frac{1-a}{a}$. For a given a, let g be the inverse function of f w.r.t. y. g is increasing and such that $g(x) \geq 2x$, $\forall x \in R_{++}$. Denoting $h(x) = g(x) - x$, a solution u' should satisfy the functional equation:

$$\tfrac{u'(x;a)}{\lambda} = u'(h(x); a), \text{ for all } x > 0. \tag{13}$$

Condition (13) may be expressed as

$$\tfrac{v'(x)}{\lambda} = v'(h(x)) \Longleftrightarrow \tfrac{v'(j(x))}{\lambda} = v'(x) \tag{14}$$

where $j = h^{-1}$. Since j is a one-to-one mapping of R_+, it is meaningful to consider iterative compositions of j. From $\frac{v'(x)}{v'(j(x))} = \frac{1}{\lambda}$ and $\frac{v'(j(x))}{v'(j \circ j(x))} = \frac{1}{\lambda}$, it follows $\frac{v'(x)}{v'(j \circ j(x))} = \frac{1}{\lambda^2}$ and, more generally:

$$\frac{v'(x)}{v'(j_{n-1}(x))} = \frac{1}{\lambda^n} \forall\ n = 1, 2, .. \tag{15}$$

where $j_0(x) = j(x)$ and $j_{n-1}(x) = j \circ .... \circ j$, for $n-1$ times. Starting from (14) and (15), a solution $u'(x; \lambda)$ of (13) may be constructed.

Step 1. Starting from an arbitrary $(x_a, v'(x_a))$, with x_a and $v'(x_a) > 0$, a second point $(x_b, v'(x_b))$ can be uniquely determined. Posing $x_b = h(x_a)$, from (14) it follows $v'(x_b) = v'(x_a)/\lambda$ (see Figure 3.a).

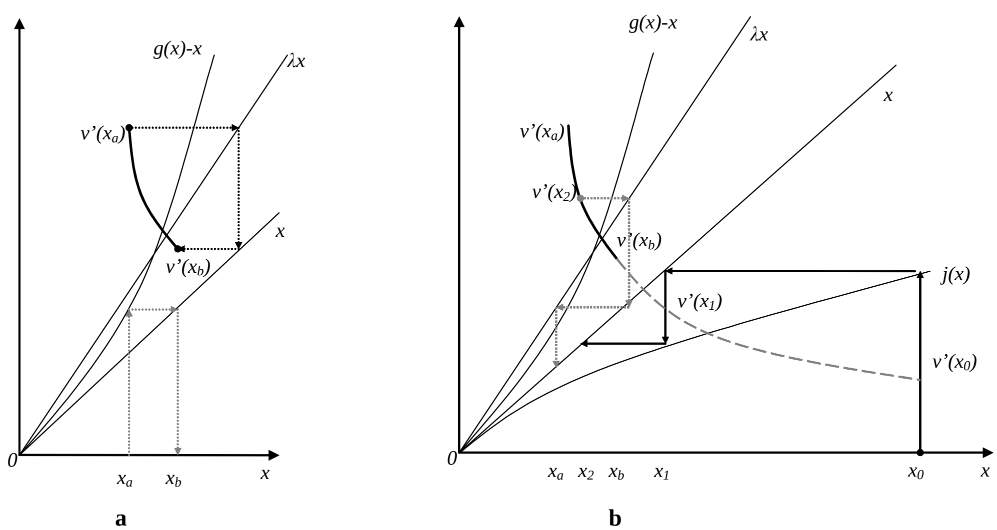

Figure 3

Step 2. Joining the points $(x_a, v'(x_a))$ and $(x_b, v'(x_b))$ by a decreasing segment belonging to an arbitrary strictly decreasing and continuous function w defined on $[x_a, x_b]$, any other value $v'(\bar{x})$, for $\bar{x} > 0$, can be determined. Two cases are possible.

a) $\bar{x} > x_b$. The function j (inverse of h) is increasing and such that $j(x) < x$ for all $x > 0$. Then the real sequence x_n, defined by $x_{n+1} = j(x_n)$ is decreasing and converges to 0. Since $x_a = j(x_b)$ and j is strictly increasing, then $j(\bar{x}) > j(x_b)$. By setting $x_0 = \bar{x}$, $x_1 > x_a$. Even if $x_1 > x_b$, there exists a k such that $x_k \in [x_a, x_b]$. Then $v'(x_k) = w(x_k)$. By (15), it follows $v'(\bar{x}) = v'(x_k)/\lambda^k$. An example with $k = 2$ is sketched in Figure 3.b.

b) $\bar{x} < x_a$. Let x_n be the real sequence defined by $x_{n+1} = h(x_n)$. It is increasing and diverges at ∞. By setting $x_0 = \bar{x} < x_a$, since $x_b = h(x_a)$, then there exists an integer k such that $x_k \in [x_a, x_b]$ and, by reasoning as in the case a, $v'(\bar{x}) = \lambda^k v'(x_k)$. In general, choosing an arbitrary strictly decreasing and continuous function w, such that $w(x_a) = q$ and $w(h(x_a)) = \frac{q}{\lambda}$, for some $q > 0$, the solution is found:

$$u'(x;\lambda,w) = \begin{array}{cc} \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda^2 w \circ h_1(x) & j_1(x_a) \leq x < j(x_a) \\ \lambda w \circ h(x) & j(x_a) \leq x < x_a \\ w(x) & x_a \leq x < h(x_a) \\ \frac{w \circ j_1(x)}{\lambda^2} & h(x_a) \leq x < h_1(x_a) \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots \end{array}$$

Now it is shown that $u'(x;\lambda)$ is right-continuous.since w and h are continuous. Let $(h_1(x_a);\lambda)$ be a possible point of non-continuity. Since $u'(h_1(x_a);\lambda) = \frac{w \circ j_1(h_1(x_a))}{\lambda^2} = \frac{q}{\lambda^2}$. By construction, $\lim_{x \to h \circ h(x_a)^-} u'(h_1(x_a);\lambda) = \lim_{x \to h \circ h(x_a)^-} \frac{w \circ j(x)}{\lambda} = \lim_{z \to x_a^-} \frac{w \circ j(h_1(z))}{\lambda} = \lim_{z \to x_a^-} \frac{w(h(z))}{\lambda} = \frac{q}{\lambda^2}$. Given the analogy with any other possible discontinuity, u' must be continuous on R_{++}. ■

Proof of Proposition 3. ■

Without loss of generality, let $a = (1 - a)$, $p_2 = 1$. Then $p_1 = \lambda$ and condition (1) becomes:

$$v'(f(y,\lambda)) = \lambda v'(y - f(y,\lambda)). \tag{16}$$

For a fixed sharing function f, with $\frac{\partial}{\partial y} f(y,\lambda) > 0$, $\frac{\partial^2}{\partial y^2} f(y,\lambda) < 0$ and $\frac{\partial}{\partial \lambda} f(y,\lambda) < 0$, we are looking for a positive function v', decreasing w.r.t. x, which solves (16). Equation (16) can equivalently be expressed as $v'(x) = \lambda v'(g(x,\lambda) - x)$, where $g(x,\lambda)$ is the inverse function of $f(y,\lambda)$ w.r.t. y. It is now shown that, for a well-defined $f(y,\lambda)$, the equation (16) does not admit solutions. Let $h(x,\lambda) = g(x,\lambda) - x$. By choosing $h(x,\lambda) = e^{\lambda x} - 1$, this function implicitly defines $f(y,\lambda)$ as strictly concave, with values on $R_+ \times [1, \infty)$ and satisfying $f(y, \lambda) \leq 1/2y, \forall \lambda > 1$, in other terms, a well-shaped sharing function. Moreover, h has a unique inverse with respect to λ. Then, following Aczel [2], p. 21, let $e^{\lambda x} - 1 = t$, and set a value of x, let us say $x = k$. It follows $\lambda = \frac{\ln(t+1)}{k}$. From (16), if a solution exists, it must fulfill the condition $v'(k) = v'(t)\frac{\ln(t+1)}{k}$, that is $v'(t) = \frac{kv'(k)}{\ln(t+1)} = \frac{c}{\ln(t+1)}$. Unfortunately, for $h(x, \lambda) = e^{\lambda x} - 1$, the function $v'(x) = \frac{c}{\ln(x+1)}$ does not solve $v'(x) = \lambda v'(h(x, \lambda))$.