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The impossibility of a Paretian egalitarian

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Abstract. In a one-good world, there is a nice correspondence between the Pigou-Dalton principle of transfer and social welfare dominance. In this paper we study the case of multiple goods (without using prices as a means to come back to one dimension), and show that many results of the one-dimensional setting carry over to the multidimensional case when individuals are assumed to have identical preferences. But the nice correspondence breaks down as soon as individual preferences display minimal differences, and multidimensional versions of the transfer principle clash with the Pareto principle. This analysis reveals an interesting connection with the theory of fair allocation, since multidimensional transfer principles are closely related to the no-domination criterion, a weak version of the no-envy criterion.

1 Introduction

Among the many issues to which Louis Gevers has contributed, comes to mind the construction of social rankings in economic environments and more precisely in the Edgeworth box (Gevers 1986). The social choice problem has more structure when one takes account of the scarcity of resources and some basic features of individual preferences like monotonicity and convexity.

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Efficiency considerations are among the most basic requirements in this setting, but the Pareto principle yields an incomplete ranking, and this makes it possible to study how it can be completed with the help of equity requirements.

The problem of allocating scarce economic resources in a fair and equitable way has received a good deal of attention and several equity criteria have been studied in the literature, such as no-envy (Kolm 1972; Varian 1974), no-domination (Moulin and Thomson 1988), egalitarian-equivalence (Pazner and Schmeidler 1978), and various solidarity notions (Thomson 1983; Moulin and Thomson 1997). The compatibility of these various equity concepts with the Pareto requirement has been examined in many papers but the bulk of the literature has focussed on the selection of first-best equitable allocations. Until recently, little was known about how to rank inefficient and inequitable allocations. Constructing fine-grained social rankings would nonetheless be important if one wanted to address issues of reform or to design second-best institutions, and the relevance of a ranking perspective has often been acknowledged (see e.g., Varian 1976; Diamantaras and Thomson 1991; Guesnerie 1995). It is indeed difficult to know a priori the restrictions faced by the decision maker which can be traced back to asymmetries of information, imperfection competition and political constraints. In recent papers, Fleurbaey and Maniquet (1997, 1999) have proposed social rankings based on equity criteria, for various models.

Our starting point in this paper, however, will not be the theory of fair allocation but the approach of welfare dominance, which directly focusses on social rankings. The core idea in this approach is to find simple criteria which guarantee that an allocation is socially preferred to another for a wide class of social welfare functions and individual utility functions. Our interest in this approach comes in particular from the idea that it may provide simple equity criteria which would enable us to compare some allocations independently of individual preferences.

The best known results in this approach deal with the one-dimensional case (i.e., when individuals consume only one good, such as income when prices are fixed), and have popularized the Lorenz curve, and the principle of transfer due to Pigou and Dalton. Interestingly, such criteria can be viewed as yielding incomplete rankings just like the Pareto principle, and this raises the question of the compatibility between these various incomplete rankings (Pareto on one side, Lorenz dominance on the other side). This issue, extended to the multidimensional case, will be our main focus in this paper.

In the one-dimensional case there is no problem, and Paretian social welfare functions may be required to satisfy the Pigou-Dalton principle of transfer without any trouble. Generalizations of Lorenz dominance to the multidimensional case have been studied by Atkinson and Bourguignon (1982), Le Breton (1986), Rietveld (1990), Mosler (1991), Koshevoy (1995,1998), Mosler (1995), Koshevoy and Mosler (1996), Tsui (1995). The progress in this matter proves to be difficult, as several authors pointed out, because very little is known about majorization in a multidimensional setting (see Marshall and Olkin 1979).

Our contribution is made of three parts. In the first part (Sect. 2), we show how many of the nice results obtained in the one-dimensional setting carry over to the multidimensional case when one restricts attention to the case when individual preferences over consumption goods are *identical* in the whole population. In particular, this analysis provides an interesting generalization of the Pigou-Dalton principle of transfer, formulated in terms of bundles. In the second part (Sect. 3), we show that the extension to the case of non identical preferences entails a severe conflict between egalitarian principles such as Pigou-Dalton and the Pareto principle. The conflict is so severe that if one insists on obeying the Pareto principle, a radical revision of the way egalitarian principles are handled is needed. This conflict appears to be closely related to a similar conflict uncovered by Kolm (1972), Suzumura (1981a,b) and Tadenuma (2002), regarding the no-envy criterion. For instance Tadenuma (2002) proves that any Paretian social ranking which relies upon the no-envy test in case of indecisiveness of the Pareto criterion fails to be acyclic. In the third part (Sect. 4), we show that the Pigou-Dalton transfer principle is closely related to the equity concept of no-domination (a weaker notion than no-envy[1]), and we rely on a methodology suggested in Fleurbaey and Maniquet (1997) to propose a solution to the conflict, that is, a particular construction of Paretian social preferences which take account of the egalitarian principle under consideration. This brings us back to the theory of fair allocation, and its extension to the ranking of all allocations in a fine-grained way.

In this way, our work establishes a connection between the theory of welfare dominance and the theory of fair allocation. This connection is somewhat surprising because the traditional source of inspiration of the theory of welfare dominance is utilitarianism, whereas fairness concepts are usually interpreted as conveying an egalitarian view, and focus on resources rather than welfare (in particular, no interpersonally comparable utilities are involved in the theory of fair allocation). But the requirements of dominance, that is, unanimity over a wide class of social welfare functions, single out egalitarian or quasi-egalitarian criteria such as the Pigou-Dalton transfer principle, and such notions display simple logical relationships with equity criteria such as no-domination. And the theory of welfare dominance is not hostage to the utilitarian philosophy, as will be explained below. We hope, therefore, that our work will contribute to clarify the content and meaning of these various theories.

[1] No-envy is obtained when no individual would rather consume another's bundle; no-domination is obtained when no individual has a smaller bundle than another.

2 Multidimensional dominance

We consider the classical model of a division economy with n individuals identified by $i = 1, \ldots, n$ and ℓ goods denoted by $j = 1, \ldots, \ell$. $N = \{1, \ldots, n\}$ is the set of individuals, and $L = \{1, \ldots, \ell\}$ the set of goods. An allocation is described by a matrix $x = (x_{ij}) \in \mathbb{R}_+^{n\ell}$ with n rows and ℓ columns. The vector x_i is the *ith* row of this matrix, and the *jth* column x_j gives the distribution of attribute j among the n persons. An allocation x is feasible when $\sum_{i=1}^{n} x_{ij} = \omega_j$ for $j = 1, \ldots \ell$, with $\omega \in \mathbb{R}_{++}^{\ell}$ being the total endowment vector. X stands for the set of feasible allocations.

Comparisons of vectors are denoted as follows: $x_i \geq y_i$ if $x_{ij} \geq y_{ij}$ for all $j, x_i > y_i$ if $x_{ij} \geq y_{ij}$ for all j and $x_i \neq y_i$, $x_i \gg y_i$ if $x_{ij} > y_{ij}$ for all j.

x is obtained from y by a increment of good j, when some individual has more of good j, given that all other attributes levels remain fixed. When $x > y$ we say that x is obtained from y by an increment.

Following Sen (1970a), we call quasi-ordering a reflexive and transitive binary relation, and ordering a complete quasi-ordering.

The individual consumption set is $\mathbb{R}_+^{\ell}$ and individual i's preferences are described by a continuous, monotonic and convex ordering $\succeq_i$ over $\mathbb{R}_+^{\ell}$, with corresponding strict preference $\succ_i$ and indifference relations $\sim_i$. We also need to define u_i a utility function representing $\succeq_i$. The upper contour set of x_i in $\mathbb{R}_+^{\ell}$ for preferences $\succeq_i$ is denoted

$$\succeq_i (x_i) = \{x_i' \in \mathbb{R}_+^{\ell} | x_i' \succeq x_i\}.$$

Let $\succeq = (\succeq_1, \ldots, \succeq_i, \ldots, \succeq_n)$ be a profile of preferences, with D the domain of profiles satisfying the above conditions.

A social quasi-ordering is a quasi-ordering over the set of allocations (which, in the sequel, will be either $\mathbb{R}_+^{n\ell}$ or X). A social quasi-ordering function (SQOF) is a mapping R defining a social quasi-ordering $R(\succeq)$ for every preference profile $\succeq$ in a given domain. The corresponding strict preference and indifference functions are denoted $P(\succeq)$ and $I(\succeq)$.

We first introduce the Paretian SQOFs over $\mathbb{R}_+^{n\ell}$.

Definition 1. *(i) x is said to Weakly Pareto dominate y ($x R_{WP}(\succeq) y$) if:*

$$\forall i \in N, \; x_i \succ_i y_i.$$

(ii) x is said to Strongly Pareto dominate y ($x\, R_{SP}(\succeq) y$) if:

$$\forall i \in N, \; x_i \succeq_i y_i.$$

Notice that $P_{WP} = R_{WP}$, and that $x\, P_{SP}(\succeq) y$ whenever $x_i \succeq_i y_i$ for all i, and $x_i \succ_i y_i$ for at least one i. The Pareto criteria are silent about the distributional equity of allocations, and to this we now turn, focussing on the dominance approach first, as explained in the introduction.

The literature on dominance and inequality measurement has been largely concerned with the one-dimensional case. Kolm (1977) was the first to study multidimensional issues in the context of welfare analysis (see for instance Hadar and Russel (1974) in the context of choice under uncertainty) and, since then, there have been many papers devoted to this topic (for example, Marshall and Olkin 1979 Ch. 15; Atkinson and Bourguignon 1982; Koshevoy 1995, 1998; Koshevoy and Mosler 1996). Apart from Atkinson and Bourguignon's article, the heart of multidimensional dominance analysis concerns "price majorization" or "expenditure majorization": An allocation x is viewed as more unequal than an allocation y if the distribution of the individual budgets px_i is less unequal than the distribution of individual budgets py_i for all p belonging to some domain. Here we are reluctant to give such a role to prices, because we would like to focus on social welfare rather than on income inequality. Moreover some important goods which matter for welfare such as health or education are often nontradable goods for many reasons. In addition the use of prices as a valuation criterion in a second-best economy is more debatable than in a first-best economy.

We define here several SQOFs which reflect various dominance concepts. These SQOFs are independent of the preference profile, and therefore we retain a simpler notation for them. The first two do not mirror a specific concern with equality but are stated for the sake of completeness. They are concerned with the size of the cake rather than its distribution, and they are inspired by a perspective of first degree stochastic dominance. In a multidimensional setting, two conceptions are possible. In the first one, a good by good perspective is adopted. In the second one, a more global perspective is favored and the basic element is the bundle allocated to an individual. For any allocation of good j, denoted $x_j = (x_{1j}, \dots, x_{nj})$, let $(x_{(1)j}, \dots, x_{(n)j})$ denote the rearrangement of x_j such that $x_{(1)j} \leq \cdots \leq x_{(n)j}$. The good by good view is encapsulated in the following definition.

Definition 2. *x is said to Rank Order dominate y ($x \succeq_{RO} y$) if:*

$$\forall i \in N, \forall j \in L, \; x_{(i)j} \geq y_{(i)j}.$$

Allocation x is obtained from allocation y by a permutation if there is a bijection σ from N to N such that: for all $i \in N$, $x_i = y_{\sigma(i)}$. The idea of justice surely has something to do with some kind of symmetry among individuals. Suppes (1966), with his grading principles of justice, was the first to put forward the permutation operator in a formal criterion of justice (see Sen 1970, Ch. 9 and 9*). For that reason Saponisk (1981, 1983) coined the term of Suppes dominance for the following dominance concept, which belongs to the bundle approach.

Definition 3. *x is said to Suppes dominate y ($x \succeq_S y$) if there exists a permutation σ of N such that:*

$$\forall i \in N, \; x_i \geq y_{\sigma(i)}.$$

Checking that the Suppes' requirement is stronger than the Rank Order one when there are at least two goods is immediate.

Remark 1. *It follows from the above definitions that* $x \succeq_S y \Rightarrow x \succeq_{RO} y$. *When* $\ell = 1$, $x \succeq_{RO} y \Rightarrow x \succeq_S y$ *(see Saposnik 1981, 1983). For* $\ell \geq 2$, *it is no longer true, as shown in the following example.*

Example 1. *Let* $x = \begin{pmatrix} 2 & 1.1 \\ 1.1 & 2 \end{pmatrix}$ *and* $y = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$. *One has* $x \succeq_{RO} y$ *but there does not exist* σ *on* $N = \{1, 2\}$ *such that* $x_i \geq y_{\sigma(i)}$ *for* $i = 1, 2$.

The ensuing definitions capture different notions of equality in a multidimensional setting. The Generalized Lorenz curve proposed by Shorrocks (1983) is a fundamental tool for drawing conclusions about welfare from individual income data. It gives the cumulative income associated to any subset of individuals provided that individuals are ranked in an increasing order. Here is an extension of this notion for the multidimensional framework.

Definition 4. *x is said to Generalized-Lorenz dominate y* ($x \succeq_{GL} y$) *if*:

$$\forall k \in N, \forall j \in L, \sum_{i=1}^{k} x_{(i)j} \geq \sum_{i=1}^{k} y_{(i)j}.$$

In the one-dimensional setting, a well known and fundamental result of Hardy-Littlewood-Polya (1952) establishes the equivalence between Lorenz dominance and the fact that the dominated distribution can be transformed into the dominant one by means of a finite sequence of Pigou-Dalton progressive transfers. We find again a dichotomy between the good by good view and the bundle view when we try to define principles of transfer in a multidimensional setting. The view that there are goods and services, such as health care, housing or education, whose availability to different individuals should not depend on their income, is known as *specific egalitarianism* (Tobin 1970). We follow here Kolm (1977) in considering commodity-specific equalizing transfers of the Pigou-Dalton type.

Definition 5. x_j *is obtained from* y_j *by a Pigou-Dalton transfer of good* j ($x_j \succeq_{PD} y_j$) *if*: $\exists h, k \in N$ *such that*:

1) $x_{ij} = y_{ij} \; \forall i \neq h, k$,
2) $x_{hj} + x_{kj} = y_{hj} + y_{kj}$,
3) $y_{hj} \geq \max(x_{hj}, x_{kj}) \geq \min(x_{hj}, x_{kj}) \geq y_{kj}$.

This definition is somewhat extensive in that it encompasses transpositions. It is worth emphasizing that the recipient of the transfer may be very well supplied in other goods than j. This notion is related to the idea of demanding an equalization of consumption in every dimension, among individuals.

The corresponding notion for the bundle perspective is the following.

Definition 6. *x is obtained from y by a Pigou-Dalton transfer of bundle if*: $\exists h, k \in N$, *such that*:

1) $x_i = y_i \, \forall i \neq h, k$,
2) $x_h + x_k = y_h + y_k$,
3) $y_h \geq x_h \geq x_k > y_k$.

Let $x \succeq_{PDB} y$ denote the fact that x is obtained from y by a Pigou-Dalton transfer of bundle or by an increment.

The third condition in the above definition implies that a prerequisite for the transfer is $y_h > y_k$. It means that in the initial allocation, individual k has to be weakly poorer than individual h in all dimensions, and strictly poorer in some dimensions. Furthermore the ranking of individual bundles is required to be preserved by the transfer, which means that transpositions are not included as a borderline case. The interest of adopting such a restrictive definition will become transparent in the next section since it makes the negative results of that section stronger. It leads to the coarsest social quasi-ordering among those which capture some notion of equality.

Definition 7. *x is said to Pigou-Dalton bundle dominate y* $(x \succeq_{\widehat{PDB}} y)$ *if*:

$$\exists \{z^0, \ldots, z^T\} \subset \mathbb{R}_+^{n\ell}, z^0 = x, z^T = y, \forall t \in \{0, \ldots, T-1\}, \;\; z^t \succeq_{PDB} z^{t+1}.$$

Combining this quasi-ordering with the Suppes'one leads to the following definition.

Definition 8. *x is said to Suppes or Pigou-Dalton bundle dominate y* $(x \succeq_{\widehat{SPDB}} y)$ *if*:

$$\exists \{z^0, \ldots, z^T\} \subset \mathbb{R}_+^{n\ell}, z^0 = x, z^T = y, \forall t \in \{0, \ldots, T-1\}, \\ z^t \succeq_{PDB} z^{t+1} \, or \, z^t \succeq_S z^{t+1}.$$

Switching positions of individuals is allowed with this last social quasi-ordering. The following remark states some obvious relations between all these SQOFs which are representative of a second order stochastic dominance point of view.

Remark 2. $x \succeq_{\widehat{PDB}} y \Rightarrow x \succeq_{\widehat{SPDB}} y \Rightarrow \forall j \in L, x_j$ *is obtained from* y_j *by a sequence of Pigou-Dalton transfers or increments of good* $j \Leftrightarrow x \succeq_{GL} y$. *The first two implications are a direct consequence of the definitions while the last equivalence is stated by Kolm (1977) in verbal terms and ensues from the Hardy-Littlewood-Polya Theorem (see Marshall and Olkin 1979).*

A double classification according to the degree of dominance and the good or bundle perspective makes it easier to memorize the various concepts of dominance introduced above.

	order 1		order 2
			$x \succeq_{\widehat{PDB}} y$
bundle			$\Downarrow$
	$x \succeq_S y$	$\Rightarrow$	$x \succeq_{\widehat{SPDB}} y$
	$\Downarrow$		$\Downarrow$
good	$x \succeq_{RO} y$	$\Rightarrow$	$x \succeq_{GL} y$

The interest of the Hardy-Littlewood-Polya theorem comes from the fact that it establishes the equivalence not only between transfers and Lorenz dominance, but also between these and welfare dominance, for a social welfare function of the additively separable kind:

$$\sum_{i=1}^{n} u(x_i).$$

This third notion is usually tied to a welfarist point of view, more specifically utilitarianism. Actually, a more neutral interpretation is possible. The utility function u may not only represent individuals' subjective satisfaction, but may also embody a social aversion to inequality. One may even consider u as the social planner's evaluation of consumption bundles, without any direct relation with individual preferences. In other words, many philosophical approaches can be subsumed under an additively separable social welfare function of the above kind.

Some restrictions about the utility functions considered are now formalized. A function $u : \mathbb{R}_+^\ell \to \mathbb{R}$ is said to be increasing if $u(x_i) > u(y_i)$ whenever $x_i > y_i$, and additive if there exist ℓ functions u_j such that $u(x_i) = \sum_j u_j(x_{ij})$. Let us now introduce several classes of utility functions:

$$\mathcal{U}_1 = \{u : \mathbb{R}_+^\ell \to \mathbb{R} \text{ continuous and increasing}\}.$$

$$\mathcal{U}_2 = \{u : \mathbb{R}_+^\ell \to \mathbb{R} \text{ continuous, increasing and additive}\}.$$

$$\mathcal{U}_3 = \{u : \mathbb{R}_+^\ell \to \mathbb{R} \text{ continuous, increasing, additive and concave}\}.$$

The two last classes of utility functions are less general than those considered in earlier works by Atkinson and Bourguignon (1982) or Hadar and Russel (1974), who do not assume utility functions to be additive. This restriction comes spontaneously when we try to find a class of utility functions which allows us to support the idea of Pigou Dalton good dominance (see Proposition 3 below for a formal statement).

The notion of welfare dominance studied here is based on additively separable social welfare functions which are symmetrical with respect to bundles: An allocation x is said to dominate another allocation y with respect to the class $\mathcal{U}$ whenever

$$\sum_{i=1}^{n} u(x_i) \geq \sum_{i=1}^{n} u(y_i)$$

for all $u \in \mathcal{U}$, and this will be denoted $x \succeq_{\mathcal{U}} y$.

The following propositions describe the links between some of the SQOFs introduced above and dominance with respect to various utility classes. The simple proofs of the first two results are omitted.

Proposition 1. *$x \succeq_S y \Leftrightarrow x \succeq_{\mathcal{U}_1} y \Leftrightarrow x$ is obtained from y by a permutation and a sequence of increments.*

Proposition 2. *$x \succeq_{RO} y \Leftrightarrow x \succeq_{\mathcal{U}_2} y \Leftrightarrow \forall j \in L$, x_j is obtained from y_j by a permutation and a sequence of increments.*

Kolm (1977) formulated a characterization of the Pigou-Dalton good dominance in the case where we want to compare allocations from a strict inequality view point. When we are interested in both dimensions – the size and the distribution of the cake – the adequate tool is the Generalized Lorenz Curve and extending Shorrocks (1983) we obtain:

Proposition 3. *$x \succeq_{GL} y \Leftrightarrow x \succeq_{\mathcal{U}_3} y \Leftrightarrow \forall j \in L$, x_j is obtained from y_j by a sequence of Pigou-Dalton transfers or increments of good j.*

To our knowledge a general characterization is not available for Pigou-Dalton bundle dominance. Nevertheless an interesting result has been obtained by Le Breton (1986) in a rather specific configuration. Let $r_j(i,x)$ define the rank of i in the increasing rearrangement of x_j.

Definition 9. An allocation x is co-monotone if and only if:

$$\forall i \in N, \forall j,k \in L, \; r_j(i,x) = r_k(i,x).$$

The restriction introduced is rather stringent. It means that the ranking of individuals according to each good is exactly the same. We need to introduce a new class of utility functions. A utility function is said to have non-increasing increments if

$$u(x+h) - u(x) \geq u(y+h) - u(y)$$

for all $x, y \in \mathbb{R}^{\ell}_{+}$ such that $x \leq y$, and for all $h \in \mathbb{R}^{\ell}_{+}$. When u is twice continuously differentiable on $\mathbb{R}^{\ell}_{+}$, then u has non-increasing increments if and only if $u''_{jk} \leq 0 \, \forall j,k \in L$, a condition known under the label of ALEP substitutability[2] (see Chipman 1977). When a person gets richer, marginal utility is

[2] ALEP stands for Auspitz-Lieben-Edgeworth-Pareto.

required to decrease in each dimension. Introducing this condition on utility functions allows us to define:

$$\mathcal{U}_4 = \{u : \mathbb{R}^\ell_+ \to \mathbb{R} \text{ continuous, increasing, quasi-concave with non-increasing increments}\}.$$

When does a Pigou-Dalton transfer of bundles improve the social welfare? It must be the case if there is no ambiguity in the ranking of individuals and if the marginal utility of the rich is smaller than the poor's one in each dimension, a proviso guaranteed by ALEP substitutability.

Proposition 4. *Let x and y two co-monotone allocations. Then* $x \succeq_{\widehat{SPDB}} y \Leftrightarrow x \succeq_{GL} y \Leftrightarrow x \succeq_{\mathcal{U}_4} y$.

Proof. The last equivalence is proved by Le Breton (1986) (Theorem 4.2.12, p 210). For the first equivalence the "if" part is already stated in Remark 2, while the "only if" part is a consequence of the fact that x and y are co-monotone. ■

It does not come as a surprise that when there is no ambiguity about the ranking of individuals, transferring bundles is tantamount to improving the evaluation of the distribution of resources according to the Generalized Lorenz criterion. When we drop the ALEP substitutability condition but maintain the restriction that the comparison is performed between two co-monotone allocations, the equivalence between the Generalized Lorenz criterion and the dominance according to a class of utility functions is lost. Nevertheless we can mention a limited positive result obtained by Le Breton (1986). Suppose that the comparison is carried out between two co-monotone allocations, which are also efficient when the efficiency test is computed for a given utility function u which is assumed to be continuous, increasing and concave. Then (Generalized) Lorenz dominance is a sufficient condition to guarantee welfare dominance for this particular utility function u.

3 The trade-off between dominance and Pareto

The results obtained in the previous section can be interpreted as meaning that the extension of one-dimensional analysis to multidimensional inequalities is not too problematic, provided that social welfare is assumed to be symmetric with respect to individual bundles. But using the same function u for all individual bundles means either that all individuals have the same preferences or that u does not represent individual preferences but the social planner's paternalistic conception of the good life. What happens to the above concepts when one accepts the possibility of heterogeneous individual preferences and insists on obeying the Pareto principle?

This question will make us shift from an analysis of dominance to an analysis of existence. In the previous section we could exhibit large classes of social welfare functions compatible with $\succeq_S$, $\succeq_{RO}$, or $\succeq_{GL}$. In this section we prove that finding even one Paretian social quasi-ordering function compatible with them, or even with the coarser $\succeq_{\widehat{PDB}}$, is problematic.

We shall use the following set of axioms requiring that the SQOF R we are looking for exhibits some compatibility with the basic SQOFs defined above.

Axiom 1. *Weak Pareto (WP)*: $\forall \succeq \in D, \forall x, y \in \mathbb{R}_+^{n\ell}$,

$$x\, P_{WP}(\succeq)\, y \Rightarrow x\, P(\succeq)\, y.$$

Axiom 2. *Strong Pareto (SP)*: $\forall \succeq \in D, \forall x, y \in \mathbb{R}_+^{n\ell}$,

$$[x\, R_{SP}(\succeq) y \Rightarrow x\, R(\succeq)\, y] \textit{ and } [x\, P_{SP}(\succeq) y \Rightarrow x\, P(\succeq)\, y].$$

Axiom 3. *Suppes Dominance (SD)*: $\forall \succeq \in D, \forall x, y \in \mathbb{R}_+^{n\ell}$,

$$x \succeq_S y \Rightarrow x\, R(\succeq)\, y.$$

Axiom 4. *Rank Order Dominance (ROD)*: $\forall \succeq \in D, \forall x, y \in \mathbb{R}_+^{n\ell}$,

$$x \succeq_{RO} y \Rightarrow x\, R(\succeq)\, y.$$

Axiom 5. *Generalized Lorenz Dominance (GLD)*: $\forall \succeq \in D, \forall x, y \in \mathbb{R}_+^{n\ell}$,

$$x \succeq_{GL} y \Rightarrow x\, R(\succeq)\, y.$$

Axiom 6. *Pigou Dalton Bundle Dominance (PDBD)*: $\forall \succeq \in D, \forall x, y \in \mathbb{R}_+^{n\ell}$,

$$x \succeq_{\widehat{PDB}} y \Rightarrow x\, R(\succeq)\, y.$$

Axiom 7. *Suppes and Pigou Dalton Bundle Dominance (SPDBD)*: $\forall \succeq \in D, \forall x, y \in \mathbb{R}_+^{n\ell}$,

$$x \succeq_{\widehat{SPDB}} y \Rightarrow x\, R(\succeq)\, y.$$

From the previous remark we can deduce the following logical relations:

$$\begin{array}{ccccc} \text{GLD} & \Rightarrow & \text{SPDBD} & \Rightarrow & \text{PDBD} \\ \Downarrow & & \Downarrow & & \\ \text{ROD} & \Rightarrow & \text{SD} & & \end{array}$$

and obviously SP $\Rightarrow$ WP.

The compatibility of all these notions in the context of identical individual preferences can be illustrated as follows. Let $\mathbb{R}_+^{n\ell}$ be the set of allocations to be ranked, and D_r be a domain of profiles of preferences on $\mathbb{R}_+^{\ell}$ such that individuals have the same preference relation representable by a utility function belonging to $\mathscr{U}_3$. Then there exists a SQOF satisfying both SP and GLD. Indeed, let $u^* \in \mathscr{U}_3$ represent the (identical) individual preference relation and define $R(\succeq)$ as follows:

$$x\, R(\succeq)\, y \Leftrightarrow \sum_{i=1}^{n} u^*(x_i) \geq \sum_{i=1}^{n} u^*(y_i).$$

$R(\succeq)$ satisfies SP and the axiom GLD as well.

The topic of this section is the compatibility of Pareto and dominance axioms over a wider domain of individual preferences, allowing for heterogeneity. Unfortunately, we obtain mostly negative results, even with the weakest axioms.

Proposition 5. *Let $\mathbb{R}_+^{n\ell}$ be the set of allocations and $D^* \subset D$ be any domain of profiles of preferences on $\mathbb{R}_+^{\ell}$ containing at least one profile with at least two individuals having different preferences. Then there exists no SQOF defined on D^* and satisfying both WP and SD, or WP and PDBD.*

Proof. Without loss of generality, assume that $R_1 \neq R_n$.

We first prove the incompatibility between SD and WP. One can find bundles x_1, x_n, y_1, y_n in $\mathbb{R}_+^{\ell}$ such that $y_1 \succ_1 x_1$, $y_n \succ_n x_n$, and $x_1 \gg y_n$ and $x_n \gg y_1$. Then just choose x_i, y_i for $i = 2, \ldots, n-1$ such that

$$x_1 \gg y_2 \gg x_2 \gg \cdots \gg y_{n-1} \gg x_{n-1} \gg y_n.$$

Let σ be a permutation defined on N as follows. Let $\sigma(1) = 2$, $\sigma(2) = 3, \ldots,$ $\sigma(n-1) = n$, and $\sigma(n) = 1$. One checks that for all $i \in N$, $x_i \gg y_{\sigma(i)}$.

Therefore one has $x\, R(\succeq)\, y$ by SD, but $y\, P(\succeq)\, x$ by WP, a contradiction.

We now prove the incompatibility between WP and PDBD. First, note that when $1, n$ have different preferences, it is always possible to find bundles $x_1, x_n, y_1, y_n, z_1, z_n, t_1, t_n$ in $\mathbb{R}_+^{\ell}$ such that $x_1 \succ_1 t_1$, $z_1 \succ_1 y_1$, $x_n \succ_n t_n$, $z_n \succ_n y_n$, with $x_n \gg y_n \gg y_1 \gg x_1$ and $z_1 \gg t_1 \gg t_n \gg z_n$, and $x_1 + x_n = y_1 + y_n$ and $z_1 + z_n = t_1 + t_n$. This fact is proved in the appendix.

Let $\varepsilon \in \mathbb{R}_{++}^{\ell}$ be such that

$$z_n \succ_n \; y_n + (n-2)\varepsilon,$$
$$x_n - \varepsilon \gg y_n + (n-2)\varepsilon.$$

For $i = 2, \ldots, n-1$, let $y_i = x_i = x_n$ and $z_i = t_i = x_n - \varepsilon/2$.

For $a = 1, \ldots, n-2$, define the allocations y^a by: $y_1^a = y_1$; $y_n^a = y_n + a\varepsilon$; and for $i \neq 1, n$, $y_i^a = y_i$ if $i > a+1$ and $y_i^a = y_i - \varepsilon$ if $i \leq a+1$ The following table summarizes these definitions:

	1	n	2	3	$\dots$	$n-1$
x	x_1	x_n	x_n	x_n	$\dots$	x_n
y	y_1	y_n	x_n	x_n	$\dots$	x_n
y^1	y_1	$y_n+\varepsilon$	$x_n-\varepsilon$	x_n	$\dots$	x_n
y^2	y_1	$y_n+2\varepsilon$	$x_n-\varepsilon$	$x_n-\varepsilon$	$\dots$	x_n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y^{n-2}	y_1	$y_n+(n-2)\varepsilon$	$x_n-\varepsilon$	$x_n-\varepsilon$	$\dots$	$x_n-\varepsilon$
z	z_1	z_n	$x_n-\varepsilon/2$	$x_n-\varepsilon/2$	$\dots$	$x_n-\varepsilon/2$
t	t_1	t_n	$x_n-\varepsilon/2$	$x_n-\varepsilon/2$	$\dots$	$x_n-\varepsilon/2$

By WP, one has $x\, P(\succeq)\, t$ and $z\, P(\succeq)\, y^{n-2}$. By PDBD, one has

$$y^{n-2}\, R(\succeq) \cdots R(\succeq)\, y^1\, R(\succeq)\, y\, R(\succeq)\, x,$$

and $t\, R(\succeq)\, z$. Hence a contradiction. ■

The first of these two incompatibilities can be viewed as a strengthening, in a particular economic environment, of a result obtained by Sen (1970a, Theorem 9*2) and Suzumura (1983, Theorem 6.3). Sen proves that Suppes' grading principle may contradict the weak Pareto principle in a universal domain. This principle relies on extended sympathy and is specific to each individual. According to person i, x is more just than y if there is a permutation of the set of individuals such that i in agreement with his own preferences prefers to be in the position of each person in x than to be in the position of the corresponding person in y. The natural order of vectors in $\mathbb{R}^{\ell}_{++}$ leads to an admissible grading principle. Suzumura strengthens Sen's negative result by showing that there exists no SQOF, defined on a universal domain, satisfying the weak Pareto principle and the justice unanimity principle. A social state x is more just than y according to this principle if each individual agrees according to his own grading principle of justice.

By virtue of the logical relationships between the axioms, there is also an incompatibility between WP and either ROD or GLD or SPDBD. It is also worth emphasizing that these incompatibilities occur even if one wants only to rank allocations in a neighborhood (where preferences differ), so that, for instance, the objective of formulating a purely local social goal for reform purposes would not be more easily achieved.

When two decision-making criteria are mutually inconsistent, one may give priority to one of them and use the second one only for breaking ties. This is the suggestion made by Tadenuma (2002) with the Pareto criterion and the equity-as-no-envy-criterion. Tadenuma formalizes two principles. In the *efficiency-first* criterion, priority is given to efficiency. An allocation x is ranked higher than an allocation y iff (i) x is Pareto-superior to y or (ii) x and y are Pareto-noncomparable and x is equity-superior to y. In the *equity-first* criterion, it is the opposite. We could think of applying these ideas with WP as the efficiency criterion and PDBD as the equity criterion. But the above proof shows that both the efficiency-first criterion and the equity-first criterion would always exhibit a cycle of length $n+2$, and therefore do not define proper social quasi-orderings.

We now limit our ambition to building a quasi-ordering on the Edgeworth hyperbox X. Indeed, an impossibility result for the entire allocation space does not imply that the impossibility holds when only the feasible set has to be socially ordered. For instance, Bordes et al. (1995) and Redekop (1991) have already shown the relevance of this remark concerning Arrow's impossibility theorem. Here it has to be noted that the incompatibility between WP and SD is quite strong since the latter axiom makes all permutations socially indifferent.

Proposition 6. *Let X be the space of social states and D the domain of profiles of preferences on $\mathbb{R}^{\ell}_{+}$. Then there exists no SQOF satisfying both WP and SD.*

Proof. It is easy to find a profile, and an inefficient allocation in X, such that by a permutation of bundles all agents are better off. ■

The picture is more intricate for the PDBD axiom.

Proposition 7. *Let $n = 2$, X be the space of social states and D the domain of profiles of preferences on $\mathbb{R}^{\ell}_{+}$. Then there exists a SQOF satisfying both SP and PDBD.*

Proof. Let R be defined by the transitive closure of SP and PDBD. If $n = 2$, the successive application of SP and PDBD cannot generate a cycle, because neither a Pareto improvement nor a Pigou-Dalton transfer of bundles can reverse the order of domination of bundles under the resource constraint $x_1 + x_2 = \omega$. ■

Proposition 8. *Let $n \geq 3$, X be the space of social states and D the domain of profile of preferences. There exists no SQOF satisfying both WP and PDBD.*

Proof. We provide the proof for $n = 3$. It is tedious but easy to extend it to any $n > 3$.

Let allocations x, y, z, t, r be defined by:

$$x = \begin{pmatrix} .1\omega_1 & .3\omega_2 & \dots & .3\omega_\ell \\ .13\omega_1 & .33\omega_2 & \dots & .33\omega_\ell \\ .77\omega_1 & .37\omega_2 & \dots & .37\omega_\ell \end{pmatrix}, \; y = \begin{pmatrix} .11\omega_1 & .31\omega_2 & \dots & .31\omega_\ell \\ .12\omega_1 & .32\omega_2 & \dots & .32\omega_\ell \\ .77\omega_1 & .37\omega_2 & \dots & .37\omega_\ell \end{pmatrix},$$

$$z = \begin{pmatrix} .26\omega_1 & .16\omega_2 & \dots & .16\omega_\ell \\ .2\omega_1 & .1\omega_2 & \dots & .1\omega_\ell \\ .54\omega_1 & .74\omega_2 & \dots & .74\omega_\ell \end{pmatrix}, \; t = \begin{pmatrix} .24\omega_1 & .14\omega_2 & \dots & .14\omega_\ell \\ .22\omega_1 & .12\omega_2 & \dots & .12\omega_\ell \\ .54\omega_1 & .74\omega_2 & \dots & .74\omega_\ell \end{pmatrix},$$

$$r = \begin{pmatrix} .25\omega_1 & .15\omega_2 & \dots & .15\omega_\ell \\ .22\omega_1 & .12\omega_2 & \dots & .12\omega_\ell \\ .53\omega_1 & .73\omega_2 & \dots & .73\omega_\ell \end{pmatrix}.$$

Notice that

$$x_2 \gg y_2 \gg y_1 \gg x_1 \text{ and } x_1 + x_2 = y_1 + y_2,$$
$$z_1 \gg t_1 \gg t_2 \gg z_2 \text{ and } z_1 + z_2 = t_1 + t_2,$$
$$t_3 \gg r_3 \gg r_1 \gg t_1 \text{ and } t_1 + t_3 = r_1 + r_3,$$

and that one can find a profile of preferences $\succeq$ such that for all $i \in \{1, 2, 3\}$, $x_i \succ_i r_i$ and $z_i \succ_i y_i$.

By WP, one has $x\,P(\succeq)r$ and $z\,P(\succeq)y$. And by PDBD, $y\,R(\succeq)x$, $t\,R(\succeq)z$ and $r\,R(\succeq)t$. This yields a cycle. ■

By virtue of the first remark in this section, there is also an incompatibility between WP and SPDBD or GLD. Trying to overcome the dilemma through a priority given to WP or PDBD is no more promising here than for the case when $\mathbb{R}_+^{n\ell}$ is the set of social states.

4 Equity criteria

In view of the previous results, it is hopeless to investigate full rankings of allocations satisfying both Pareto and dominance requirements. In this section we show that this does not preclude other ways of relying on the equity notions under consideration. In particular, one might argue that PDBD can still be useful to rank Pareto-efficient allocations[3]. More specifically, we exploit the connection between the PDBD axiom and the no-domination equity criterion.

Definition 10. *An allocation x satisfies no-domination if*: $\nexists\, h, k \in N$, $x_h > x_k$.

The link with our dominance criteria is given by the following result, stating that, in the feasible set, maximal elements for the PDB quasi-ordering are precisely the no-domination allocations.

Proposition 9. *The allocation* $x \in X$ *satisfies no-domination if and only if*: $\nexists\, y \in X$, $y \succeq_{PDB} x$.

The straightforward proof is omitted.

For any economy with resources ω and profile $\succeq$, let $PND(\succeq, \omega)$ denote the set of Pareto-efficient allocations satisfying no-domination. This defines *PND* as a correspondence. One can then consider that it is a good property for a SQOF to have allocations in *PND* as maximal elements. Formally:

[3] An alternative route, more favorable to equity considerations, would be to resort to WP to rank the set of allocations not dominated according to PDBD. We have not explored this way since, from the very beginning, our aim was to complement the Weak Pareto principle with some basic equity considerations.

Definition 11. *A SQOF R rationalizes the PND correspondence for* $\omega \in \mathbb{R}^{\ell}_{++}$ *if*: $\forall \succeq \in D, \forall x, y \in X$,

$$x \in PND(\succeq, \omega) \Rightarrow x\,R(\succeq)\,y,$$
$$x \in PND(\succeq, \omega) \text{ and } y \notin PND(\succeq, \omega) \Rightarrow x\,P(\succeq)\,y.$$

It is clear that if a SQOF is complete, has only Pareto-efficient maximal elements, and satisfies PDBD, then it rationalizes *PND*, and this holds for any $\omega \in \mathbb{R}^{\ell}_{++}$. Unfortunately, we have seen that PDBD is incompatible with Weak Pareto, and similarly one has the following.

Proposition 10. *No SQOF satisfies Weak Pareto and rationalizes PND for all* $\omega \in \mathbb{R}^{\ell}_{++}$.

Proof. Choose $\omega, \omega' \in \mathbb{R}^{\ell}_{++}$ such that $\omega \neq \omega'$. Choose allocations x, y, z, t such that $\sum_{i \in N} x_i = \sum_{i \in N} y_i = \omega$, $\sum_{i \in N} z_i = \sum_{i \in N} t_i = \omega'$, and both x and t satisfy no-domination. And choose differentiable preferences such that for all $i \in N$, $z_i \succ_i x_i$ and $y_i \succ_i t_i$ and marginal rates of substitution are equal across agents in x and t but not in y and z. By WP, the SQOF R must be such that $z\,P(\succeq)\,x$ and $y\,P(\succeq)\,t$. But since R rationalizes PND for ω and ω', one must have $x\,P(\succeq)y$ and $t\,P(\succeq)z$. ■

Notice that the result would still hold with a weaker notion of rationalization that would only retain any one of the two conditions in the above definition.

In view of the basic difficulty to rationalize PND for all economies, it is somewhat reassuring to notice that one can at least rationalize it in the restricted context of resources belonging to a given ray. Let $\omega_0 \in \mathbb{R}^{\ell}_{++}$ be a given vector of resources. And consider the following SQOF, which ranks all allocations of $\mathbb{R}^{n\ell}_{++}$:[4]

Definition 12. $x\,R(\succeq)\,y$ *if*: *either* $V(x) > V(y)$, *or* $V(x) = V(y)$ *and card* $\{(h,k) \in N^2 | x_k > x_h\} \leq card\{(h,k) \in N^2 | y_k > y_h\}$,[5] with

$$V(x) = \sup\{\lambda \in \mathbb{R}_+ \mid \exists z \in PND(\succeq, \lambda\omega_0), \forall i \in N,\ x_i \succeq_i z_i\}.$$

where card $\{.\}$ *is the cardinality of the set* $\{.\}$.

[4] This definition is an application of a general method proposed by Fleurbaey and Maniquet (1997) in order to rationalize any correspondence. They actually propose two methods, which are both generalizations of Debreu's coefficient of resource utilization.

[5] This definition introduces the idea of counting the number of domination relations for an allocation. Then comes to mind the following SQOF. Let say that x is said to weakly cardinal-dominate y if $\forall h, k \in N^2$ such that $x_h > x_k$ then $y_h > y_k$. It can be shown that requiring to a SQOF to exhibit compatibility with the above requirement enters in conflict with Pareto principles along the same arguments developed in section 3.

The SQOF is based on a real-value function computing what could be called the "value" of an allocation. The value V attached to some allocation x is given by a scale factor λ applied to the resources of the economy. This scale factor is related to the resources needed to support a "cousin" allocation z which belongs to PND and is weakly worse than x for all agents. Notice that if $x \in PND(\succeq, \lambda\omega_0)$ for some λ, then $V(x) = \lambda$, since x is its own "cousin" allocation.

As can be easily checked, this SQOF satisfies Weak Pareto (but not Strong Pareto because rationalizing *PND* is incompatible with Pareto-indifference) and rationalizes *PND* in all economies with ω proportional to ω_0.[6]

This restricted form of the rationalizibility property can be viewed as an alternative to the PDBD axiom. More precisely, it is intuitive that when two allocations are feasible under the same resources ω (proportional to ω_0), and one is close to the selection made by *PND* for this economy whereas the other is far from it, either because of inefficiency or because of substantial domination, then the former will have a higher value V. The same holds for any pair of allocations which are Pareto-indifferent to these two allocations, since V is invariant for Pareto-indifferent allocations. Compared to the PDBD axiom, this notion of equity introduces a substantial amount of Pareto-indifference, and also the reference to the ray of ω. This seems to be the price to pay in order to satisfy the Paretian requirements.

In conclusion, it may be worth noticing that a similar analysis can be made with the no-envy criterion.

Definition 13. *An allocation x satisfies no-envy if*: $\nexists\, h, k \in N,\ x_h \succ_k x_k$.

The equivalent to PDBD would be something like the following condition:

Axiom 8. $x\ R(\succeq)\ y$ *whenever* $\exists h, k$ *such that 1)* $\forall i \neq h, k\ x_i = y_i$; *2)* $y_h R_h x_h R_h x_k$, $x_h R_k x_k P_k y_k$, $y_h R_h y_k$, $y_h P_k y_k$; *3)* $x_h + x_k = y_h + y_k$.

This axiom says that a transfer from h to k, or an exchange between them, is socially good if k envies h, h does not envy k, and the change is good for k and not for h. This axiom is stronger than PDBD, and therefore is also incompatible with Paretian conditions. Similarly, any maximal allocation in X for the SQOF implicitly defined in this axiom will satisfy the no-envy criterion, or at least display only symmetrical envy relations (h envies k, and k envies h).[7] And again this suggests the alternative route of defining a SQOF

[6] See Fleurbaey and Maniquet (1997) for a discussion of other properties satisfied by this kind of SQOF.

[7] Under symmetrical envy, a permutation of bundles is an improvement for the two involved agents. Therefore any Paretian SQOF will eliminate symmetrical envy relations.

which rationalizes the efficient no-envy correspondence for all resources on a given ray, similarly as above.

5 Concluding remarks

The starting point of this work was the idea that it would be convenient if, in some cases, social judgments could be based on simple comparisons of bundles, independently of individual preferences. It is well known that in the one-dimensional case of distributions of income, a lot can be said on the comparison of distributions without knowing precisely the agents' utility functions. The case of multiple goods is more complex, but, for instance, it is tempting to say that there is social improvement when all bundles are increased, possibly with some reshuffling among agents, or when dominations between bundles are reduced by transfers. We have obtained here a combination of positive and negative results.

When all agents are assumed to have the same preferences, it is indeed possible to rely on such criteria bearing directly on bundles, and results derived from the literature on Lorenz dominance show the link between these criteria and additive social welfare functions. Unfortunately when agents are allowed to have different preferences, then it is essentially impossible, in most cases, to rely on the bundle criteria if one wants to satisfy the Pareto principle. This difficulty is not too amazing, and reminds us of other famous examples where non-welfarist judgments prove to be incompatible with the Pareto principle (e.g., Sen 1970 b). In some sense, our negative results can also be viewed as providing another illustration of the trade-off between efficiency and equity.

This kind of negative results may warrant reservations about the use of multidimensional inequality tools. It seems clear that dominance analysis will not be highly recommended when two conditions are met: 1) some information about preferences is available, and 2) it is considered ethically acceptable to use it. If either the former or the latter condition is not fulfilled, then there is still room for multidimensional dominance analysis. It is important to distinguish between these two types of potential applications. As an example of the former, there are many goods which are private goods but for which information about preferences is not truly reliable. For many reasons including distributional ones, a market has not been organized for post-graduate education in many countries. In these countries, we have no information about preferences and we cannot perform the efficiency test about the allocation of resources between post-graduate education and other goods. As an example of the latter kind of application, there is a black market for kidneys in some developing countries. According to standard microeconomic theory, it means that we have some information about the marginal willingness to sell a kidney from the poorest part of the population and about the marginal willingness to buy one from the richest part of the population. In accordance with common wisdom, it is dubious that we would like to take into account this information about preferences in any welfare analysis.

Furthermore, it has been shown that there is a close connection between one of the bundle dominance criteria (Pigou-Dalton bundle dominance) and the equity condition of no-domination. Suppose now that we are in a context where the Pareto principle is relevant. A way to construct social preferences which embody the no-domination criterion and the Pareto requirements has been suggested. Unfortunately, the latter requirements loosen the link between the domination relations between particular bundles in a particular allocation and the social value of that allocation.

This raises the question of whether the social preferences proposed in the last section in order to rationalize the Pareto non dominated correspondence do satisfy equity properties related to the no-domination idea, apart from the rationalization itself. Or if other social preferences exist which would satisfy such properties. From our results, it seems difficult to define these new equity properties, because they would have to take account of preferences in a clever way in order to guarantee compatibility with Pareto requirements.

Another way out would consist in weakening the Pareto conditions. But one may conjecture that as soon as the weakened Pareto condition is not merely based on comparisons of bundles dominating each other (which would make it a consequence of Suppes dominance) but allows for comparisons of non-dominating bundles, the incompatibility would arise again.

Appendix

We prove here the following lemma, which is used in the proof of Proposition 5.

Lemma 1. *When two agents i, i' have different preferences, it is possible to find bundles $x_i, x_{i'}, y_i, y_{i'}, z_i, z_{i'}, t_i, t_{i'}$ in $\mathbb{R}^{\ell}_{+}$ such that $x_i \succ_i t_i$, $z_i \succ_i y_i$, $x_{i'} \succ_{i'} t_{i'}$, $z_{i'} \succ_{i'} y_{i'}$, with $x_{i'} \gg y_{i'} \gg y_i \gg x_i$ and $z_i \gg t_i \gg t_{i'} \gg z_{i'}$, and $x_i + x_{i'} = y_i + y_{i'}$ and $z_i + z_{i'} = t_i + t_{i'}$.*

Proof. One can select bundles in a given hyperplane, so that one can work in $\mathbb{R}^2_+$. By monotonicity and convexity of preferences, the indifference curves in $\mathbb{R}^2_+$ are almost everywhere differentiable. There must be a bundle $a \in \mathbb{R}^2_{++}$ with marginal rates of substitution $0 < s_i < s_{i'}$, respectively. Select a sequence of bundles $b^t, c^t, d^t, e^t, f^t, g^t$ such that $\lim_{t\to\infty} b^t = a$, and for all t, $b^t_1 < a_1$, $b^t \sim_i a, c^t_2 = b^t_2$, $c^t \sim_{i'} a, d^t = (b^t + c^t)/2$, $e^t \sim_i d^t$, $f^t \sim_{i'} d^t$, $g^t_1 = e^t_1 = f^t_1$, $g^t = 2a - f^t$. One can see that $\lim_{t\to\infty} c^t = \lim_{t\to\infty} d^t = \lim_{t\to\infty} e^t = \lim_{t\to\infty} f^t = \lim_{t\to\infty} g^t = a$. More importantly, one also has by convexity of preferences:

$$\lim_{t\to\infty} \frac{b_2^t - a_2}{a_1 - b_1^t} = \lim_{t\to\infty} \frac{d_2^t - e_2^t}{e_1^t - d_1^t} = s_i,$$

$$\lim_{t\to\infty} \frac{c_2^t - a_2}{a_1 - c_1^t} = \lim_{t\to\infty} \frac{d_2^t - f_2^t}{f_1^t - d_1^t} = s_{i'}.$$

As a consequence, one computes

$$\lim_{t\to\infty} \frac{d_2^t - g_2^t}{g_1^t - d_1^t} = \frac{s_i}{s_i + s_{i'}}(3s_{i'} - s_i).$$

Since $\frac{s_i}{s_i+s_{i'}}(3s_{i'} - s_i) < s_{i'}$, one has $g^t \succ_{i'} f^t$ for t high enough. Fix this t.

Now, take $x_{i'} = c^t$, $x_i = b^t$, $y_{i'} = d^t + (\varepsilon, 0)$, $y_i = d^t - (\varepsilon, 0)$, $z_i = e^t$, $z_{i'} = g^t - (0, 3\varepsilon)$, $t_i = a - (0, \varepsilon)$, $t_{i'} = a - (0, 2\varepsilon)$. By construction, one has $x_i \succ_i t_i$, $z_i \succ_i y_i$, $x_{i'} \succ_{i'} t_{i'}$, and for ε small enough, one also has $z_{i'} \succ_{i'} y_{i'}$. And by construction, $x_i + x_{i'} = y_i + y_{i'}$ and $z_i + z_{i'} = t_i + t_{i'}$. ■

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