Protecting Minorities through the Average Voting Rule*

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Abstract

Properties of an average voting rule - the outcome being some weighted average of votes - are investigated, with particular attention to its ability to protect minorities. The unique average voting outcome is characterized with a median formula which depends on the voters' preferred allocations and some parameters constructed from the voters' weights. We provide necessary and sufficient conditions for the average outcome to be above the majority outcome. A minority is said to be protected by a switch in voting rule if the voting outcome becomes closer to the median bliss point of the minority. A sufficient condition for minority protection is that, either the minority's weight is sufficiently large or the majority outcome is too unfavorable to the minority. Applications to the composition of public goods and to public expenditures level are considered. We end by exploring the combined use of average and majority voting in a two-stage procedure for determining both the level and the composition of public expenditures.

Keywords: minority, majority voting, public goods, Nash equilibrium.
(**JEL**: D74, H41, I22)

1 Introduction

If a minority's aspirations are insufficiently taken into account by the collective decision process, the resulting tensions can only be settled through collective action outside the legitimate institutions; in the most extreme cases, it may involve violence. For instance, it has been found that countries where representatives are elected under the majority rule face more political violence than those with proportional representation (see Powell (1981)). The history of Europe throughout the twentieth century provides numerous tragic examples of

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how minority issues may undermine the unity of a nation. One such example is Northern Ireland where Catholics and Protestants tend to systematically oppose on all issues. Some observers like Emerson (1998) strongly disapprove of the use of the majority rule on the grounds that it is ineffective in solving the conflict and may actually reinforce it. North American societies are also confronted with recurrent unrest caused by the dissatisfaction and frustrations of some ethnic, religious or language minority (see for instance Guinier (1994)). When a decision is not unanimous, some community members are bound to lose from it. If the losing members' dissatisfaction is excessive, they will be less willing to cooperate and it might become necessary to deplete resources to ensure, either through enticement or coercion, that they comply with the collective decision.

Majority voting often comes under attack for providing a poor representation of minorities. The issue arises most strikingly in the case of a bimodal distribution, with one group clustered around one extreme and the other clustered around the other. If one group is larger than the other, the median voter will be at one extreme, and the smaller group's preferences are completely ignored in the majority voting outcome. For instance if the issue is how to allocate public funds among two competing uses, such as Arabic and Hebrew schools, and all individuals only care about one of the two types, majority voting will provide only one type of school. The fairness of majority voting is dubious in this case, since both communities must pay taxes but only one receives the good it wants. The present paper investigates the properties of an alternative to the majority rule, the average voting procedure, with particular attention to circumstances under which it may ensure an improved representation of minorities.

Some countries' tax systems have provisions which are a good illustration of what is meant by an average voting rule. In these countries a "forced to pay yet free to choose mechanism" is used to determine the distribution of public expenditures among several uses. In France, for instance, corporations must pay a "training tax", the amount of which is based on their wage bill. They may however decide on which teaching institution or training program receives the money. In Canadian provinces of Ontario and Saskatchewan, the tax system allows for the existence of publicly financed separate school boards along with the public school boards; households may decide on whether their property taxes should be used to finance the public or separate school board. Bilodeau (1994) argues that the provisions for financing school boards in Canada have helped to limit conflicts resulting from the existence of a catholic minority. The efforts of the Ontarian government to remove the school-board system have been deemed illegal by a court on the basis that such a decision would hurt the constitutional rights of the catholic minority[1]. In Spain, tax payers may devote up to 3% of their income tax to financing the catholic church and similar provisions can be found in other European countries such as Italy[2] or Germany.

These tax systems are formally equivalent to weighted average voting rules for determining the allocation of public expenditures. If there are only two possible uses of public funds (public and private education for instance), the vote of a tax payer is the fraction of his taxes which he chooses to allocate to one of them (say public education). Then the outcome of the vote (the proportion of public funds going to public education) is a weighted average of the votes, where the weight of each voter is his share in total tax contributions.

Obviously such a rule cannot be used for every purpose and requires that the choice space be continuous. Fortunately many social choices, notably those concerning economic issues, have a truly quantitative feature. For instance, the average procedure could be used when the issue at hand is the fraction of total wealth that should be allocated to the provision of a public good as in the voting problem studied by Bowen (1943)[3].

The average is quite an intuitive alternative to the majority rule. Once again the bimodal population example illustrates the point quite nicely. In contrast with majority voting, every minority voter's ballot counts and contributes to shifting the outcome closer to the middle of the interval. However, as the present paper shows, there is a potential for strategic manipulation and average voting does not usually yield an average opinion. Hence a precise assessment of how minorities may benefit from such a procedure requires a specific characterization of

[1] Source: "*The Globe and Mail*", Friday, July 24,1998.

[2] In Italy the percentage of the income tax which may be devoted to financing the church is up to 0.8.

[3] See Section 4.2 for a formal argument.

the outcome of the vote taking into account strategic behavior.

The paper presents a simple average voting model in which the issue is one-dimensional. The problem considered here is that of choosing an allocation fully described by a real number in some closed interval (e.g., share of total wealth devoted to public uses, share of property taxes devoted to public schools). Voting consists in announcing a value for the allocation, the result of the vote being some weighted average of announcements. Voters are taken to have a non cooperative behavior. It is first shown that the Nash equilibrium allocation resulting from average voting is unique. Most voters behave strategically by choosing a vote at either end point of the interval. We provide two alternative characterizations, one of which expresses the equilibrium allocation as the median of a set comprised of the voters' bliss points and parameters that are functions of weights and on the vote cap. The latter characterization allows for a direct comparison with the majority outcome. We also show that the outcome for a large population may be approximated by a simple fixed point formula.

In order to evaluate how well average voting performs in protecting minorities, its outcome is compared to that of majority voting, which, in a one dimensional environment with single-peaked preferences, is the median voter's preferred allocation[4] (see Black, 1948). We identify a minority as a subgroup whose members' preferred allocations are on one side (henceforth on the right) of the median voter's preferred allocation. We say that a minority is *protected* by a switch from majority to average voting if the outcome of the voting game is moved closer to the median bliss point of the minority. Ensuring a gain for the minority's median voter undermines the support for political activism within the minority: in particular it reduces the risk that an attempt at secession is successful. We show that a sufficient condition for minority protection is that the weight of the minority exceeds the majority outcome. This corresponds to a situation where the minority is relatively strong (e.g., because of its share in overall population or its share in total wealth), or where the majority outcome is sufficiently close to zero. In the first interpretation, minority protection is all the more needed that the minority could use its power to destabilize political institutions. In the second interpretation,

[4] Alternatively this allocation can be viewed as the outcome of the competition between two downsian political parties (see Downs (1957)).

the minority's frustration with the majority outcome is so severe that fairness considerations may vindicate a change in the decision rule. It is also shown that it may be necessary to impose a cap on votes, as in Spain or Italy, to mitigate the minority's strategic power and prevent the outcome from moving too far to the right. We then turn to investigating the possibility of using a lower bound (a floor) on votes to protect the minority even when its weight is less than the median outcome. We find that the restriction on votes that is needed under average voting is less severe than what it would be under majority voting. In order to illustrate the empirical relevance of the above results, two public goods applications are considered: the choice of an allocation of public funds between two alternative uses (the "Forced to Pay yet Free to Choose" model) and the choice of the fraction of total wealth allocated to public uses (Bowen's model). Finally we explore the joint use of majority and average voting in a context where public expenditures and their allocation among different uses are chosen sequentially. More specifically, we introduce average voting in the framework of Alesina *et al.* (1999) who study a *two stages* procedure where majority voting is used at both stages. Public good spending is chosen in the first stage, while public good composition is determined in the second stage.

Section 2 provides a characterization of the average voting outcome. A discussion of its merits in protecting minorities relative to majority voting is offered in Section 3. Applications are presented in Section 4, while Section 5 is devoted to the combined use of average voting and majority voting applied to a sequential choice of public good spending and its composition. Some final remarks are gathered in Section 6.

2 The Average Voting Outcome

The social choice problem under consideration is as follows. The social state y belongs to some bounded interval normalized to $[0, 1]$ without loss of generality. Boundedness may reflect a budget constraint or, more generally, that resources are scarce. There are n voters indexed by i. Each voter's preferences are single-peaked and represented by a continuous utility function, u_i with b_i denoting the bliss point. Individual i has a given weight, $w_i \geq 0$,

$\sum_{i=1}^{n} w_i = 1$. Apart from the equal weight case where all voters are treated anonymously, these weights may have various interpretations: individual share in total wealth or in total tax contribution or, if i represents some collective entity (constituency, country, company in a shareholder assembly...), the weight may be related to the importance of the group among the overall population assessed on some criterions. For instance, the weight may indicate the population share of group i in the overall population[5]. The game under consideration is as follows. Each voter i chooses a vote denoted s_i in $[0, c]$ with $0 < c \leq 1$ and voting involves no costs. Allowing for a vote cap $c < 1$ is meant to account for actual situations such as church financing in Spain and Italy where the strategic space does not coincide with the social choice space. The value cw_i is referred to as the corrected weight. Votes are cast simultaneously and the allocation is

$$y = \sum_{i=1}^{n} w_i s_i. \tag{1}$$

Since the strategic space $[0, c]$ is a subset of the space of feasible allocations and since the latter space is convex, the average outcome is always feasible. Tastes as well as weights are common knowledge. It is now shown that the game has a unique equilibrium allocation.

In this context, a voter's optimal behavior is quite simple. Other player's choices only matter to player i in so far as they affect the aggregate vote, S_{-i}, which is the weighted sum of votes by the rest of the population, that is, $S_{-i} = \sum_{j \neq i} w_j s_j$. Then agent i's best response is given by

$$r_i(b_i, S_{-i}) = \begin{cases} 0 & \text{if } b_i < S_{-i} \\ \frac{b_i - S_{-i}}{w_i} & \text{if } S_{-i} \leq b_i < S_{-i} + w_i c \\ c & \text{if } b_i > S_{-i} + w_i c. \end{cases} \tag{2}$$

The behavior described by r_i is based on a comparison between the bliss point of voter i, b_i, and the aggregate vote of the rest of the population S_{-i}. If the aggregate vote by others yields a value that is beyond the bliss point (first line in Equation (2)), it is optimal to vote 0 since any non zero vote would make the situation worse. If the aggregate vote by others is

[5] The demographic interpretation however is not appropriate in the analysis of subsequent sections, where the average outcome is compared to an unweighted median.

below the bliss point, two situations are possible depending on the size of the discrepancy. If it is not too large (middle line in Equation (2)), agent i's corrected weight, $w_i c$, may enable him to make up for the difference, in which case he obtains his exact bliss point as the final outcome. If the difference is too large (last line in Equation (2)), the voter shifts the final allocation upwards and makes it closer to his preferred outcome without reaching it. In the latter case, it is optimal to pick the largest possible vote which is c.

The best response is clearly increasing in b_i, which suggests that the equilibrium vote is also increasing in b_i. It is now useful to rank individuals according to *decreasing* values of b_i.[6] Let us define the cumulative weight of the first i individuals:

$$W_i = \sum_{j=1}^{i} w_j.$$

The value cW_i is referred to as the corrected cumulative weight. Now let

$$i^* = \min\{i \in \{1, ..., n\}; cW_i \geq b_{i+1}\},$$

with $b_{n+1} = 0$.

The following proposition provides a characterization of the unique equilibrium allocation.

Proposition 2.1 *The average voting game has a Nash equilibrium. Furthermore, the equilibrium allocation, y^*, is unique and is given by:*

$$y^* = \min\{b_{i^*}, cW_{i^*}\}. \tag{C1}$$

Proof. Since preferences are single-peaked, existence is an immediate consequence of Debreu's theorem (1952). Let y be an equilibrium allocation. Note that, if for individual i, $b_i > y$, we must have $s_i = c$. If not, individual i can modify the allocation in his favor by increasing s_i. A similar argument shows that, if $b_i < y$, then $s_i = 0$. Let $E = \{i \in \{1, \cdots, n\} : b_i > y\}$, with $e = \#\{i : i \in E\}$. Then $y \geq cW_e$. It is now useful to distinguish two cases.

[6] Note that if a group of individuals share the same bliss point the sequence of individuals is not uniquely defined and it depends on the order in which individuals within the group are ranked. However characterizations (C1), (C2) in Propositions 2.1 and 3.1, respectively, and Proposition 3.2 below are independent of the selected ranking.

Case 1 : $y = cW_e$.

Since $e+1 \notin E$, $b_{e+1} \leq y = cW_e$. Moreover, for $i < e$, $cW_i \leq cW_e = y < b_{i+1}$, since $i+1 \in E$. Thus $cW_i < b_{i+1}$ and $e = i^* = \min\{i \in \{1,...,n\} : cW_i \geq b_{i+1}\}$. We deduce $y = cW_{i*} = \min\{cW_{i*}, b_{i*}\}$ from the definition of E.

Case 2 : $y > cW_e$.

We know that $b_{e+1} \leq y$. Note that, if $b_{e+1} < y$, all individuals beyond e vote 0. Then $y = cW_e$, a contradiction. Thus, $y = b_{e+1}$. It follows that, if i is such that $b_i < b_{e+1}$, we have $s_i = 0$. Since this is true for all individuals beyond i, we have $y \leq cW_{i-1}$, and therefore, $b_i < cW_{i-1}$. Hence i^* is such that $b_{i^*} \geq b_{e+1}$. Now take i such that $b_i > b_{e+1}$; then we must have $i < e$ and therefore $cW_i \leq cW_e < y = b_{e+1} \leq b_{i+1}$. Thus i^* is such that $b_{i^*} \leq b_{e+1}$. It follows that $b_{i^*} = b_{e+1} = y$.

Finally, if $b_{i^*+1} < b_{i^*}$, all voters beyond i^* vote 0 and we have $b_{i^*} = y \leq cW_{i^*}$. If $b_{i^*+1} = b_{i^*}$, we also have $b_{i^*} \leq cW_{i^*}$ from the definition of i^*. Hence $y = \min\{b_{i*}, cW_{i*}\} = y^*$. This completes the proof. ■

The bliss point b_{i^*} constitutes a cut point for the equilibrium strategy: all voters with bliss point strictly below b_{i^*} vote 0, while those with bliss points strictly greater than b_{i^*} vote c. Only voters with bliss point at b_{i^*} may choose to vote strictly between 0 and c. If i^* is the only such voter, he votes c if $b_{i^*} > cW_{i^*}$ and otherwise, he votes $\frac{b_i^* - cW_{i^*-1}}{w_{i^*}}$ thus enjoying his bliss point in equilibrium. If more than one individual share a bliss point of b_{i^*}, the equilibrium strategies are unique only for those whose bliss points differ from b_{i^*}. In the case where the equilibrium allocation is cW_{i*} and differs from b_{i*}, all the votes are extreme, either 0 or c.

Given the equilibrium strategies, any redistribution of weights among individuals with bliss points strictly below b_{i^*}, among individuals at b_{i^*} or among individuals strictly above b_{i^*} leaves the Nash outcome unchanged. This is reminiscent of Warr's neutrality property in the private provision literature (see Warr (1983) and Bergstrom, Blume and Varian (1986)).

The solution may be depicted graphically by drawing the decreasing sequence of bliss points and the increasing sequence of cumulative weights on the same picture. It is illustrated in Figure 1 in a simple example involving four individuals with a cap of 1 and demographic weights. The bliss points are respectively .8, .6, .4, .2. In the picture $i^* = 2$ and $y^* = .5$.

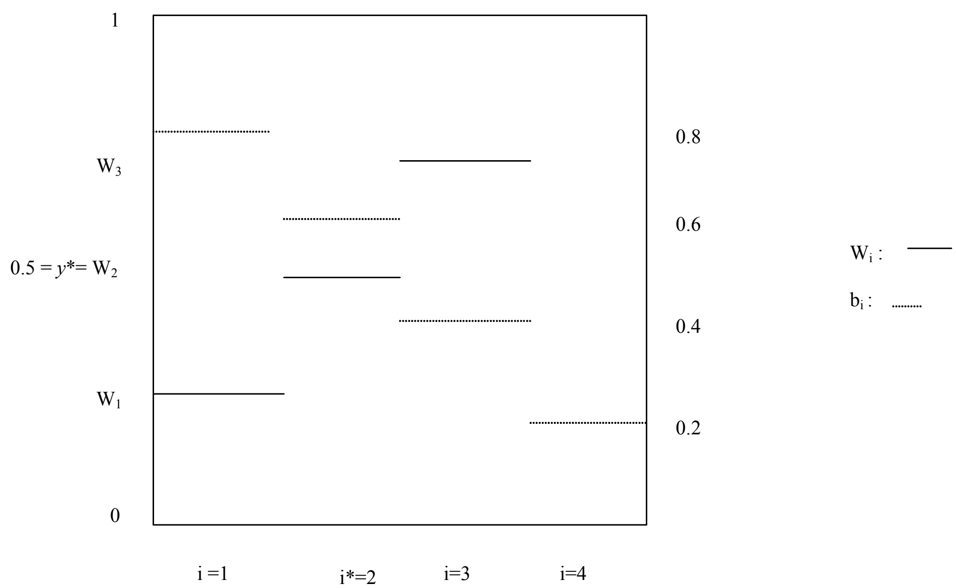

Figure 1: Illustration of Proposition 2.1

We now discuss the relative merits of average voting and majority voting from the point of view of a minority.

3 Protecting Minorities

Prior to discussing how average voting may be used to achieve minority protection, we start with a general discussion of the relative position of the average outcome and the majority outcome in the space of feasible allocations. Due to the single-peakedness assumption, the latter outcome is given by the unweighted median opinion which precludes the weight w_i to be interpreted as the population share of i in the overall population.

3.1 Comparison of Majority and Average Outcomes

In order to perform a comparison with the majority outcome, we first provide an alternative characterization of the average outcome when the vote cap is not too small. To this end we use the following definition. The median of a finite set of real numbers A with N elements, is defined as the smallest number $med(A) \in A$ which satisfies

$$\frac{1}{N}\#\{a \in A : a \leq med(A)\} \geq \frac{1}{2} \text{ and } \frac{1}{N}\#\{a \in A : a \geq med(A)\} \geq \frac{1}{2}. \tag{3}$$

If N is odd, condition (3) defines a unique number while if it is even, there are 2 such numbers. We adopt the convention that the median is the smallest.[7]

Note that from Proposition 2.1, if the vote cap is strictly below the smallest bliss point, the outcome is merely c. In the more interesting case, where at least one voter is not constrained by the vote cap, we obtain a second characterization of the equilibrium allocation.

Proposition 3.1 *Suppose* $c \geq b_n$.

(i) *The equilibrium allocation* y^**may be written as*

$$y^* = med(b_1, ..., b_n, cW_1, ..., cW_{n-1}). \tag{C2}$$

(ii) *If* $med(cW_1, ..., cW_{n-1}) \geq med(b_1, ..., b_n)$, *then* $y^* \geq med(b_1, ..., b_n)$.

(iii) *If there is a unique* $j = 1, ..., n$ *such that* $b_j = med(b_1, ..., b_n)$, *then* $y^* \geq med(b_1, ..., b_n)$ *iff* $med(cW_1, ..., cW_{n-1}) \geq med(b_1, ..., b_n)$.

Proof. (i) It is shown that (C1) and (C2) are equivalent. Let $A = \{b_1, ..., b_n, cW_1, ..., cW_{n-1}\}$. Again we distinguish two cases.

Case 1: $y^* = b_{i^*}$. On the one hand, since voters are ranked in decreasing order, $b_i \leq b_{i*}$ $\forall i \geq i^*$. The number of such voters is $n - i^* + 1$. On the other hand, $\forall i < i^*$ we have $cW_i \leq cW_{i^*-1} < b_{i^*}$ (from the definition of i^*). There are $i^* - 1$ such values of cW_i. This shows that $\frac{1}{2n-1}\#\{a \in A : a \leq y^*\} = \frac{n}{2n-1} \geq \frac{1}{2}$. On the one hand, from the definition of y^*, we have $cW_i \geq y^* \geq b_{i^*}$ for all $i \geq i^*$. There are $n - i^*$ such values of cW_i. On the other

[7] If N is odd, $\#\{a : a \geq med(A)\} = \frac{N+1}{2}$ while if N is even, it is $\frac{N}{2} + 1$.

hand, since voters are ranked according to a decreasing order of bliss points, $b_i \geq b_{i^*}$ for all $i \leq i^*$. There are i^* such voters. Thus $\frac{1}{2n-1}\#\{a \in A : a \geq y^*\} = \frac{n}{2n-1} \geq \frac{1}{2}$.

Case 2 : $y^* = cW_{i^*}$. Since $c \geq b_n$, if $i^* = n$, then $y^* = b_n$ and the proof of case 1 applies. We assume that $i^* < n$. Since W_i is non decreasing in i, $cW_i \leq cW_{i^*}$ for all $i \leq i^*$. There are i^* such values. Furthermore, from the definition of i^*, since b_i is non increasing in i, $b_i \leq b_{i^*+1} \leq cW_{i^*}$ for all $i > i^*$. There are $n-$ i^* such values. This shows that $\frac{1}{2n-1}\#\{a \in A : a \leq y^*\} \geq \frac{1}{2}$. Finally, since W_i is non decreasing in i we have $cW_i \geq cW_{i^*}$ for $i \geq i^*$. There are $n-$ i^* such values. Moreover, from the definition of y^* and since b_i is non increasing in i, $b_i \geq b_{i^*} \geq cW_{i^*}$ for all $i \leq i^*$. There are i^*such values. This shows that $\frac{1}{2n-1}\#\{a \in A : a \geq y^*\} \geq \frac{1}{2}$.

(ii) It suffices to note that the median of the union of two populations lies in the interval delimited by each of the medians of the two initial populations.

(iii) The if part is already proved. For the only if part, we prove that $\text{med}(b_1, ..., b_n) > \text{med}(cW_1, ..., cW_{n-1}) \Rightarrow \text{med}(b_1, ..., b_n) > y^*$. Since $y^* \in [\text{med}(cW_1, ..., cW_{n-1}), \text{med}(b_1, ..., b_n)]$, it suffices to show that $y^* \neq \text{med}(b_1, ..., b_n)$. Suppose that $y^* = \text{med}(b_1, ..., b_n)$. Case 1: n is odd. By applying (i), $\#\{a \in A : a \leq \text{med}(b_1, ..., b_n)\} = n$. By assumptions, $\#\{i : b_i \leq \text{med}(b_1, ..., b_n)\} = \frac{n+1}{2}$. Therefore $\#\{i : cW_i \leq \text{med}(b_1, ..., b_n)\} = \frac{n-1}{2}$. This contradicts the facts that $\#\{i : cW_i \leq \text{med}(cW_1, ..., cW_{n-1})\} = \frac{n-1}{2}$ and $\text{med}(b_1, ..., b_n) > \text{med}(cW_1, ..., cW_{n-1})$. Case 2: n is even. By applying (i), $\#\{a \in A : a \geq \text{med}(b_1, ..., b_n)\} = n$. By assumptions, $\#\{i : b_i \leq \text{med}(b_1, ..., b_n)\} = \frac{n}{2}$. Therefore $\#\{i : cW_i \geq \text{med}(b_1, ..., b_n)\} = \frac{n}{2}$. This contradicts the facts that $\#\{i : cW_i \geq \text{med}(cW_1, ..., cW_{n-1})\} = \frac{n}{2}$ and $\text{med}(b_1, ..., b_n) > \text{med}(cW_1, ..., cW_{n-1})$. ■

Remarkably, characterization (C2) shows that the average outcome may be expressed with an extended median formula which facilitates comparison with the majority outcome. To illustrate this characterization, it is easily verified that, in the numerical example of Figure 1, the equilibrium outcome 0.5 is the median of bliss points 0.2, 0.4, 0.6, 0.8 and cumulative weights 0.25, 0.5, 0.75.

Provided that there is only one median voter, the average voting outcome will be larger than[8] the majority outcome if and only if the median of the corrected cumulative weights is above the median of bliss points (part *iii*). In the simple example where all voters are weighted equally and $c = 1$, the median of corrected cumulative weights is one half (or tends to one half for large populations). If the median bliss point is lower than one half, the outcome of average voting is always closer to one half than the majority outcome, and both outcomes lie on the same side of one half. Hence the outcome of average voting is always less extreme than that of majority voting.

When there are more than one median voter, the necessary and sufficient condition expressed above is only sufficient for a weak inequality (part ii) or necessary for a strict inequality[9]. It is easy to build examples, for instance in a fully bipolarized society, where the two voting outcomes coincide, although the median of corrected cumulative weights is strictly larger than the median vote.

This alternative characterization may be restated in a way that allows for a neat graphical interpretation using a picture where the axes in Figure 1 are reversed. Let F_n be the cumulative distribution of bliss points, namely,

$$F_n(y) = \frac{1}{n} \# \{i \in \{1, ...n\} \mid b_i \leq y\}$$

and G_n be the cumulative distribution of corrected cumulative weights, i.e.,

$$G_n(y) = \frac{1}{n-1} \# \{i \in \{1, ..., n-1\} \mid cW_i \leq y\}.$$

From Proposition 3.1, there are exactly n bliss points or corrected cumulative weights which are at most as large as the equilibrium outcome y^*. Thus y^* satisfies the following conditions

$$nF_n(y^*) + (n-1)G_n(y^*) = n.$$

This equation may be rewritten to yield the following result.

Proposition 3.2 *Suppose $c \geq b_n$. The equilibrium allocation $y^*$is defined by*

$$y^* \in \{b_1, ..., b_n, cW_1, ..., cW_{n-1}\}$$

[8] Moreover (iii) holds with the reversed weak inequalities, and thus (iii) with strict inequalities is valid.

[9] Indeed, $\mathrm{med}(cW_1, ..., cW_{n-1}) > \mathrm{med}(b_1, ..., b_n)$ is necessary for y^* to be strictly larger than the majority outcome.

and

$$G_n(y^*) = \frac{n}{n-1}(1 - F_n(y^*)). \tag{C3}$$

Figure 2 depicts the numerical example of Figure 1 with reverse axes, so that the increasing step function represents G_4 and the decreasing step function represents $1 - F_4$. At the equilibrium outcome, $y^* = 0.5$, $G_4(y^*) = 2/3$ and $1 - F_4(y^*) = 1/2$, which provides an illustration of (C3).[10]

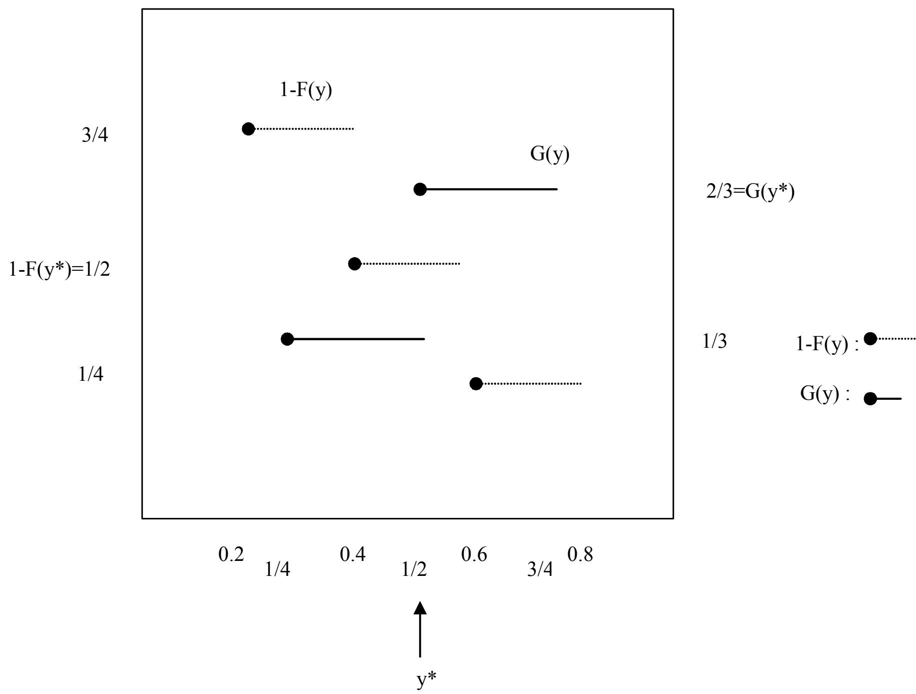

Figure 2: Illustration of Proposition 3.2.

Now for large enough populations and cumulative weights and bliss points scattered throughout the [0,1] interval, the steps in the graphs of the two functions $1 - F_n$ and G_n become very narrow. Then the equilibrium outcome is approximated as the value of y at which the two graphs cross. This approximation is confirmed by taking n to infinity in (C3). This suggests that the equilibrium can be studied using a continuous version of the two functions.

[10] Among the values listed on the horizontal axis, 0.5 is the only one satisfying (C3).

This intuition is borne out by the following formal argument, which provides a characterization of the limit equilibrium outcome as the number of voters goes to infinity.[11] Let F and G be two strictly increasing and continuous distribution functions. From now on y^* denotes the unique solution to

$$G(y^*) = 1 - F(y^*) \tag{4}$$

and y_n^* denotes the equilibrium allocation when the population size is n. We have the following result.

Proposition 3.3 *If $\{F_n\}$ converges pointwise to F and $\{G_n\}$ converges pointwise to G, then $\{y_n^*\}$ converges to y^*.*

Proof. Note that since $c > 0$ and F is strictly increasing $F(c) > 0$. Thus for n large enough, $F_n(c) > 0$, which implies $b_n \leq c$. Then we may apply Proposition 3.2. Now note that since [0,1] is a compact set, the functions F_n and G_n are monotone and F and G are continuous, pointwise convergence of F_n and G_n towards F and G respectively implies uniform convergence (see Rudin p.167). Thus from (C3) we have

$$\lim_{n \to \infty} F(y_n^*) + G(y_n^*) = 1.$$

Since F and G are continuous and strictly increasing, $\{y_n^*\}$ should converge to the unique y^* satisfying (4). ■

Since G is invertible, (4) may be rewritten as the following fixed point relation

$$y^* = G^{-1}[1 - F(y^*)].$$

In this limit situation, the weight of those with a bliss point of exactly y^* vanishes to 0, so that the entire weight is concentrated on those who vote either 0 or 1. The average vote is therefore given by the cumulative weight of those voting 1 (i.e. $G^{-1}[1 - F(y^*)]$) which in turn must be equal to the bliss allocation of the marginal individual, y^*. In Section 4, some applications are discussed in which, in the weighted case, the distribution G has an

[11] See Proposition 3.4 in a companion paper by Renault and Trannoy (2003) for an alternative argument.

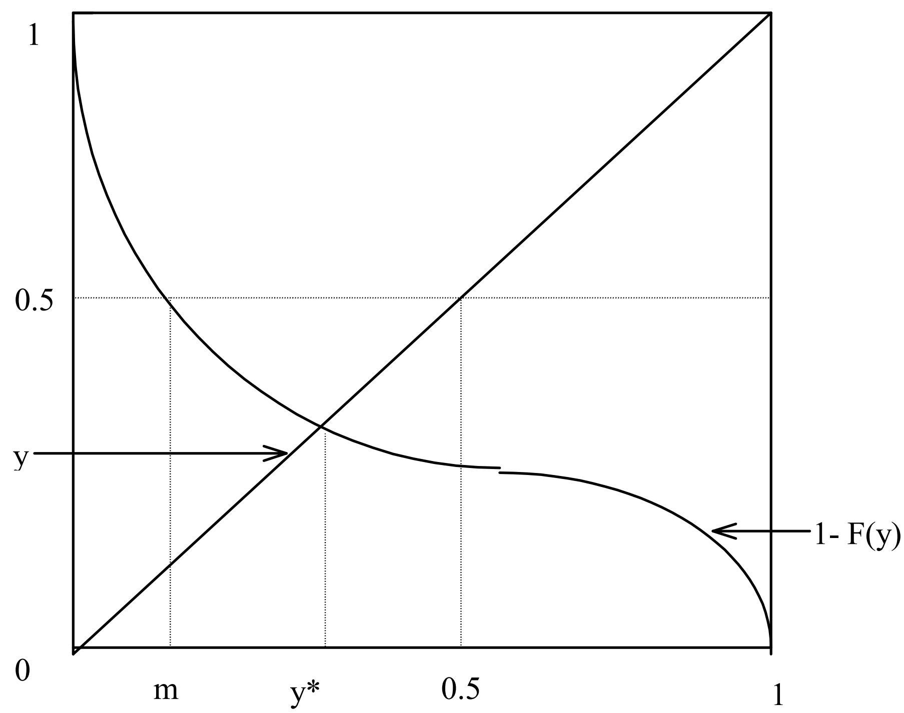

Figure 3: Equilibrium outcome in the anonymous case.

actual economic interpretation. In the unweighted case, $G(y) = \frac{y}{c}$ so that y^* must satisfy the simple fixed point relation

$$y^* = c(1 - F(y^*)),$$

which is illustrated in Figure 3 with $c = 1$.

Here the average vote is the proportion of the population voting 1 (i.e. $1 - F(y^*)$) which must be equal to the bliss allocation of the pivotal voter. It may be easily compared with the outcome of majority voting denoted m in the figure which satisfies $1 - F(m) = 0.5$. In the application below in Section 4.1, an equivalent figure for the weighted case (Figure 4) provides some useful insights as to the impact of a change in the weights distribution on the equilibrium outcome.

3.2 The Case for Average Voting

We start our analysis of the protection of minorities by stating some formal definitions. Defining a minority is an intricate question. If the definition is too specific, it cannot be used in a wide range of applications. A rather general definition proves to be more appropriate. There is no loss of generality in taking the minority to be on the right[12] of the median choice.

Definition 3.1 *A minority is any subset $M = \{1, ..., m\}$ with $b_m > med(b_1, ..., b_n)$, individuals still being ranked according to decreasing values of b_i.*

Clearly the value of b_m, the bliss point of the least extreme member of the minority, is somewhat arbitrary. In real world applications, it should be expected to be remote enough from the median so that all minority members are truly unhappy with a majority outcome.

In any case, the mere fact that such a minority exists does not imply that it should be protected. Indeed the actual motives for protecting a minority do not stem from the distribution of preferences itself. We pointed out in the introduction that the protection of a minority is as much a political matter as it is an ethical one. Both ethical and political considerations are concerned with the discrepancy between minority tastes and the majority outcome. As a result, protection should definitely require that the switch to average voting moves the outcome rightward. If this move were too large however, it might only benefit the most extreme members of the minority, leaving a large fraction of the minority less satisfied than before. Political considerations provide a neat criterion for selecting an upper bound on how far the outcome should be allowed to move. Avoiding political unrest requires that potential activists receive little enough support among minority members. In particular, the support of the minority's median voter seems critical. For instance, if secession is at stake majority voting within the minority is a likely decision rule. Then from a political standpoint, there is no need to go beyond the minority's median bliss point, and protecting the minority should consist in making the median member of the minority happier as well

[12] All the following discussion would apply just as well for a minority located "on the left" of the median, once the inequalities have been reversed.

as all individuals to his right. The following definition ensures that a majority of minority members benefits from the switch.[13]

Definition 3.2 *The minority is protected by a switch from majority to average voting if* $med(b_1, ..., b_n) \leq y^* \leq med(b_1, ..., b_m)$. *Protection is strict if the first inequality is strict.*

To illustrate, let us again consider the situation, where voters have equal weights, $c = 1$ and the majority outcome is below 1/2 which is the median of cumulative weights. Proposition 3.1 (ii) tells us that a switch from majority to average voting would weakly increase the outcome but it would remain less than 1/2. If the median minority bliss point is above 1/2, the minority is protected.

In the general case, still using Proposition 3.1 *(ii)* the equivalent sufficient condition for minority protection would be that the median of corrected cumulative weights is between the majority outcome and the median taste of the minority. Such a condition may be difficult to check in practice, for computing the median of corrected cumulative weights requires much information about the distribution of tastes and weights.[14] As we now show, it is possible to establish a similar sufficient condition that applies to the corrected weight of the minority alone, which is presumably easier to observe.

Proposition 3.4 *If* $med(b_1, ..., b_n) < cW_m < med(b_1, ..., b_m)$, *then the minority is strictly protected by a switch from majority to average voting.*

Proof. Let $A = \{b_1, ..., b_n, cW_1, ..., cW_{n-1}\}$. Since $cW_m > \text{med}(b_1, ..., b_n)$, there are at most $m-1$ corrected cumulative weights which are smaller than or equal to $\text{med}(b_1, ..., b_n)$. By assumption there are at most $n-m$ bliss points which are smaller than or equal to $\text{med}(b_1, ..., b_n)$. Therefore there are at most $n-1$ elements in A out of $2n-1$, which are smaller than or equal to $\text{med}(b_1, ..., b_n)$. By applying Proposition 3.1 (i), we deduce $y^* > \text{med}(b_1, ..., b_n)$. Now since $\text{med}(b_1, ..., b_m) > cW_m$, there are at most $n-m$ corrected

[13] Propositions 3.4 and 3.5 of this section would go through if we selected any critical minority member other than the median voter. The statement should however be modified replacing $med(b_1, ..., b_m)$ by whichever critical bliss point larger than or equal to b_m has been selected.

[14] Furthermore, from Footnote 9, if $med(W_1, ..., W_{n-1}) \leq med(b_1, ..., b_n)$, there is no c such that the minority could be strictly protected.

cumulative weights which are strictly larger than $\text{med}(b_1, ..., b_m)$. By assumption there are at most $m/2$ bliss points which are strictly larger than $\text{med}(b_1, ..., b_m)$. Therefore there are at most $n - (m/2) \leq n - 1$ (for $m > 1$, the case $m = 1$ being obvious) elements in A out of $2n - 1$ which are strictly larger than $\text{med}(b_1, ..., b_m)$. By applying Proposition 3.1 (i), we deduce $y^* \leq \text{med}(b_1, ..., b_m)$. ■

The first inequality ensures that the equilibrium outcome moves in the right direction whereas the second inequality guarantees that it does not move too far. The result would be obvious if we considered a situation where, in equilibrium, all minority members vote 1 and all majority members vote 0 (for instance, tastes are polarized at each end of the segment). However, in typical configurations, there will be either minority members voting 0 or majority members voting 1, so that the average outcome may exceed or fall short of the corrected weight of the minority.

The condition in Proposition 3.4 may seem stringent since the first inequality requires that the uncorrected weight of the minority exceeds the median bliss point (if it does not hold for $c = 1$, it would not hold for any lower cap). Note however that it is only a sufficient condition since there are situations where some voters outside the minority choose to vote 1.

Still the reader may justly wonder whether this proposition allows for making a strong case in favor of average voting as a tool for minority protection. Does it apply to relevant situations where there is a need for minority protection that could be achieved by a switch from majority to average voting? We now discuss this point with particular attention to the first inequality in Proposition 3.4.

A first approach is to think of a given distribution of tastes, letting the minority weight vary. Then the first inequality means that the minority has a large enough weight. This is quite appealing in the anonymous case when the concern is about political unrest. In the example of Section 4.1, we briefly discuss another instance where a large minority weight makes minority protection particularly desirable.

There are however many instances where one would like to protect a poor or small minority, and yet it would be worse off under average voting than under majority voting.[15] In order to see how average voting may nevertheless be an effective shield for the minority, we now take the weight distribution as given letting tastes vary, so as to identify situations where the need for minority protection results from the distribution of tastes rather than from the distribution of weights. Assume an initial situation where the minority would actually lose in a switch to average voting. Consider a shift to the left of the bliss points of those outside the minority, keeping the minority tastes and the weight distribution unchanged. Then the majority outcome moves closer to zero, while the minority's weight is fixed. The minority is confronted with a majority outcome which becomes more extreme and more remote from all the preferred outcomes of its members. For a large enough shift in taste, the majority outcome drops below the minority's weight, so that the first inequality in Proposition 3.4 eventually holds for $c = 1$. This ensures that if the dissatisfaction of minority members with the majority outcome is too severe, then a switch to average voting will help reduce the problem to some extent, no matter what the weight of the minority might be. Example 4.1 below provides an illustration of this principle.

The second inequality in Proposition 3.4 does not imply a restriction on the distribution of weights or tastes to the extent that the cap may always be chosen appropriately so that it holds. More specifically whenever W_m is larger than $\text{med}(b_1, ..., b_n)$, minority protection may be achieved by picking $c \in (\frac{\text{med}(b_1,...,b_n)}{W_m}, \min\{1, \frac{\text{med}(b_1,...,b_m)}{W_m}\})$. A vote cap strictly below one is only needed when, for $c = 1$, the switch would take the allocation beyond the minority's median bliss point. The introduction of a cap to curtail the strategic power of the minority is coherent with what is done in actual applications. Note however that introducing a cap is costly from an ethical point of view. It violates the non imposition condition introduced by Arrow (1963) which requires that, whatever the social state, there must exist a profile of preferences such that the outcome of the voting rule is precisely this social state. It violates

[15] For instance, as we show in Section 4.1, in the "forced to pay yet free to choose" setting, a combination of a poor minority and a progressive tax system may result in a worse outcome for the minority under average voting.

the Pareto requirement as well.

3.3 Introducing a Vote Floor

Despite the above objection, we end this section by investigating how an extended use of restrictions on the voting space may enhance the protecting power of average voting for minorities. Propositions 3.1 and 3.4 assume that the only restriction that can be imposed on the vote is a cap. This is somewhat arbitrary and it is intuitive that a vote floor would be more appropriate to protect the minority. With a vote floor, those who have voted 0 must cast a strictly positive vote which will move the outcome upwards, in a direction which is favorable to the minority.

The comparison of the average outcome with the majority outcome must be fair so that if a floor is allowed in the average rule, it should also be allowed in the majority rule[16]. Let us call this new rule the *restricted* majority rule. Then the relevant comparison between the two rules should be based on the ethical costs associated with the introduction of a floor. The interesting question is whether the restriction on the domain of voting choices is more stringent with the restricted majority rule or with the average rule. Other things equal, the smaller the floor, the better. Now let f_m (respectively f_a) be the smallest floor ensuring minority protection in a switch from majority to restricted majority (resp. average) voting.

Proposition 3.5 *Suppose* $W_m \leq med(b_1, ..., b_n)$*. Then* $f_a < f_m$*.*

Proof. Clearly $f_m = \text{med}(b_1, ..., b_n)$. We now show that $f_a < \text{med}(b_1, ..., b_n)$.

Organizing the vote on y with a floor f is equivalent to organizing the vote on $z = 1 - y$, with a cap $c = 1 - f$. When the social choice is z the minority is to the left of the median of bliss points.

By assumption $1-\text{med}(b_1, ..., b_m) < 1-\text{med}(b_1, ..., b_n) < 1-W_m$. Then it is always possible to choose c in $(0, 1)$ such that $1 - \text{med}(b_1, ..., b_m) < c(1 - W_m) < 1 - \text{med}(b_1, ..., b_n)$. Let $B = \{1 - b_1, ..., 1 - b_n, c(1 - W_1), ..., c(1 - W_{n-1})\}$. From Proposition 3.1, $z^* = \text{med}B$. Since

[16] This is irrelevant under the assumptions of Proposition 3.4. There, a cap in majority voting either leaves the outcome unchanged or makes it worse for the minority.

$c(1-W_m) < 1-\text{med}(b_1,...,b_n)$, there are at most $m-1$ corrected cumulative weights which are larger than or equal to $1-\text{med}(b_1,...,b_n)$.[17] Since the minority is to the left of the median, there are at most $n-m$ bliss points which are larger than or equal to $1-\text{med}(b_1,...,b_n)$. Therefore there are at most $n-1$ elements in B out of $2n-1$ which are larger than or equal to $1-\text{med}(b_1,...,b_n)$. Thus $z^* < 1-\text{med}(b_1,...,b_n)$ or $y^* > \text{med}(b_1,...,b_n)$.

Now since $1-\text{med}(b_1,...,b_m) < c(1-W_m)$, a similar argument shows that there are at most $n-1$ elements in B out of $2n-1$ which are strictly smaller than $1-\text{med}(b_1,...,b_m)$. Thus $z^* \geq 1-\text{med}(b_1,...,b_m)$ or $y^* \leq \text{med}(b_1,...,b_m)$.

Note finally that $c \geq 1-f_a$ may be arbitrarily close to $\frac{1-\text{med}(b_1,...,b_n)}{1-W_m} > 1-\text{med}(b_1,...,b_n)$, so that $f_a < \text{med}(b_1,...,b_n)$. ■

If the weight of the minority is too small[18], the only way to guarantee minority protection is to impose some restrictions on allowed votes. However, the needed restrictions under average voting are milder than those which should be imposed under majority voting, so that the associated ethical cost is smaller.[19]

Whenever a floor is needed, one may wonder how it is determined. The appropriate floor depends on the minority's weight and the median taste in the overall population as well as within the minority, all of which may be reasonably well approximated. This would typically be the outcome of a constitutional stage where the majority relinquishes some of its future influence to ensure social cohesion. This however would be at the expense of giving up flexibility in adjusting to changes in the median taste of the population or the minority's weight.

To conclude this section, it should be emphasized that there are important cases where a restriction on votes is not needed to achieve minority protection. From Proposition 3.4, we know that this is the case when the minority's weight is strictly between the majority

[17] Note that here $1-b_i$ is increasing in i, while $1-W_i$ is decreasing in i.

[18] Combining the results in Propositions 3.4 and 3.5 and allowing for vote floors as well as vote caps, it is always possible to protect a minority by a switch from majority voting to average voting.

[19] In the case covered by Proposition 3.4, the restriction on votes is a cap under average voting while it would be a floor under the restricted majority rule. Thus the choice spaces may not be compared in terms of one being included in the other. Nevertheless, the two rules may be compared in terms of the number of voters affected by the restriction. A simple reasoning shows that it is smaller under average voting than in the restricted majority rule.

outcome and the median taste of the minority. We now turn to discussing an existing as well as a potential application of the average rule in the context of public goods provision.

4 Public Goods Applications

We now consider two applications. The first is concerned with the allocation of public expenditures between two alternative uses. In the second, the issue at hand is the fraction of total wealth allocated to the provision of public goods.

4.1 The Forced to Pay yet Free to Choose Model

The introduction describes several instances of actual applications of the average voting rule. All involve allocating a fixed amount of public resources among several uses and each tax payer is allowed to choose how his individual contribution should be split, while the amount of the contribution is imposed to him. These collective choice procedures have been studied in Bilodeau (1994).

The economy is as follows. There are n consumers, one private good and two pure public goods. Agent i's preferences are represented by a strictly quasiconcave utility function v_i whose first argument is private good consumption. Total private good endowment is denoted Ω and consumer i's share is $\alpha_i \in [0, 1]$. The amount of private resources used to produce public goods is denoted T and t_i is the fraction of tax burden borne by individual i. Here, contrary to Bowen's model below, the amount of public expenditure T is exogenous. Thus the collective decision has no bearing on individual disposable income. Although each individual is forced to pay his tax contribution, he may choose the fraction s_i which is affected to the production of public good 1. Then the collective choice variable y is the resulting fraction of public expenditure dedicated to producing public good one and we have $y = \sum_{i=1}^{n} s_i t_i$. Public good quantities are given by $f_1(yT)$ for good 1 and $f_2(T - yT)$ for good 2, where f_1 and f_2 are concave. The functions v_i, f_1 and f_2 are increasing in each argument. Thus i's utility as a function of y is given by

$$u_i(y) = v_i(\alpha_i \Omega - t_i T, f_1(yT), f_2(T - yT)).$$

Our assumptions on v_i, f_1 and f_2 ensure that u_i is single-peaked[20] and its maximum point is denoted b_i.

This mechanism is akin to a private provision of public goods procedure in which an individual's contributions to the various public goods are constrained to add up to the amount of taxes he is required to pay (see Bergstrom, Blume and Varian (1986) for an unconstrained private provision model). In this interpretation, the outcome results from aggregating private decisions. As it is the case for market mechanisms, the weight of an individual in the allocation process is closely related to his wealth (tax contributions are typically correlated with wealth). However, letting $w_i = t_i$, the above model may be viewed as a special case of the average voting model of Section 2 in which the weight of a voter is determined by his tax contribution. Viewing this procedure as a voting scheme, fairness may appear as the appropriate criterion for selecting weights, especially when voters are households or individuals. Then the selected weights depend upon whether we favor fairness in taxation or fairness in voting. In the former case the "equal sacrifice" principle would prescribe that the wealthy should pay more taxes, which would mechanically translate into a larger weight for the wealthy in voting. In the latter case, the "one man, one vote" principal would prescribe equal weights.

One interesting special case is that of a fully polarized society where each voter cares about only one public good. The amount of resources devoted to the minority's preferred good under average voting would then be the same as what the minority could afford on its own, whereas it would be 0 with majority voting. This is an instance where the need for protecting the minority is all the more critical that its weight is large since, under majority voting, minority members would be required to pay taxes without having the benefit of enjoying the public good they care about. This is also a case where Proposition 3.4 obviously applies.

Now consider the school financing system used in some Canadian provinces which allows tax payers to earmark their property taxes either to a separate school board running catholic

[20] u_i is quasiconcave function as a composition of a strict quasi-concave function which is increasing in each argument with concave functions.

schools or to a public school board (for institutional details, see Bagnoli and McKee (1992) and McKee (1988)). It is now possible to investigate, by comparing the outcome with that of majority voting, whether Ontario's Catholics are right to defend their school-board system.[21] Here the collective decision at hand is the fraction of property taxes devoted to catholic schools financing (good 1). The actual system only offers an all or nothing choice, since each household must devote the full amount of its taxes to one school board. However, as pointed out in the comment on Proposition 3.3, the restriction that votes should be 0 or 1 has little effect on the outcome when the population is large, since the weight of voters picking an intermediate value tends to 0. Clearly, Catholics are those with bliss points closer to 1. Since they are a minority, the median of bliss points is smaller than the highest bliss point among non Catholics. Furthermore, it is unlikely that a non catholic would want the percentage of catholic school financing to exceed the proportion of Catholics in the population. Hence the median of bliss points and therefore the outcome with the majority rule is likely to be below one half.

According to Proposition 3.1, in order to determine the position of the average vote relative to the median of bliss points, it is necessary to figure out the cumulative distribution of weights, where voters are ranked according to the decreasing order of bliss points. The position of the median of cumulative weights depends on the wealth distribution as well as on the tax system. In particular, if the tax used was a poll tax (the same amount being paid by all households), then the median of cumulative weights would be one half, independent of the wealth distribution. In this case, the average voting outcome is unambiguously more favorable to the minority than that of majority voting.

However, most tax systems use proportional or even progressive taxes for redistribution purposes. Then the weight of a voter is positively related to his wealth. The two main determinants of the outcome are the correlation between wealth and the relative taste for good 1 (in particular the relative wealth of the minority) and the progressiveness of taxes.

[21] Since in the actual system parents must finance the school they send their children to, the voting choice that is modelled here only concern parents with no child in school age. However, it is likely that other parents would not behave differently, if they were given the choice. A last difference with the average procedure we study is that only catholics may contribute to catholic schools.

For a given tax scheme, the lowest possible median for cumulative weights is obtained when wealth is perfectly and negatively correlated with bliss points, in which case it is less than one half (the minority being then comprised of the poorest households). Symmetrically, the highest possible median for cumulative weights is obtained when wealth is perfectly and positively correlated with the bliss points, in which case it is above one half (the minority being then comprised of the richest households). In both of these two extreme cases, the more progressive the taxes are, the more distance there is between the median of cumulative weights and one half.

As we argue in Proposition 3.3, in a large population, provided that wealth and tastes are sufficiently diverse, the outcome of average voting may be approximated as the intersection of two continuous functions $1 - F$ where F is the cumulative distribution of bliss points and G the cumulative distribution of cumulative weights. The two cases may then be contrasted as shown in Figure 4 which illustrates how tax progressiveness affects the outcome of average voting.

From panel (a) in Figure 4, it is clear that if the minority is rich, it is unambiguously favored by average voting compared to what it would obtain with the majority rule and this is exacerbated by the progressiveness of taxes. On the contrary, there is no clear cut relationship between the two outcomes when the minority is poor. Panel (b) shows a situation in which the minority is favored by average voting when taxes are proportional but is not when taxes are progressive.

Applying the above analysis to the Canadian example yields mixed conclusions as to how much the catholic minority may benefit from the school financing system. Since Catholics are on average poorer than the rest of the population, the situation is closest to that of panel (b) and it is possible that, even though the property tax is proportional, Catholics are penalized by the financing procedure.

This conclusion, however, should be qualified by two remarks. First, the possibility that Catholics are penalized becomes unlikely when the median bliss point is close to zero, that is, when the majority rule would hurt Catholics the most, their confessional schools receiving

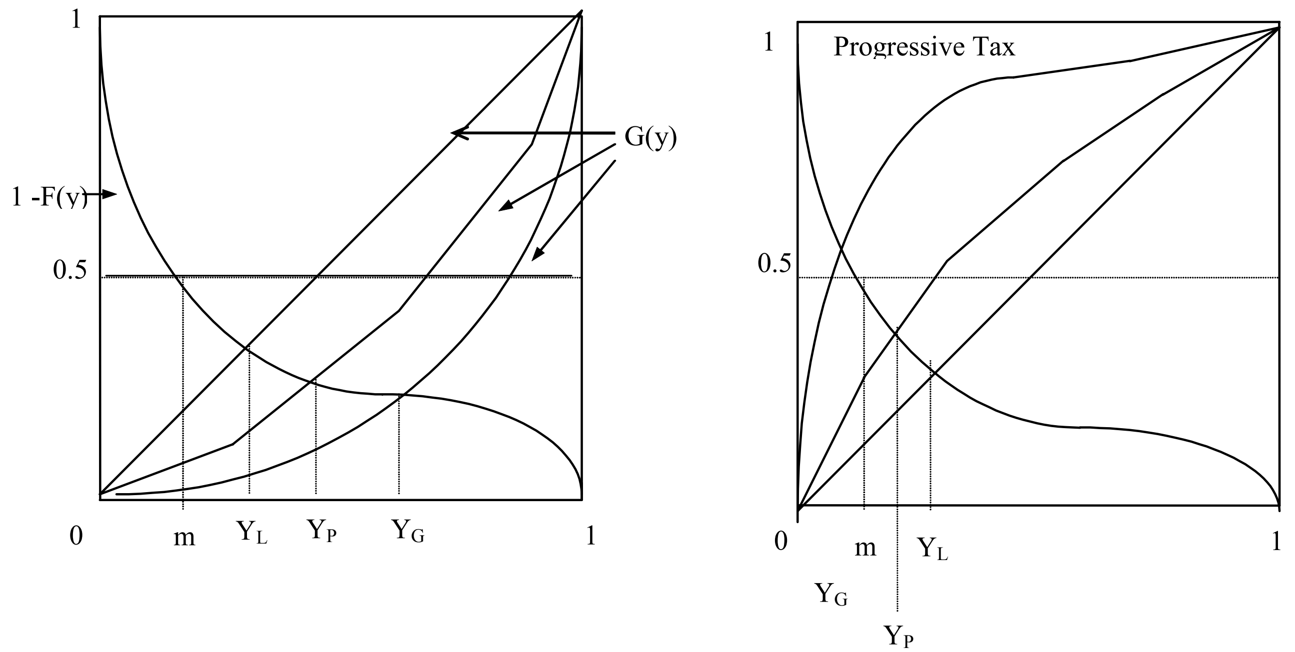

Figure 4 Panel (a): Equilibrium outcome with a rich minority.

Figure 4 Panel (b): Equilibrium outcome with a poor minority.

Y_L^*: Equilibrium outcome with a poll tax
Y_P^*: Equilibrium outcome with a proportional tax
Y_G^*: Equilibrium outcome with a progressive tax;
m:median

Figure 4: Comparing the median outcome and the average one for different assumptions regarding the relative wealth of the minority.

hardly any money. Second, Catholics being poorer than average does not mean that there is a perfect negative correlation between the taste for public financing of confessional education and wealth. In particular, the relationship might be reversed among non Catholics, those with higher income being the most supportive of public financing of catholic schools. Then the cumulative distribution of weights would not be concave throughout, which would allow for the median to lie further to the right on the panel (b) graphic and, even, to exceed one half.

The above analysis has an obvious limitation due to the exogeneity of tax rates. In particular it should be expected that if tax rates are set in a first stage, the greater weight given to minorities in the second stage may influence voting on the total amount of taxation in the first stage. Before tackling this more complex issue in the next section, we discuss the application of the average voting rule to the simple choice of a tax rate.

4.2 The Average Bowen Model

Bowen (1943) (see also Bergstrom (1979)) studies the allocation of total wealth between public and private goods under majority voting, for a given tax share distribution. This type of procedure has actually been used in the Florida system for school financing studied in Holcombe (1977).

We now present an "average Bowen model" in which the share of public expenditures is determined by the average rather than the median expressed opinion and for which the model of the current paper is a reduced form.

The economy is as follows. There are n consumers, one private good and one pure public good. Agent i's preferences are represented by a quasiconcave utility function v_i whose first argument is the private good consumption. Total private good endowment is denoted Ω and consumer i's share is $\alpha_i \in [0,1]$. If a fraction $y \in [0,1]$ of total private good endowment is allocated to public good production, the quantity produced is $f(y\Omega)$, where the production function f is concave. Public expenditures are financed through taxes and t_i denotes individual i's tax share in public good financing. Thus his private good consumption is $(\alpha_i - t_i y)\Omega$ and his utility is given by[22]

$$u_i(y) = v_i((\alpha_i - t_i y)\Omega, f(y\Omega)).$$

Our assumptions on v_i and f guarantee that u_i is single-peaked and b_i denotes the bliss point. It follows that Bowen's problem may be studied within our framework, where voters have equal weights.

The paper by Alesina *et al.* (1999) provides a real life example of a bimodal distribution of preferences regarding public good spending with a well defined minority. Both income and race play a role in this example. As pointed out by Alesina *et al* "each ethnic group's utility level for a given public good is reduced if other groups also use it[23]".

Example 4.1 *Black White Racial Clash over Public Schooling*

We assume that there is only one public good: public schooling which is financed on a property tax as in the Florida experience described in Holcombe (1977). Hence the vote concerns the rate of an earmarked property tax. Let us redefine Ω to be total real estate wealth and let W_i denote individual i's remaining wealth net of all other taxes. Taking the amount of other public good as given, agent i's utility may be written as:

$$u_i(y) = v_i(\alpha_i(1 - y)\Omega + W_i, f(y\Omega)).$$

A poor black minority faces a rich white majority. Because of their poverty, blacks cannot consider the alternative of private schooling. Then they are supportive of public education, the only way for their children to get a good education. Because of their wealth, whites can consider the alternative of private schooling. Furthermore they fear bad peer effects in public schooling due to black children attendance. Then they favor a low public spending level. Therefore the ethnic division and the differences in wealth jointly explain the bimodal character of the distribution of preferences. It is difficult to know whether this kind of pattern

[22] This expression implicitly assumes no deadweight loss from taxation. This is consistent with taxes based on exogeneous endowments.

[23] p 1244.

existed in the Florida experience but Alesina *et al.* (1999) provides anecdotal evidence of similar conflicts over public school financing in other parts of the US.

In the Florida example the outcome was decided within a Bowen referendum process. Holcombe alludes to the possibility of using an average procedure instead, and surmises that there would be strategic manipulation. The results of the previous section apply and show under which circumstances the black minority would profit from the introduction of the average procedure. In particular, Proposition 3.4 ensures that if the median bliss point, i.e. the tax rate chosen with majority voting, is less than the weight of the black minority, namely the proportion of black voters, the black minority would be protected with average voting, provided that the cap imposed is not too large. For the sample of cities considered by Alesina *et al.* (1999) the weight of the black minority is on average 10,4% (descriptive statistics in their appendix 1), so that school financing would be increased by a switch whenever the tax rate chosen under majority voting falls below this figure.

The level of public expenditure is also an issue for a club of political entities who run joint public programs. The EU budget provides our second example and we now investigate how the present model may shed some light on the European Union's decision procedures. There is much debate over the extension of the use of majority rule in the European Council (see for instance Kirman and Widgren (1994)).

Example 4.2 *Dispute over the size of structural funds in the EU budget*

The debate over the size of the EU budget which took place during Jacques Delors' Presidency of the European Commission is a case of a bimodal distribution of tastes. A minority of relatively poor southern countries opposed a majority of relatively rich northern countries: poor countries wanted an increase in the budget, since this increase meant more public good in these countries financed by the EU, while rich countries resisted this idea since it meant an increase in taxes for their taxpayers. It is a classical problem of conflict over vertical redistribution, where the redistribution in this case takes the form of a financing of public goods (in particular transportations facilities), that will be used mainly by locals (though a Danish tourist will also enjoy landing in a modern airport on the Costa del Sol).

To be more specific, let us briefly describe the debate that took place around what came to be known as the Delors II package. The question debated was the value of the ceiling on the EU budget (in terms of the percentage of the EU GDP). The EU budget has four main uses, 1) financing the Common Agricultural Policy, 2) internal policy (R and D), 3) external Policy, 4) structural funds and cohesion funds. It turns out that there was more or less an agreement about the budget to be devoted to the first three issues during the 90s. In particular all countries agreed that the budget devoted to the CAP could be reduced somewhat. Then the question of the size of the overall budget was tantamount to the issue of the size of the structural funds and cohesion funds.

The main objective of structural funds since 1988 has been to promote the development and structural adjustment of regions which are lagging behind (with a per capita Regional GDP under 75% of per capita EU GDP). All countries are eligible but obviously poor countries will gain more in this game. The figures in the first row of Table 1 show indeed that it is the case. The first 7 countries which belong to the north and central part of Europe receive a relatively small share, while the last 4 countries Spain, Ireland, Greece and Portugal obtain more than 57%. Italy is in a medium position. Over the whole period 1994-1999 the structural funds amount to ECU 141,5 billion at 1992 prices (roughly the same in US dollars). These in-kind transfers to poor countries are far from negligible: for example for Ireland they represented more than 5% of GDP and for Portugal more than 3%. The question which was discussed in December 1992 at the European Council Meeting in Edinburgh was the amount of structural funds needed to back the single market. Looking at the second row in Table 1 and comparing with the first one shows that 7 countries were losing, from a purely selfish point of view, in an increase in structural funds while 5 countries were gaining (the four cohesion countries plus Italy due to the presence of Mezzogiorno).

	B	DK	F	G	L	NL	UK	I	IRL	GR	SP	P
1.SF	1.4	0.6	10	14.5	0.07	1.7	8.7	14.6	4.1	10.1	23.1	10.1
2.%B	4.3	2.0	19	32.7	0.3	6.7	12.3	11.7	1	1.6	6.7	1.6
3.w	5	3	10	10	2	5	10	10	3	5	8	5
4.gdp	104.2	100.8	108.3	114.6	142.3	99.3	97.3	99.9	71,1	57.4	74.1	60.0

Meaning and Sources 1: Distribution of structural funds among countries for the period 1994-1999: table at the bottom of page 31 in "L'Europe en chiffres" Documentation Française 1999. The % are given for the EU12.

1 the total is equal to 99% of the budget, the last percent being not shared *a priori* among nations

2: % contributions to European Budget : table at the bottom of page 440 "Eurostat annuaire : Vue statistique sur l'Europe. Données 1987-1997 1999". The figures are the average for the years 1994,1995,1996 computed for the EU 12.

3 : weights in the European Council.

4 : per capita GDP in 1990 (PPS EU15 = 100): table at the bottom of page 232 "Eurostat annuaire: Vue statistique sur l'Europe." Data 1987-1997.

Table 1: Rough Data.

We now compare the outcome of the majority and average rules under the following assumptions. The outcome of the game is the proportion of the EU GDP devoted to structural funds. The actual outcome which has been decided in the Delors II package was about 0.4% of the EU GDP (33.6% of the EU budget which itself represents 1.19% of the EU GDP). There is some evidence that at least one country at the time (the UK of Margaret Thatcher) would not have been unhappy with a 0 % outcome. It is then reasonable to assume that the lower bound of the strategic space is 0. There is no clear basis for picking an upper bound. The results below depend on a high enough upper bound : roughly it should be one percent or above. We assume that the bliss points of the five winning countries are strictly above the value decided in the Delors II package, while the bliss points of the 7 losers are strictly below. A casual observation supporting the latter assumption is that before the entrance of Ireland, Greece, Portugal and Spain in 1986, the proportion of EU budget devoted to structural funds was far lower (about 15% in 1985). For both the majority and average rules, we will use the weights in the Council of the European Union which is the main decision body in the European institutions (see third row in table 1). The outcome of the majority rule would be the bliss point of one of the loser countries, for they represent 59.8% of the votes. Since by assumption their bliss points are strictly below 0.4%, so is the median taste. With a cap of 1 %, the corrected weight of the minority is 0.408%. Then from Proposition 3.4, we are in a situation where average voting would ensure a strict protection of the minority of poor countries compared to what they would obtain if the majority rule was used.

It turns out that with a cap of 1%, the budget size obtained with the average rule is close to the actual size which has been determined through a compromise. This provides a real world illustration of the conjecture by Gerber and Ortuno (1998) that the average outcome could also be interpreted as a reduced form for a compromise solution.

The next Section presents an extension where both public expenditure and its composition are determined through a sequential voting procedure involving both majority and average voting.

5 Combined Use of Majority and Average Voting

We consider a situation where there is a conflict between a minority and a majority over the allocation of public funds. The issue could for instance be the split of education spending between Serb and Albanian schools in Kosovo. We now wish to investigate whether a minority (in our example the Serbs) will benefit from the introduction of the average rule in an institutional arrangement where voting is organized first on education spending and second on its composition. The work of Alesina *et al.* (1999) provides a useful framework for this investigation. These authors find evidence using US data that a conflict over public good composition affects the amount of public good spending.

In their model, the generic individual i's utility function is given by:

$$u_i(y,g) = g^{\alpha}(1- \mid y - b_i \mid) + \omega - g \qquad \text{with } 0 < \alpha < 1,$$

where ω denotes individual wealth, $g \in [0, \omega]$ denotes the overall public spending, $y \in [0,1]$ is the public spending composition and b_i is the ideal individual composition. As in Alesina *et al.* (1999) we normalize population size to one, abstract from differences in wealth and assume that all individuals pay the same lump sum tax[24]. They consider a situation where

[24] Given the quasilinearity of preferences, the results will be unaffected, if wealth is heterogeneous and taxation was allowed to depend on wealth.

individuals vote first on public good spending and second on its composition, where majority voting is used at both stages; henceforth we call this a *two stages* majority procedure. Then for any positive amount of public good, g, the composition chosen, y, is the one most preferred by the median voter, $\text{med}(b_1, ..., b_n)$, and they show that the amount of public spending provided in equilibrium is given by:

$$g_M = [\alpha(1 - l_M)]^{1/(1-\alpha)}$$

$$\text{with } \; l_M = med(\mid \text{med}(b_1, ..., b_n) - b_1 \mid, ..., \mid \text{med}(b_1, ..., b_n) - b_n \mid).$$

Here l_M is the median distance from the median preferred composition. Note that the median voter in the first stage is one of the individuals, whose bliss point in term of composition is one of the two values: $\text{med}(b_1, ..., b_n) \pm l_M$. It is assumed that the minority median in term of composition is to the right of the median bliss point and moreover it is supposed that the conflict over composition translates into a conflict over public good expenditure: namely the minority median is strictly greater than $\text{med}(b_1, ..., b_n) + l_M$. Hence the minority median finds g_M too large.

Let us first consider the introduction of the average rule at the second stage, while still using the majority rule to determine public expenditures: the *majority-average* procedure. We compare the outcome of this procedure with that of the two-stage majority procedure. We assume that, in the second stage, the switch to average voting strictly protects the minority so that the outcome is larger than the majority outcome but does not exceed the minority's median bliss point. Using backward induction we find that the amount of public good spending is given by

$$g^* = [\alpha(1 - l^*)]^{1/(1-\alpha)}$$

$$\text{with } l^* = med(\mid y^* - b_1 \mid, ..., \mid y^* - b_n \mid).$$

That is, l^* is the median distance to the average voting outcome. Once again we use the utility of the minority's median voter as criterion for evaluating whether the minority is protected.

Definition 5.1 *A minority is protected by a switch in voting rule, if all minority members whose bliss points are located at the minority median bliss point in term of composition or beyond, see an increase in utility, i.e.,*

$$u_i(y^*, g^*) \geq u_i(med(b_1, ..., b_n), g_M) \qquad \text{for all } i \text{ such that } b_i \geq med(b_1, ..., b_m). \tag{5}$$

Under the assumption that the minority is protected at the second stage, overall protection depends on how the switch in voting rule affects public expenditures. Whether the amount of public good spending g^* increases or not relative to that of majority voting g_M depends on how l^* relates to l_M. Public expenditures strictly increase if and only if $l^* < l_M$. Unfortunately in the general case, the relation between the two median distances is ambiguous. However, if the spread between the two composition outcomes, y^*- med$(b_1, ..., b_n)$ is small enough, it is possible to determine whether the minority is protected by a change in voting rule depending on the direction in the change in public expenditure.

Proposition 5.1 *Let us assume that the minority is protected in the composition stage, i-e, the condition in Proposition 3.1* (ii) *is verified. There exists an upper bound δ such that, if y^*- $med(b_1, ..., b_n) < \delta$ and $g^* < g_M$ hold, the minority will be strictly protected by a switch from a two stages majority procedure to a majority-average procedure.*

Proof. Let us consider the members of the minority such that their b_i is larger than or equal to the minority median. Since by assumption this latter is larger than both the equilibrium average composition and med$(b_1, ..., b_n)+ l_M$, it follows that $b_i \geq \sup(y^*, \text{med}(b_1, ..., b_n)+ l_M)$. By way of consequence, their bliss point in terms of expenditure level is strictly smaller than g_M. Thus u_i is strictly decreasing in g in a neighborhood of $(\text{med}(b_1, ..., b_n), g_M)$. If y^*- med$(b_1, ..., b_n) < \delta$, for δ sufficiently small, then by continuity, g^* is close to g_M. Since by assumption $g^* < g_M$, $u_i(\text{med}(b_1, ..., b_n), g^*) > u_i(\text{med}(b_1, ..., b_n), g_M)$ and since u_i is strictly increasing in y on $[\text{med}(b_1, ..., b_n), y^*]$, $u_i(y^*, g^*) > u_i(\text{med}(b_1, ..., b_n), g^*)$. Therefore $u_i(y^*, g^*) > u_i(\text{med}(b_1, ..., b_n), g_M)$ for all i such that b_i is larger than or equal to the minority median. The result follows. ■

In contrast, if public spending increases, it is ambiguous whether minority voters benefit from the change. On the one hand the public good composition is more to their liking. On the other hand they have to pay more taxes for a public good mix which remains remote from the one that suits them best. In the general case, it is difficult to obtain a condition warranting the decrease of public good spending after introducing the average rule at the second stage.

For the sake of further discussion, we now consider a double-spikes bliss points distribution. Minority members share bliss point b_1, while majority members have bliss point b_n. The median composition is b_n and we focus on the most interesting case where the average voting composition is cW_m, that is when $b_1 > cW_m > b_n$. Public good spending is always smaller after introducing the average rule. We now derive a sufficient condition ensuring that this fall benefits the minority.

Claim 5.1 *Assume a double-spikes bliss points distribution such that* $(b_1 + b_n)/2 > cW_m > b_n$. *The minority will be strictly protected by a switch from a two stages majority procedure to a majority-average procedure.*

Proof. $l_M = 0$ and $l^* = cW_m - b_n > 0$ by assumption imply $g^* < g_M$. The minority bliss point in terms of public expenditure is determined by the distance $l_1 = b_1 - cW_m$. If $cW_m \leq (b_1 + b_n)/2$, then $l_1 \geq l^*$ and therefore g^* exceeds the level of public expenditures favored by the minority. Thus under the assumption, u_i is strictly decreasing in g on the interval $[g_M, g^*]$. Then $g^* < g_M$ implies $u_i(b_n, g^*) > u_i(b_n, g_M)$. Furthermore, since u_i is strictly increasing in y on $[b_n,\, b_1]$, $u_i(cW_m, g^*) > u_i(b_n, g_M)$ for all minority members. ■

When the corrected weight of the minority exceeds the arithmetic mean of bliss points, the benefit for the minority of having a better mix of public good is somewhat mitigated and may be overturned by the reduction in public expenditure imposed by the majority which is unwilling to finance a public good combination mainly enjoyed by the minority.

This simple example illustrates that under average voting on the public good composition, the distribution of tastes over public expenditure may be switched around if the minority weight is large enough. If the latter weight is low enough, then its increase may induce a convergence of tastes and thus a reduction of conflicts over public expenditure.

It is also interesting to note that contrary to what happens in a one stage game, it is no longer true that a minority always benefits from having an increased weight. It remains unambiguously true only if the weight of the minority is not too large. Along the same line of argument as in the above claim, we state

Remark 5.1 *Assume a double-spikes bliss points distribution and consider a majority-average procedure. An increase in the corrected weight* cW_m *will induce an increase in utility for minority members as long as* $cW_m \leq (b_1 + b_n)/2$.

One may wonder whether introducing average voting at the first stage as well, would render the outcome more favorable to the minority. We now compare the *two stages average* procedure to the *majority-average* procedure. Once again the double-spikes example provides a convenient framework to get a flavor of what might happen. We focus on the situation stated in claim 5.1 so that the minority unambiguously gains when the average vote is introduced at the second stage only. We find it sensible to select an unweighted average procedure for the first stage, which is a Bowen game. The assumption in claim 5.1 implies that the majority favors more expenditure than the minority. We consider realistic values for bliss points on public expenditure and therefore assume that they do not exceed 50%. A straightforward application of part (ii) of Proposition 3.1 implies that the level of public expenditures under the *two stages average* procedure will exceed the level obtained in the *majority-average* voting game. The minority would therefore not gain from substituting average voting to majority voting in the choice of public expenditures, given that an average procedure is used to determine composition.

Finally, a line of reasoning very similar to that just followed shows that the minority would prefer the original *two-stage majority* procedure to an average-majority procedure. With an average procedure at the first stage only, minority members would pay more taxes for a good they dislike. The main message here is that, in a sequential game, the interests of the minority are better preserved if the average rule is only used for solving a conflict over the public good mix.

6 Conclusion

Much of our analysis relies on our second characterization according to which the equilibrium outcome can be expressed as the median of bliss points and corrected cumulative weights. It is striking that the outcome of an average procedure should be expressed as a median. This turns out to be convenient to compare majority and average outcomes.

We exhibit conditions under which average voting may be effective in protecting minority interests. These interests are identified with those of the median voter in the minority, whose bliss point is larger than that of the median voter in the overall population. As should be expected, a larger minority weight facilitates protection. For instance in Bowen's model, if the conflict is over the rate of a property tax earmarked at public school financing, a sufficient condition for the minority to benefit is that its share in the electorate should exceed the tax rate prevailing under majority voting. In the "Forced to Pay yet Free to Choose" model, where weights are based on tax contributions, the effectiveness of average voting in minority protection depends upon the correlation between wealth and tastes and the progressiveness of taxes. A minority would be protected with a poll tax, provided that the majority outcome and the minority's median bliss point fall on each side of 50%. In any case, independent of the minority weight, a switch in voting rule always provides some degree of protection, whenever the majority outcome is small enough.

When public expenditure is chosen first and its allocation is chosen next, the interest of the minority is best protected by using an average rule in the second stage with a majority rule in the first stage. Moreover the use of the average rule to determine public good composition tends to reduce conflict over the amount of public good spending. Once again, the weight of the minority proves to be a key parameter to find a sufficient condition to get minority protection. If the minority is not too powerful, it is favored by a switch from a two-stages majority procedure to a majority-average procedure, at least in simple examples.

One remaining question is whether our characterization of the average outcome strongly depends upon tastes being common knowledge among voters. In our companion paper, Renault and Trannoy (2001), we show that the asymmetric information outcome for a large population is very close to the symmetric information outcome. Another remaining question, which could be addressed in future research, is whether it is possible to characterize the outcome of the average voting game in a multidimensional setting.

The average rule is well suited for protecting minorities when the set of alternatives is continuous in a direct democracy context. Other voting rules have been proposed and sometimes used for the same purpose such as proportional voting, voting with quotas, the Borda rule (see Emerson (1998)). These rules are especially appropriate in a representative democracy setting (see Myerson (1993) or Laslier (2002) for the treatment of minorities in such a context), or when the set of alternatives is finite. Our definition of minority protection could be used to evaluate the relative merits of these various rules.

References

[1] Alesina, A., R. Baqir and W. Easterly (1999): Public Goods and Ethnic Divisions, *The Quarterly Journal of Economics*, vol.114, pp 1243-1284.

[2] Arrow (1963): Social Choice and Individual Values, 2d ed. New York: Wiley.

[3] Bagnoli, M. and M. McKee (1992): Controlling the game: Political Sponsors and Bureaus, *Journal of Law Economics and Organization*, vol.7, pp 229-247.

[4] Bergstrom, T. (1979): When does Majority Rule Supply Public Goods Efficiently, *Scandinavian Journal of Economics,* pp 216-226.

[5] Bergstrom, T., L. Blume and H. Varian (1986): On the Private Provision of Public Goods, *Journal of Public Economics,* vol.29, pp 25-49.

[6] Bilodeau, M. (1994): Tax Earmarking and Separate School Financing, *Journal of Public Economics,* vol.54, pp 51-63.

[7] Black, D. (1948): On the Rationale of Group Decision Making, *Journal of Political Economy,* vol.56, pp 23-34.

[8] Bowen, H. (1943): The Interpretation of Voting in the Allocation of Economic Resources, *The Quarterly Journal of Economics,* vol.58, pp 27-48.

[9] Debreu, G. (1952): A social Equilibrium Existence Theorem, *Proceedings of the National Academy of Sciences*, vol.38, pp 886-893.

[10] Downs, A (1957): An Economic Theory of Democracy. New York: Harper and Row.

[11] Emerson, P.J. (1998): Beyond the Tyranny of Majority Rule: Voting Methodologies in Decision-Making and Electoral Systems, Belfast: The De Borda Institute.

[12] Gerber, A. and I. Ortuno-Ortin (1998): Political Compromise and Endogenous Formation of Coalitions, *Social Choice and Welfare*, vol.15, pp 445-454.

[13] Guinier, L. (1994) : The Tyranny of the Majority: Fundamental Fairness in Representative Democracy. The Free Press.

[14] Holcombe R. (1977) : The Florida System: A Bowen Equilibria Referendum Process. *National Tax Journal*, vol.30, pp 77-84.

[15] Kirman, A. and M.Widgren (1995): European Economic Decision-making : Progress or Paralysis? *European Economic Review,* vol.21, pp 423-460.

[16] Laslier, J.F.(2002): How two-party competition treats minorities, *Review of Economic Design*, Vol.7, pp 297-307.

[17] McKee, M. (1988): Political Competition and the Roman Catholic Schools: Ontario, Canada, *Public Choice,* vol.56, pp 57-67.

[18] Myerson, R. (1993): Incentives to cultivate favored minorities under alternative electoral systems, *American Journal of Political Science* vol. 87, pp 856-869.

[19] Powell, G. B.(1981): Party Systems and Political System Performance: Voting Participation,Government Stability and Mass Violence in Contemporary Democracies, *American Political Science Review,* vol.75, pp 861-879.

[20] Renault, R. and A. Trannoy (2003) : The average rule in a large population. *Mimeo.* THEMA.

[21] Rudin,W (1976): Principles of Mathematical Analysis, 3^{rd} edition, McGraw Hill.

[22] Warr, P. (1983): The Private Provision of a Public Good is independent of the Distribution of Income, *Economic Letters*, vol.13, pp 207-211.