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NOTE

ON THON'S AXIOMATIZATION OF THE GINI INDEX

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We give a short proof of Thon's characterization of the Gini index.

Key words: Income inequality; Gini index.

1. Introduction

In a past issue of this journal, Thon (1982) has provided an interesting axiomatization of the Gini index. First, we give a simple proof of this main result with a slight strengthening of one of Thon's axioms. Then we comment on Thon's most important axiom, the equidistance one.

2. A simple proof

We use the notations of Thon's paper and only strengthen the first part of the constant population comparability axiom (Axiom CPC):

Axiom CPC*.

$$\min\{I(Y): Y \in D(n,S)\} = 0, \quad \forall S, n.$$

Second part of Axiom CPC.

Combined with the transfer axiom it means that the value of the inequality index must be zero if all individuals have an income equal to the average. This property is satisfied by all inequality indices and it is considered as a minimal requirement (see Shorrocks, 1980).

Theorem. *Axioms E, T, S, and CPC* hold altogether iff $I(Y)$ is the Gini index up* *a positive linear transformation:*

$$I(Y) = CG(Y), \quad \text{with } C \in \mathbb{R}_{++},$$

with

$$G(Y) = \frac{2}{n^2 \bar{y}} \sum_{i=1}^{n} \left(i - \frac{n+1}{2}\right) y_i, \text{ with } y_1 \leq y_2 \cdots \leq y_i \cdots \leq y_n.$$

Proof. The sufficient part is obvious; for the necessity part, recall that Y is always ordered such that $y_1 \leq y_2 \leq \cdots \leq y_n$.

Let us define:

$$w(\bar{y}, n) = w_2(\bar{y}, n) - w_1(\bar{y}, n),$$

$$w_0(\bar{y}, n) = w_1(\bar{y}, n) - w(\bar{y}, n).$$

Then, with Axiom E, we can write:

$$w_i(\bar{y}, n) = w_0(\bar{y}, n) + i w(\bar{y}, n), \quad i = 1, \ldots, n.$$

Now the first part of Axiom CPC* and Axiom T imply that:

$$\sum_{i=1}^{n} w_i(\bar{y}, n) = 0 \Leftrightarrow n w_0(\bar{y}, n) + \frac{n(n+1)}{2} w(\bar{y}, n) = 0.$$

Or

$$w_0(\bar{y}, n) = -\frac{n+1}{2} w(\bar{y}, n).$$

Therefore,

$$w_i(\bar{y}, n) = w(\bar{y}, n)\left(i - \frac{n+1}{2}\right), \quad i = 1, \ldots, n. \tag{1}$$

By Axiom T, the maximum value of $I(Y)$ for all Y of dimension n and given $\bar{y}$ is, for a distribution, such that someone receives all the income and everyone else has zero income, and therefore is equal to

$$w(\bar{y}, n)\left(\frac{n+1}{2}\right) n\bar{y}.$$

Axiom CPC* implies, for all Y and Y':

$$w(\bar{y}, n) = w(\bar{y}', n)\bar{y}'/\bar{y}.$$

Then, we can write

$$w(\bar{y}, n) = \frac{s(n)}{\bar{y}}, \tag{2}$$

where $s(n)$ is some function of n.

With (1) and (2), we can write:

$$I(Y)=\frac{s(n)}{\bar{y}}\sum_{i=1}^{n}\left(i-\frac{n+1}{2}\right)y_i \qquad (3)$$

and

$$I(g(Y,k))=\frac{s(kn)}{\bar{y}}\sum_{i=1}^{kn}\left(i-\frac{n+1}{2}\right)y_i.$$

Using the fact that the Gini index is also equal to

$$\frac{1}{2\bar{y}n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}|y_i-y_j|$$

(Sen, 1973, p. 31), we can write:

$$I(g(Y,k))=\frac{s(kn)}{\bar{y}}k^2\sum_{i=1}^{n}\left(i-\frac{n+1}{2}\right)y_i,$$

and Axiom S holds if and only if:

$$s(n)=s(kn)k^2$$

or

$$\frac{s(n)}{4}=s(2n)=\frac{s(2)}{n^2},$$

which gives $s(n)=C/n^2$, with $C=4s(2)$. With Axiom T, C belongs to $\mathbb{R}_{++}$ and this completes the proof. □

3. The equidistance axiom

Let us recall the equidistance axiom:

Axiom E

$$I(Y)=\sum_{i=1}^{n}w_i(\bar{y},n)y_i,$$

with

$$w_{i+1}(\bar{y},n)-w_i(\bar{y},n)=w_{j+1}(\bar{y},n)-w_j(\bar{y},n),\ \forall i,j\leq n-1.$$

The economic content of the second part of this axiom is that an order-preserving equalizing transfer between any two successive people has the same effect on $I(Y)$. But we have a comment to make on the first part of the axiom. Thon writes that any index can be written as $I(Y)=\sum_{i=1}^{n}w_i(y)y_i$. We agree with him; it suffices to define:

$$w_i(y)=\frac{I(Y)}{ny_i},\quad \text{for } y_i\neq 0.$$

But we cannot say the same thing for $I(Y)=\sum_{i=1}^{n}w_i(\bar{y},n)y_i$. Therefore, the first part of the equidistance axiom is not innocuous at all and it would be worth it to put the following axiom in the place of Axiom E:

Axiom E*.

$$I(f_i(Y,r)) = I(f_j(Y,r)),$$

$$0 \leq r \leq \min\left(\frac{y_{i+1}-y_i}{2}, \frac{y_{j+1}-y_j}{2}\right), \ \forall i,j \leq n.$$

It remains an open problem to know whether axioms E*, T, S, and CPC* characterize the Gini index.

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