Optimal Student Loans and Graduate Tax under Moral Hazard and Adverse Selection[*]

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Abstract

We characterize the set of second-best "menus" of student-loan contracts in an economy with risky labor-market outcomes, adverse selection, moral hazard and risk aversion. We combine student loans with optimal income taxation. Second-best optima provide incomplete insurance because of moral hazard. Optimal repayments must be income-contingent, or the income tax must comprise a graduate tax. Individuals are ex ante unequal because of differing probabilities of success, and ex post unequal, because taxation trades off incentives and redistribution. In addition, second-best optima exhibit an interim equalization property: the post-study but pre-work expected utilities of newly graduated student types must be equal. KEYWORDS: Student Loans, Graduate Tax, Adverse Selection, Moral Hazard, Risk Aversion.

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1   Introduction

The importance of student loans for the accumulation of human capital, economic growth and welfare cannot be overestimated. In the United States, the total amount of outstanding student debt has reached $1 trillion at the end of 2011 (see Avery and Turner, 2012). In Great Britain, the rise of tuition fees seems to have caused a sharp increase in average student debt[1]. Student loans pose interesting financial engineering and regulation problems (see, *e.g.*, Palacios, 2004). There are many discussions on the optimal design of these loans: for instance, the UK and Australia have a form a income-contingent repayment system, since loan repayments are based on the graduate's earnings, just like income tax, and interest rates are subsidized[2]. In many continental European countries, student loans play a negligible role but, given the severe shortage of public funds, they could go hand in hand with a substantial raise in tuition fees, and become a new source of funds for universities (see, *e.g.*, Jacobs and Van der Ploeg, 2006).

There is an important econometric literature on the impact of credit constraints on university or college attendance. See, for instance, Carneiro and Heckman (2002), Keane and Wolpin (2001), Stinebrickner and Stinebrickner (2008). For a survey, see Lochner and Monge-Naranjo (2012). For recent quantitative studies of alternative student-loan policies in the US, see, *e.g.*, Ionescu (2009), Lochner and Monge-Naranjo (2011). These questions are hotly debated, yet, to the best of our knowledge, the micro-economic theory of student loans is still underdeveloped[3]. In the following, we propose a simple model of student loans, under the combined effects of risk aversion, moral hazard and adverse selection. We explore the structure of the set of second-best optimal allocations of credit to risk-averse students in an economy in which individual talents and efforts are not observed by the lender, future earnings are subject to risk and incomes can be taxed. The student-loan problem is compounded with an elementary form of the Mirrleesian optimal income-tax problem.

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[1] The average student debt is predicted to be around 50,000 pounds, on leaving the university, for those starting in 2012 (for details, see *http://www.slc.co.uk/statistics*). The number of students experiencing difficulties to repay their loans reached 8.8% in the United States, after the economic downturn of 2008.

[2] On these questions, see, *e.g.*, Chapman (1997), Chapman (2006), Johnston and Barr (2013). New Zealand and Sweden are also providing income-contingent student loans.

[3] See our discussion of the literature below.

We consider an economy with unobservable, two-dimensional individual *types*. Students are characterized by an *ex ante* type and an *ex post* type. The ex ante type represents cognitive (and other) skills that influence academic success. In contrast, the ex post type plays a role once the student finds a job, and can be interpreted as representing some job-market skills, or job-market opportunities that are random, but correlated with the quality of education[4]. Students choose the length (or the quality) of their studies in a "menu"; they know their ex ante type before the choice of how much to invest in education. In contrast, the ex post type, which is private information too, is revealed later, after graduation.

Students exert two kinds of effort. While attending college, students exert some *study effort* that is not observable. This determines an *ex ante moral hazard* problem. Ex ante effort (or study effort) influences the probability of graduation. Students can be either *successful*, or *fail* to obtain a certificate. The distribution of individual earnings is also affected by unobservable effort exerted at work. Effort at work may also be a called *ex post* effort. The government does not observe ex post effort but observes education, success and individual earnings. A successful student with high ex post skills can choose to behave like a student with low ex post skills and save on ex post effort costs. This is the source of an *ex post* moral hazard problem, with the consequence that the most productive students can decide to earn less if tax rates are too high.

The student's success, education and ex post type jointly determine her potential wage. There are two sources of risk: random academic success and the drawing of ex post types. Thus, individual wages are risky. Because of risk aversion, students would like to be insured against these random shocks.

We characterize the set of second-best Pareto-optima of this model. These optima can be implemented by a combination of structured loans and income taxes.

Our main results are the following. The optimal menu of contracts exhibits incomplete insurance: this is mainly due to moral hazard. Students obtain the maximal amount of income insurance compatible with the exertion of high study effort among the less-talented. As a consequence, the talented ex ante types bear more risk than strictly necessary to ensure that they choose a high level of study effort. The planner cannot reduce the risk borne by

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[4]They may also be correlated to the ex ante type, without changing the main results.

the talented, because that would violate the self-selection constraints.

As a by-product, we find that second-best optimal loan repayments are always income-contingent, even in the presence of an income tax. More precisely, we find a *non-decomposition property*: the second-best cannot be implemented by means of the sum of a loan repayment, that depends only on the observed years of education, and an income tax, that does not depend on education. It must be that, either the income tax depends on education, or the loan repayments depend on income. Since the budget is by construction balanced (we did not explore subsidies that would be financed by means of external sources of funds), the second-best optima typically exhibit cross-subsidies between types: the talented repay more and subsidize the less-talented, but the former are not fully exploited. The second-best solution can be interpreted either as an income-contingent loan, or as a progressive *graduate tax*.

In the standard utilitarian case, we find that the second-best optimal menu of loan contracts exhibits a form of pooling, called *equal treatment*: the students' expected utilities, net of repayments, must be equal, *conditional* on random academic success. In other words, the post-study but pre-work expected utility of net earnings should be independent of the student's unobservable ex ante type. This property can also be defined as *interim* equalization, because equalization takes place, in conditional expected utility terms, just after graduation and before the revelation of job market skills. Under interim equalization, all succesful students leave school with equal expected utilities, but their net wages will not be equal. This means that the most talented students will be exactly compensated for ex post efforts. In spite of being treated equally in this particular sense, students are *ex post* unequal, since the optimal income tax trades off the provision of insurance (and redistribution) against that of incentives: the monetary rewards for ex post efforts guarantee that the talented are never fully exploited. In addition, students are *ex ante* unequal in utility terms, since the talented types have a greater probability of success. As a consequence, the talented types are happier on average, and, to the extent that they work hard, get a higher net compensation than the less talented ones.

The equal treatment property results from the interplay of self-selection and effort incentives constraints: we show that it solves the screening problem in the presence of moral

hazard. In a nutshell, this equal treatment property is the only way to satisfy two weak inequalities that pull in opposite directions. On the one hand, without an adverse selection problem, each type of student would be subjected to the smallest amount of risk needed to guarantee high effort. This takes the form of a gap between the net income earned in case of success and that earned in case of failure. The appropriate gap between the utilities of success and failure must be larger for less-talented types of individual, that is, a relatively smaller amount of risk is enough to induce high effort from the talented types. On the other hand, self-selection constraints require a larger gap between success and failure when the student is high-talented, that is, the opposite of what moral hazard requires. Hence, the optimal difference between success and failure is the same for each type. Equal differences in turn imply the equal treatment property, because under equal differences, self-selection constraints also require the utility to be the same, across types, in case of failure. To sum up, interim equalization is the best way of providing insurance to good students, compatible with the revelation of academic talent, and given that complete insurance against failure is precluded.

Another contribution of the present article is to provide a characterization of the set of second-best optima in an economy with wage risk, risk aversion, moral hazard and adverse selection. In other words, we provide a study of the structure of second-best optima in a screening model with the added complication of moral hazard. The equal treatment result described above is not the textbook allocation of insurance under adverse selection, *à la* Rothschild-Stiglitz.[5] To obtain the familiar separating menu of loan contracts as a second-best optimum, we show that the weight of the talented types must exceed their natural frequency in the social planner's objective function, which is not the common assumption in public finance. The complete characterization result is presented in Section 6.

Our theory of student loans combines four main ingredients: risk aversion, ex ante moral hazard, a screening problem and ex post moral hazard. To be posed in a meaningful way, the student-loan problem should embody these ingredients, and none of them can be removed without losing an important aspect of our results. *(i)* Without risk aversion, there is no demand for insurance and redistribution. *(ii)* Absent the ex ante moral hazard problem,

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[5]See Rothschild and Stiglitz (1976), Wilson (1977), Miyazaki (1977), Spence (1978) and Hellwig (1987).

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students may be perfectly insured against the consequences of academic failure. It would not be reasonable to assume that study effort is ineffective (or that it can be perfectly monitored). *(iii)* If ex ante types are observable, the self-selection problem is replaced with an education planning problem. In other words, when ex ante self-selection constraints are removed, the social planner will force all talented students to study longer. We model a society in which, to a certain extent, individuals choose their individual investment in education, and, to predict a person's career, it matters a great deal, for instance, if this person went to college, and if so, if she completed two or four years, etc. This is why we should consider a menu of educational choices. *(iv)* In the absence of an ex post moral hazard problem, the optimal solution would exhibit an unpalatable form of "exploitation of the talented": the marginal rate of income taxation could reach 100%. The specific impact of the "ingredients", working together or in isolation, is studied in Section 5.

Student loans are a very natural application for the theory of incentives or Mechanism Design under hidden actions and hidden types[6]. The general theory of optimum (or equilibrium) contracts under moral hazard, adverse selection *and* risk aversion is known to be a very hard problem (see Arnott (1991) for comments and further references to unpublished essays on this question; see also the recent synthesis of Boadway and Sato (2012) on optimal taxation with uncertain earnings). Solutions can be exhibited when principal and agent are both risk-neutral[7]. An extension of Rothschild and Stiglitz's insurance market model to moral hazard is proposed in the often quoted, but unpublished manuscript of Chassagnon and Chiappori (1997). Our model is close to that of the latter contribution, but Chassagnon and Chiappori did not study the set of Pareto-optima[8].

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[6]It is well-known that microeconomic models of insurance and models of banking are formally close. Rothschild and Stiglitz's approach to screening in insurance markets has been applied to banking, albeit with adaptations (see, *e.g.*, Bester (1985)). Classic theories of credit contracts typically treat adverse selection and moral hazard separately (see Freixas and Rochet (1998)). The structure of second-best optima in insurance markets with pure adverse selection has been studied by Crocker and Snow (1985) and Henriet and Rochet (1990).

[7]See, *e.g.*, Picard (1987) and Caillaud, Guesnerie and Rey (1992); see also the discussion in Laffont and Martimort (2002, chapter 7). In the field of optimal regulation theory, a few contributions have dealt with special cases; see, McAfee and McMillan (1986), Baron and Besanko (1987), Laffont and Rochet (1998).

[8]Recent work on the Principal-Agent model in the case at hand required advanced mathematical optimisation techniques (see Faynzilberg and Kumar (2000)) or used stochastic calculus, as in the asset-pricing, continuous-time finance literature (see, *e.g.*, Sung Jaeyoung (2005)). These intimidating technicalities mainly

The question of student loans is certainly not new in economics; see Friedman and Kuznets (1945), Friedman (1955), Shell *et al.* (1968), Nerlove (1975), Krueger and Bowen (1993). A few papers have recently proposed a theoretical study of income-contingent loans, or of the graduate tax, comparing alternative ways of subsidizing higher education, and performed numerical explorations of their welfare properties. See, *e.g.*, Garcia-Penalosa and Wälde (2000), Hanushek *et al.* (2004), Del Rey and Racionero (2010). A much smaller number of papers come to grips with asymmetric information. De Fraja (2001) studies the interaction of unobservable skills and risk aversion in the allocation of higher education. Cigno and Luporini (2009) consider a microeconomic model of student loans with pure moral hazard. Chatterjee and Ionescu (2012) propose a quantitative analysis of a model of college loans with moral hazard, in which insurance is provided in the form of partial loan forgiveness in the event of student failure, but they do not rely on Mechanism Design techniques as we do here.

Some contributions were devoted to the allocation of education in an optimal income-taxation model. This question was initially studied in static settings, see, *e.g.*, Arrow (1971), Hare and Ulph (1979), Tuomala (1986), Fleurbaey *et al.* (2002), De Fraja (2002), Bovenberg and Jacobs (2005). These papers mainly studied the question of optimal education subsidies, in cash or in kind. In the past decade, the New Dynamic Public Finance literature analyzed multiperiod extensions of Mirrlees' (1971) optimal taxation model, in which the individuals' unobserved productivity is exogenous and varies with time (see, *e.g.*, the surveys of Golosov *et al.*, 2006, Kocherlakota, 2010), but a few articles considered human capital (and hence, endogenous productivity) in this framework. Bohacek and Kapicka (2008) studied second-best allocations in an economy with private information and education and showed that human capital investments should be subsidized (in other terms, a positive human-capital *wedge* is second-best optimal) while marginal income taxes are positive. As a general principle, the second-best allocation can be implemented by a tax system where taxes in each period depend on the entire past history of incomes and human capital levels. In the same type of model, Anderberg (2009) and Grochulski and Piskorski (2010) study how human capital

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explain why we study a simple textbook model here, but it conveys, we think, the essential intuitions and ideas (and yet, some of the proofs are not straightforward).

changes the risk properties of future labor income at the optimum, and the consequences in terms of optimal taxation. Education may increase or reduce wage risk in the future. Human capital investment therefore interacts with the insurance and redistribution problems posed by unobserved stochastic productivities during the individuals' life-cycles[9]. The literature on this topic is currently burgeoning. Closer to the present contribution, Findeisen and Sachs (2012), Stantcheva (2014) and Koeniger and Prat (2014) explicitly studied the combination of an income-tax with income-contingent loans as a way of implementing second-best allocations in an optimal taxation model with human capital; they use a more complicated model than us, with a continuum of types, and have recourse to numerical simulations. There are many other differences in our ways of modeling the problem, but they reach similar conclusions about the usefulness of income-contingent reimbursement. None of these papers consider the combination of privately known types and genuine moral hazard (unobservable learning effort) that we study in the present article.

In the following, Section 2 describes the model and studies first-best optima. Section 3 is devoted to a preliminary analysis of asymmetric information and incentive constraints. Section 4 presents the main results, characterizing the second-best optimum and discusses implementation by means of income-contingent student loans, the graduate tax, and the income tax. Section 5 studies what happens if some ingredients are removed in turn from the theory. Section 6 provides a complete characterization of the set of second-best optima. Section 7 gathers concluding remarks. The proofs are in the appendix.

2   A Simple Model

Basic Assumptions

We consider a population of students with the same von Neumann-Morgenstern utility $u(.)$. Students have two-dimensional unobservable characteristics: an *ex ante* type, denoted i and an *ex post* type, denoted k. To fix ideas, the reader may interpret the *ex ante* type as representing the individual's *cognitive skills*, while the *ex post* type represents *job-market skills and*

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[9]The empirical aspect of this question is studied in, *e.g.*, Brodaty *et al.* (2014).

opportunities. Both factors influence the distribution of a given individual's future wages, albeit in different manners. The important difference between the two type components i and k is the timing of information revelation. Both components are private information of individuals. The ex ante type i is known when the student decides how much to invest in education, while the ex post type is revealed only later, when the student enters the labor market. Thus, some important individual characteristics will be revealed in the future, when the student comes to grips with a real job. An individual is characterized by the pair (i, k). We assume that each component can take one of two values only, i, $k = 1, 2$. We assume that student types are independent draws from the same distribution. The frequency of the *ex ante* type i in the student population is denoted λ_i and of course, $\lambda_1 + \lambda_2 = 1$.[10]

The model has three building blocks: the education phase, the job-market phase and the loan contract. We describe each of these blocks in turn (see Fig. 1 for an illustration of the model's timing). There are three consecutive lotteries: type i, *success* or *failure* and type k. At the end, the individual chooses ex post effort.

Education

During the education phase, each student first learns her type i, then chooses a quantity (or quality) of education q in a set $\{1, 2\}$. There are long (or highly-demanding) studies, *i.e.*, $q = 2$ and short (or less demanding) studies, *i.e.*, $q = 1$.[11] The cost of education is simply γ_q for quality q; and we assume $\gamma_2 \geq \gamma_1$. The chosen education depends on the ex ante type i only, since k is unknown when the decision is made. The educational choice q is assumed observable. Let q_i be the education of (or chosen by) type i. The efficient choice of q for type i will be $q_i = i$, due to Assumption 4, presented below in sub-section 2.2. At the end of the schooling period, each student is *successful* or *fails*. To fix ideas, success may be interpreted as *graduation, i.e.,* the student is granted a degree or not (but success may have slightly different interpretations). The individuals of both types have independent probabilities of success, that depend on individual effort.

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[10] λ_i is also the marginal probability of i in the distribution of $(i, k) \in \{1, 2\}^2$.

[11] If q represents a quantity, it measures, for instance, years of education, and if it's a quality, it may be a level in a hierarchy of certificates, or the quality of the educational institution.

Each student chooses an *ex ante effort* denoted e_i. The ex ante effort level is exerted before the realization of success. To fix ideas, we can view e_i as *study effort*. For simplicity, we assume that e_i may be equal to 0 or to 1. The *study-effort* cost is $c_i e_i$ for type i, where $c_i > 0$ is a parameter. Study effort influences the probability of success in a type-dependent way. The probability of success of a student exerting ex ante effort, e_i, is denoted $\mathbf{p}_i(e_i)$, where

$$\mathbf{p}_i(e_i) = \Pr(\text{ success } \mid i \text{ and } e_i).$$

To simplify notation, define $P_i = \mathbf{p}_i(1)$, the probability of success under high ex ante effort, and $p_i = \mathbf{p}_i(0)$, the probability of success under low ex ante effort. Type 2, the "talented type", is more likely to succeed given the effort level. We assume that success is observable. This is how we model *ex ante moral hazard*.

Formally, we assume,

Assumption 1. $0 < p_i < P_i < 1$, $i = 1, 2$, and $P_2 > P_1$; $p_2 > p_1$.

To sum up, types i and efforts e_i are not observable, but the public authority observes the individual's wage w, the quality of education q_i, which is recorded, and the event of success or failure.

Job market

We now describe the job-market phase. In case of failure, the student gets a basic job with the basic wage $\underline{w}$. Success, the second chance move, will allow the student to find a job with a salary greater than the basic wage[12].

In case of success, once the student graduated from university, a third chance move determines the student's ex post type k, or job-market skills. Depending on the realization of these skills, students will be able to occupy either a *top job* or a *middle-range job*. The probability of ex post type k, given success and education q, is defined as follows,

$$\Pr(k = q \mid q \text{ , success }) = 1 - \pi,$$

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[12] The government observes success (*i.e.*, graduation). This assumption leads to some simplifications, because it allows us to reduce the number of *ex post* incentive constraints taken into consideration, again without losing anything essential in the analysis.

and obviously, $\Pr(k \neq q \mid q$, success $) = \pi$. We assume that $\pi < 1/2$. This may be interpreted as follows: with a small probability π, an individual choosing q in $\{1, 2\}$ will be endowed with ex post characteristics $k \neq q$. In other words, a successful type $i = 2$ who chose $q = 2$ "normally" meets expectations and becomes a type $(i, k) = (2, 2)$, with high realizations of both types, and can occupy a top job, but some individuals, namely, the types $(i, k) = (2, 1)$, lack the necessary characteristics, in spite of being high quality graduates. These students will find a middle-range job. In the same fashion, a successful type $i = 1$ with education $q = 1$ would "normally" become a type $(i, k) = (1, 1)$ and obtain a middle-range job, but some of these students, namely, types $(i, k) = (1, 2)$, "escape their fate" because they possess the necessary skills or characteristics to occupy the top jobs.

To sum up, the quality of education increases the probability of drawing a high job-market type $k = 2$. This assumption is reasonable. The increased probability may be due to knowledge, training, to peer effects, or constitutes a return to network building during college years. It may also be due to some complementarity between job-market skills and higher education, these skills being in a sense catalyzed by education.

The assumption about π is a simplification, because we in fact assumed that the distribution of k is independent of the distribution of i. The ex ante and ex post types will be correlated at the optimum, but only to the extent that $q = i$ is required by incentive constraints. However, this simplification is innocuous. More generally, we could have assumed that π depends on i and q, as follows: $\Pr(k \neq q \mid q$, i , success $) = \pi_{iq}$. It happens that under this more general assumption, our results would be exactly the same (this is explained below).

During the job-market phase, successful students choose *effort at work*, or *ex post effort*. The ex post effort of a type (i, k) is denoted ε_{ik}, and the ε_{ik} may be equal to 0 or to 1. For simplicity, we assume that ex post effort plays no role in case of failure. High effort at work is required from individuals occupying top jobs and low effort is the norm for middle-range jobs[13]. The wage earned by a successful individual depends on education q and effort at work ε_{ik} only. In other words, wages depend on type i through q only (academic talent is crystallized in q when $q_i = i$), and type k determines effort at work only through

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[13]The fact that zero ex post effort is the "norm" for middle-range jobs is just a convenient normalization.

its impact on disutility (as indicated below).

Let $\boldsymbol{\omega}(q, \varepsilon)$ be a function measuring the wage of a graduate of quality q when ex post effort ε is exerted. We set,

$$\boldsymbol{\omega}(q, 1) = W_q,$$
$$\boldsymbol{\omega}(q, 0) = w_q,$$

where W_q corresponds to a top job and w_q to a middle-range job, $q = 1, 2$. This means that a successful type (i, k) with education $q = i$ can earn a top salary W_q, but only at the cost of high ex post effort, $\varepsilon_{ik} = 1$; she obtains a middle-range salary w_q otherwise. The above specification is a simple way of representing the fact that, to a certain extent, a type $i = 1$ with education $q = 1$ can make up for a low expected productivity w_1 by means of high ex post effort and obtain $W_1 > w_1$. Conversely, a type $i = 2$ with education $q = 2$ can decide to exert low effort and will in that case content herself with a middle-range salary $w_2 < W_2$. To sum up, there are 5 different wage levels: the base wage $\underline{w}$, and w_1, w_2, W_1, W_2.

We now state a natural assumption on $\boldsymbol{\omega}(q, \varepsilon)$.

Assumption 2.

(a) $\boldsymbol{\omega}(q, \varepsilon) \geq \underline{w}$ for all q and ε;

(b) $\boldsymbol{\omega}(2, 0) > \boldsymbol{\omega}(1, 0)$;

(c) $\boldsymbol{\omega}(2, 1) - \boldsymbol{\omega}(2, 0) \geq \boldsymbol{\omega}(1, 1) - \boldsymbol{\omega}(1, 0) > 0$;

(d) $\boldsymbol{\omega}(1, 1) \geq \boldsymbol{\omega}(2, 0)$.

Assumption 2b guarantees $w_2 > w_1$. Assumption 2c implies $W_q > w_q$. The return on ex post effort is higher when education is longer. As a consequence of Assumptions 2b and 2c, we also have,

$$W_2 - W_1 \geq w_2 - w_1 > 0, \tag{1}$$

and by 2d, the smallest W_q is larger than the highest w_q, *i.e.*, $W_1 \geq w_2$.[14]

The disutility of *effort-at-work* is equal to $\beta_{qk}\varepsilon_{ik}$ in state (i, k). Ex post types differ in this disutility. The total cost of effort is additively separable and equal to $c_i e_i + \beta_{qk}\varepsilon_{ik}$. In

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[14]Assumption 2d is not crucial.

a nutshell, the disutility of effort at work of a type $(i, 1)$ is assumed to be huge. We choose the simplest structure for the β_{qk} terms; precisely,

$$\begin{aligned}
\beta_{22} &= \beta_{12} = b > 0, \\
\beta_{11} &= \beta_{21} = B > 0.
\end{aligned}$$

The meaning of these assumptions is that a type $(i, 2)$ incurs a disutility b to get a top job, and a type $(i, 1)$ incurs disutility $B > b$ while trying to mimic a type $k = 2$. We assume that B is very large, in such a way that no individual endowed with a low ex post type will ever try to exert high effort and earn the highest wage.[15]

The public authority does not observe k and ε_{ik}, and will not be able to infer k and ε_{ik} from the observation of wages in all cases. The tax authority will face an informational problem, because a successful person with observed education $q = q_i$ and observed wage w_q may be, either a type $(i, 1)$, or a type $(i, 2)$ that chose to behave as a type $(i, 1)$ ex post. By assumption, the latter deviation is not observable. It follows that if the government taxes the income of types $(2, 2)$ too much, these individuals will choose to occupy middle-range jobs (the same is true of types $(1, 2)$ who can also occupy top jobs). This specific structure is a way of combining a student-loan model with an elementary form of the optimal income-tax problem in a simple and relatively tractable manner, as will be seen below.

Loans

We now turn to the menu of student loans and the distribution of income. Consider a public (or publicly regulated) lending system and assume that the public lending authority distributes all loans. By assumption, a student loan covers the cost of education[16]. So, the amount of a loan to an individual choosing education q is γ_q. Students initially choose in a menu of loan contracts. Reimbursement is contingent on earnings and on the quality of

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[15] We could have assumed that b depends on q, and the results would not change.

[16] This is a restriction because we could have considered partial funding, but this would have required consideration of a third dimension of types: initial wealth or parental wealth. Under the simplifying assumption that all students have the same initial wealth, we would have then studied the possible impact of credit rationing on incentives.

education, both observed by the government. Let (R_{q1}, R_{q2}, r_q) denote the repayment profile of a loan of size γ_q. A student choosing education q must repay,

(i), r_q in case of failure (and earnings are $\underline{w}$),

(ii), R_{q1} if earnings are equal to w_q, and finally,

(iii), R_{q2}, if earnings are W_q.

There are 6 possible repayment levels. This framework creates a Mirrleesian taxation problem, because the most lucky types can decide to reduce their effort *ex post* if their income is taxed too heavily through repayments R_{22} and R_{12}.

A standard bank loan would be of the form $(R_{q1}, R_{q2}, r_q) = (R_q, R_q, 0)$ with $R_{q1} = R_{q2} = R_q = (1 + \rho)\gamma_q$, assuming that ρ denotes the interest rate and that the student does not reimburse the loan in case of failure: the ordinary bank loan is just a particular case in a much larger set of admissible contracts.

By definition, in this economy, an allocation is an array $\{(e_i, q_i, r_i, R_{ik}, \varepsilon_{ik})\}_{i,k=1,2}$. A menu of contracts is an array $\{(q, R_{q1}, R_{q2}, r_q)\}_{q=1,2}$. Given the above assumptions, we now write the resource constraint when (q, R_{q1}, R_{q2}, r_q) is chosen by type $i = q$ only, $i = 1, 2$.

(i) With probability $\mathbf{p}_i(e_i)(1 - \pi)$, the public banker collects the amount

$$X_i = \varepsilon_{ii} R_{i2} + (1 - \varepsilon_{ii}) R_{i1}$$

from type i students;

(ii), with probability $\mathbf{p}_i(e_i)\pi$, the public banker collects the amount

$$Y_i = \varepsilon_{ik} R_{i2} + (1 - \varepsilon_{ik}) R_{i1},$$

with $k \neq i$, from type i students; and

(iii), with probability $1 - \mathbf{p}_i(e_i)$, the public banker collects the amount r_i from type i students.

Using the notation just defined, the resource constraint can be written,

$$\sum_i \lambda_i \{\mathbf{p}_i(e_i)[(1 - \pi)X_i + \pi Y_i] + (1 - \mathbf{p}_i(e_i))r_i - \gamma_i\} \geq 0, \qquad (RC)$$

with $i = q$. We assume that the lending system is self-financed, but of course, it would be easy to add an external source of funds: that would simply add a constant in RC.[17]

Finally, we formally assume that students are risk averse.

Assumption 3. The utility function $u(.)$ is strictly increasing, strictly convex and continuously differentiable.

First-Best Optimality

Let V_i denote the *ex post* utility of a successful student with education $q_i = i$, when the ex post type is $k = i$. Let v_i denote the *ex post* utility of a student with education $q_i = i$, when $k \neq i$. The *interim* expected utility of a successful student who chose education $q_i = i$ is by definition,

$$U_i = \pi(v_i - \beta_{ik}\varepsilon_{ik}) + (1 - \pi)(V_i - \beta_{ii}\varepsilon_{ii}), \tag{2}$$

where

$$v_i = u[\boldsymbol{\omega}(q_i, \varepsilon_{ik}) - Y_i],$$
$$V_i = u[\boldsymbol{\omega}(q_i, \varepsilon_{ii}) - X_i].$$

with $i \neq k$. The utility of an unsuccessful type i is denoted u_i, and $u_i = u(\underline{w} - r_i)$.

The *ex ante* expected utility of a type i student, net of expected effort costs, is defined as follows,

$$\mathbf{p}_i(e_i)U_i + (1 - \mathbf{p}_i(e_i))u_i - c_ie_i, \tag{3}$$

where $i = 1, 2$.

If we assume that welfare is higher when all types i choose education $q = i$, $i = 1, 2$, the first-best *utilitarian* optimum can be obtained as a solution of the following problem,

$$\text{Maximize} \quad \sum_i \lambda_i[\mathbf{p}_i(e_i)U_i + (1 - \mathbf{p}_i(e_i))u_i - c_ie_i] \tag{4}$$

with respect to $\{(e_i, r_i, R_{ik}, \varepsilon_{ik})\}_{i,k=1,2}$, subject to the resource constraint RC, and each effort level is chosen in the set $\{0, 1\}$. In the above formulation of the first-best problem, it is

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[17]The government may collect some taxes on tuition payments γ_i, so that the appropriate definition of γ should be net of any such taxes.

understood that type i is assigned to education $q_i = i$, for all i. This assumption is justified below.

We call the solution of this problem a *standard* utilitarian optimum because the expected utility of each type is weighted by its frequency in the population. There are of course other optima that can be generated by varying the weight of type i in the social objective. These optima are studied in Section 6 below.

To determine the first-best effort vector $(e_i^*, \varepsilon_{i1}^*, \varepsilon_{i2}^*)$, we need to compute the optimal allocation of utility in $2^6 = 64$ cases, that is, consider in turn each possible vector of efforts and compare the value of welfare for each of these combinations. But the only really inter-esting case is when $(e_1^*, e_2^*) = (1, 1)$, *i.e.*, high study effort is required from both types, and $(\varepsilon_{i1}^*, \varepsilon_{i2}^*) = (0, 1)$, *i.e.*, ex post effort is high if and only if job-market skills are high. In doing so, we are discarding 63 relatively uninteresting cases. If, in a given case, effort is very costly for a student type, it is socially inefficient to exert it; society's and the student's interests are aligned, so this case can be neglected. However, the reader should keep in mind that the case singled out here has a domain of validity, in parameter space.

It can be shown that $e_i = 1$ is optimal for all i if the effort costs c_1, c_2 are small enough and if the difference $P_i - p_i$ is large enough, that is, if effort is sufficiently effective in increasing the probabilities of success. We assume that this is indeed the case. In addition, if b is not too large, while B is very large, and if W_q is sufficiently large as compared to w_q, it will always be optimal to let the types $(i, 2)$ choose high effort ex post. In other words, under these assumptions, it is socially efficient to require that all individuals endowed with $k = 2$ occupy high-pay jobs. It follows that we have,

$$
\begin{aligned}
V_2 &= u(W_2 - R_{22}), \quad v_2 = u(w_2 - R_{21}), \\
V_1 &= u(w_1 - R_{11}), \quad v_1 = u(W_1 - R_{12}).
\end{aligned}
\tag{5}
$$

Given the optimal choice of efforts, the expected wage Ew_q of a student with education q is defined as follows,

$$
\begin{aligned}
Ew_1 &= (1 - \pi)w_1 + \pi W_1, \\
Ew_2 &= (1 - \pi)W_2 + \pi w_2.
\end{aligned}
$$

The social surplus S_i is defined as follows:

$$S_i = P_i E w_i + (1 - P_i)\underline{w} - \gamma_i.$$

S_i gives the expected social benefit of education for a type i student choosing $q = i$ and the "right" amounts of effort, ex ante and ex post. We assume the following,

Assumption 4.

    (a) $S_i \geq \underline{w}$ for all i;

    (b) $P_2[Ew_2 - Ew_1] \geq \gamma_2 - \gamma_1 \geq P_1[Ew_2 - Ew_1]$.

Thus, we assume that an education of quality $q_i = i$ is profitable, on average, for type i, as compared to the no-education alternative, with value $\underline{w}$. Assumption 4a will be satisfied, in essence, if $\boldsymbol{\omega}(q, 0)$ is high enough as compared to $\underline{w}$, and if γ_q is not too large. Assumption 4b says that assigning type i to education $q = i$ generates more surplus than choosing $q \neq i$, given the probabilities of success P_i.[18] This assumption eliminates some uninteresting corner solutions.

Now, define the inverse utility function,

$$z(x) = u^{-1}(x). \tag{6}$$

By definition, $z(u)$ is the minimal amount of resources needed to provide utility u. Using $z(.)$, we obtain

$$R_{22} = W_2 - z(V_2); \quad R_{21} = w_2 - z(v_2); \qquad R_{12} = W_1 - z(v_1); \quad R_{11} = w_1 - z(V_1); \tag{7}$$

and $r_i = \underline{w} - z(u_i)$, $i = 1, 2$.

With this interpretation in mind, the expected amount of resources needed to provide expected utility $P_i U_i + (1 - P_i)u_i$ to a type i with education $q = i$ is given by the expression,

$$\mathbf{E}(z_i) = P_i[(1 - \pi)z(V_i) + \pi z(v_i)] + (1 - P_i)z(u_i). \tag{8}$$

With the help of these definitions, the first-best optimality problem can be rewritten as follows. Eliminating R_{ii}, R_{ij} and r_i from the objective and the resource constraint RC, and

---

[18]Note that $Ew_2 > Ew_1$ since under Assumption 2, $(1-\pi)W_2 + \pi w_2 > (1-\pi)W_1 + \pi w_1 > (1-\pi)w_1 + \pi W_1$.

recalling that $P_i = \mathbf{p}_i(1)$, we obtain an equivalent expression for the first-best utilitarian optimum problem,

$$\text{Maximize} \quad \sum_i \lambda_i [P_i U_i + (1 - P_i) u_i - c_i] \tag{9}$$

with respect to $(V_i, v_i, u_i)_{i=1,2}$, subject to the resource constraint,

$$\sum_i \lambda_i \{S_i - \mathbf{E}(z_i)\} \geq 0. \tag{$\overline{RC}$}$$

Given that the values of (e^*, ε^*) are now given, the cost of effort terms are just additive constants that play no role in the above maximization problem.

We can state the following result.

Proposition 1. *Under Assumptions 1-4, first-best Pareto optimality implies* full insurance, *that is, for all i,*

$$V_i^* = v_i^* = u_i^*,$$

and the standard utilitarian first-best optimum exhibits full equality, *i.e.,*

$$V_1^* = V_2^*, \quad v_1^* = v_2^* \quad \text{and} \quad u_1^* = u_2^*.$$

In addition, first-best efficiency implies that the resource constraint is binding.

Proposition 1 is a standard consequence of risk aversion. Students are fully insured against labour market risk, but also against the risk of being of type 1.

This proposition describes an extremely idealized situation in which any degree of redistribution is possible, and politically acceptable.

We can also show the following.

Corollary 1. *Under Assumptions 1-4,*

(i), there does not exist an optimum with $r_i^* \geq 0$ for all i;

(ii), if, in a first-best Pareto optimum, individuals obtain weakly more in case of success than if they were not educated, i.e., if $W_2 - R_{22}^* \geq \underline{w}$, and $w_1 - R_{11}^* \geq \underline{w}$ then, first-best efficiency implies $r_i^* \leq 0$ for all i, i.e., the students who fail receive insurance payments.

Corollary 1 shows that the public banker should also be an insurer. If we do not permit negative repayments (*i.e.*, if the banker cannot be an insurer), it can be shown that optimality implies $r_i^* = 0$: we find a *contingent reimbursement loan*, in the ordinary sense that no repayment is required in case of failure. In the analysis below, negative repayments are permitted.

3   Asymmetric Information and Incentive Constraints

Let us now study the case in which types are not observed by public authorities. By definition, second-best optimal (or *interim efficient*) allocations maximize a weighted sum of the students' expected utilities, subject to resource-feasibility and incentive-compatibility constraints. Students self-select in a menu of contracts proposed by the public authorities. The allocation determines *ex post* utility values (V_i, v_i, u_i) and a quality of education q_i for each type i. Although in principle, second-best effort levels could be different from their first-best counterparts, we now assume that second-best effort levels are equal to the first-best effort levels. More precisely, we assume that the second-best optimal study effort is high for all types, that is, $e_i = 1$ for all i, and that second-best ex post effort is high ($\varepsilon_{ik} = 1$) if and only if $k = 2$. This is at the same time a reasonable assumption and the interesting case here, given that effort variables are discrete. Other cases, in which some of the types exert zero ex ante effort could be studied in a very similar way, and they may arise under some parameter configurations. But high levels of study effort will be second-best optimal if the ratios $c_i/(P_i - p_i)$ are not too large, for otherwise, the social cost of providing incentives could be higher than the benefits of effort in terms of aggregate surplus. The social benefits of effort are clearly the increased probabilities of success and the increased productivity of agents on the labour market. In this model, the social cost of providing incentives is due to the addition of a dose of inequality and risk, imposed by incentive constraints, on top of the direct disutility of effort itself.

Incentives

We first consider the *ex post* incentives. Students know their ex ante types. They will reveal their ex ante types by choosing the quality of their education: type i students choose education $q_i = i$. This result will be obtained if the menu of loans $\{(q, R_{q1}, R_{q2}, r_q)_{q=1,2}\}$ is *ex ante* incentive compatible. In the ex post stage, the students with education level $q = 1$ and ex post type $k = 2$ will not choose a job with a middle-range salary *ex post* if and only if,

$$v_1 - b \geq V_1. \tag{ICX_1}$$

Similarly, a student with education level $q = 2$ and type $k = 2$ will not be tempted to behave as a type $k = 1$ *ex post* if and only if,

$$V_2 - b \geq v_2. \tag{ICX_2}$$

We will see below that these constraints must be binding at a second-best optimum. The other constraints, that is, $V_1 \geq v_1 - B$ and $v_2 \geq V_2 - B$ will always be satisfied since B is sufficiently large, as assumed above.

Given values of (V_i, v_i) satisfying ICX_1 and ICX_2, we now have a *generalized Principal-Agent problem* in the sense of Myerson (1982).[19] We apply the *extended revelation principle*. The constraints bearing on *ex ante* utilities and efforts are *revelation* and *obedience* constraints: the students should simultaneously self-select by choosing the right contract in the menu of loans *and* exert the right amount of effort ex ante. The *interim* expected utility (*i.e.*, the expected utility knowing success), can now be written as follows,

$$U_1 = (1 - \pi)V_1 + \pi(v_1 - b), \tag{EU_1}$$

$$U_2 = (1 - \pi)(V_2 - b) + \pi v_2. \tag{EU_2}$$

Since we assume that high ex ante effort is efficient, incentive constraints can easily be written. First, the *ex ante self-selection* constraint $\overline{IC}_i$ says that type i should not be tempted to mimic type j *ex ante* while exerting high effort,

$$P_i U_i + (1 - P_i)u_i \geq P_i U_j + (1 - P_i)u_j, \tag{$\overline{IC}_i$}$$

---

[19]On this notion, see also Laffont and Martimort (2002)

for all $i = 1, 2$ and $j \neq i$. The cost of ex ante effort being c_i on both sides, it cancels from $\overline{IC}_i$.[20] Second, the *ex ante moral hazard* constraint MH_i says that type i should prefer to exert high study effort over low study effort *ex ante*,

$$P_i U_i + (1 - P_i)u_i - c_i \geq p_i U_i + (1 - p_i)u_i \qquad (MH_i)$$

(Recall that $p_i = \mathbf{p}_i(0)$ and $P_i = \mathbf{p}_i(1)$). In addition, the incentive constraint $\underline{IC}_i$ says that type i prefers high study effort to low study effort *and* mimicking type j *ex ante*, that is,

$$P_i U_i + (1 - P_i)u_i - c_i \geq p_i U_j + (1 - p_i)u_j, \qquad (\underline{IC}_i)$$

with $j \neq i$.

By definition, the *second-best optimality problem* is the following:

$$\text{maximize} \sum_i \lambda_i [P_i U_i + (1 - P_i)u_i - c_i]$$

subject to $\overline{RC}$, $\overline{IC}_i$, MH_i, $\underline{IC}_i$, ICX_i and EU_i.

Ex Post Moral Hazard

When the second-best optimality problem is posed as above, it is immediate that the variables (V_i, v_i) appear only in ICX_1, ICX_2 and in the resource constraint $\overline{RC}$. We can therefore decompose the problem as follows. Fix the value of U_i. Then, ex post utility levels (V_i, v_i) must be chosen in such a way that they minimize the expected cost $(1 - \pi)z(V_i) + \pi z(v_i)$ subject to ICX_i, $i = 1, 2$ and constraints EU_i. This can be proved easily, using standard techniques. Intuitively, this minimization operation is just finding the least costly way of providing incentives, given the *ex post* effort-incentive constraints. In other words, the social planner should choose the minimal level of risk compatible with ICX_i. It follows that both ICX_i constraints must be binding at the second-best optimum. We can state this result formally.

Lemma 1. *The ex post incentive constraints ICX_i, $i = 1, 2$, must be binding at any second-best optimum, that is,*

$$v_1 = V_1 + b, \qquad and \qquad v_2 = V_2 - b. \qquad (10)$$

---

[20] The cost of ex ante effort could depend on q_i. For instance, we could easily handle the following more general formulation: $C_i(q, e_i) = d(q) + c_i e_i$, but this would not change the conclusions.

Given this result and constraint EU, the (V_i, v_i) variables can be completely eliminated from the welfare maximization problem. We find convenient expressions for ex ante utility, namely,

$$U_1 = V_1, \qquad and \qquad U_2 = V_2 - b. \tag{11}$$

Eliminating (V_i, v_i) from $\overline{RC}$, we obtain a modified resource constraint, in which V_i and v_i are functions of U_i, as follows.

$$V_1 = U_1, \quad V_2 = U_2 + b, \quad v_1 = U_1 + b, \quad v_2 = U_2. \tag{12}$$

As a consequence, the resource constraint and all the incentive constraints are expressed in terms of (U_i, u_i) only.

Ex Ante Moral Hazard

We now study the role played by ex ante moral hazard or study-effort incentives. It is not difficult to see that MH_i can be rewritten as, $(P_i - p_i)(U_i - u_i) \geq c_i$, or

$$U_i - u_i \geq K_i \tag{MH_i}$$

where by definition,

$$K_i = \frac{c_i}{P_i - p_i}. \tag{13}$$

Moral hazard will thus force a gap between the reward of success and that of failure. It is natural to assume that type $i = 2$ is more efficient than type $i = 1$ while exerting effort[21]. Formally, we assume the following.

Assumption 5. $\quad K_1 \geq K_2 \geq 0.$

Can we now ignore at least one of the two moral hazard constraints? The answer is yes, under $\overline{IC}$.

Lemma 2. *Under Assumption 5, if $\overline{IC}_1$, $\overline{IC}_2$ and MH_1 hold, then MH_2 is satisfied.*

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[21] $1/K_i$ measures the efficiency of effort.

Adverse Selection

We now study the implications of the *ex ante* incentive constraints. Since $\overline{IC}_i$ can be rewritten $P_i(U_i - U_j) \geq (1 - P_i)(u_j - u_i)$, we get the string of inequalities,

$$\frac{P_2}{1 - P_2}(U_2 - U_1) \geq u_1 - u_2 \geq \frac{P_1}{1 - P_1}(U_2 - U_1). \qquad (\overline{IC})$$

Using $\overline{IC}$, we then derive the following useful result.

Lemma 3. *$\overline{IC}$ constraints imply the conditions,*

$$U_2 - u_2 \geq U_1 - u_1, \qquad (D)$$

$$U_2 \geq U_1, \qquad and \qquad u_1 \geq u_2. \qquad (D')$$

(a) If $\overline{IC}_1$ and $\overline{IC}_2$ are simultaneously binding, then $U_2 = U_1$ and $u_1 = u_2$: we get equal treatment *(but not necessarily* full insurance*).*

(b) Under $\overline{IC}_1$ and $\overline{IC}_2$, we have $u_1 = u_2$ if and only if $U_2 = U_1$.

Lemma 3 says that if equal treatment does not hold, then, either $\overline{IC}_1$ or $\overline{IC}_2$ is binding, or none of them, but not both. Assumption 5 is not needed for the proof of Lemma 3. The $\underline{IC}_i$ constraints are an added difficulty, but we can in fact ignore them, as shown by Lemma 4.

Lemma 4. *Under Assumption 5,*

(a) if $\overline{IC}_i$, $i = 1, 2$ and MH_1 hold, then $\underline{IC}_1$ is satisfied.

(b) if $\overline{IC}_2$ is satisfied, and if, in addition, $\overline{IC}_1$ and MH_1 are binding, then, $\underline{IC}_2$ is satisfied.

The proofs of all Lemmas are in the appendix. Being equipped with these preliminary results, we can now solve the optimal student-loan problem under asymmetric information.

4 The Graduate Tax Problem

The second-best optimality problem described above can be called *the graduate tax problem*, since we can easily derive the optimal loan repayment schedule from the optimal solution, and

a graduate tax can be used to implement this schedule. We then inquire whether the optimal graduate tax can be decomposed as the sum of a standard loan repayment, that depends only on education, and an income tax, not depending on education. The answer to this question is no, in general. To implement the optimal graduate tax, we need a loan repayment schedule depending on education *and* income, that is, an *income-contingent repayment schedule*, or else, we need an income tax that depends also on education, *i.e.*, a *graduate tax*.

A further study of first-order conditions for optimality yields the following result, that may be called the *equal treatment* result.

Proposition 2. (Equal treatment as a second best under moral hazard and adverse selection.) *Under Assumptions 1-5, for all $\pi < 1/2$, the second-best optimal, standard utilitarian solution has the following properties:*

$$U_1 = U_2 = \mathbf{U}, \qquad u_1 = u_2 = \mathbf{u} \quad (equal \quad treatment),$$

$$\mathbf{U} = \mathbf{u} + K_1 \quad (incomplete \quad insurance),$$

$\overline{RC}$, MH_1, $\overline{IC}_1$, $\overline{IC}_2$, ICX_1 *and* ICX_2 *are all binding. If, in addition, $K_1 > K_2$, then, MH_2, $\underline{IC}_1$ and $\underline{IC}_2$ hold as strict inequalities.*

Proposition 2 shows that the second-best optimal solution exhibits *equal treatment*, in the limited sense that $U_1 = U_2$ and $u_1 = u_2$, and *incomplete insurance*, since $U_i = u_i + K_1$. The *equal treatment property* means that the post-study but pre-work expected utilities are equalized. In particular, the *interim* expected utilities U_i of both ex ante types i are the same when students are successful. They are also the same conditional on failure.

The intuition for the equal treatment result is relatively easy to convey. Suppose for a moment that the planner observes the ex ante types i. If moral hazard is the only problem in the economy, the planner should impose the minimal amount of risk compatible with high ex ante effort on each type, that is, $U_i - u_i = K_i$ for all i.[22] The talented types being more efficient, we have $K_1 > K_2$ and therefore, $U_1 - u_1 > U_2 - u_2$. This contradicts condition D, which is a consequence of $\overline{IC}$, *i.e.*, of self-selection constraints, as shown by Lemma 3. If

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[22] This is shown in Section 5 below.

we now re-introduce the $\overline{IC}$ constraints, it is intuitively clear that the best is to choose the smallest admissible $U_2 - u_2$, that is, to set "equal differences" $U_2 - u_2 = U_1 - u_1 = K_1$. Under $\overline{IC}$, equal differences in turn imply $u_1 = u_2$, as can easily be checked[23]. So, we derived the equal treatment property. This shows why equal treatment is the result of the interplay of moral hazard and adverse selection.

However, inequalities will be created ex post, when the ex post types k are drawn and revealed. Indeed, the optimal solution typically involves *ex post inequality*, since $U_1 = U_2$ implies $v_1 = V_2 > V_1 = v_2$ (as explained by Proposition 3 and Corollary 3 below).

The solution also entails *ex ante inequality.* To see this, consider the ex ante expected utility of type i, namely, $\mathbf{E}u_i = P_i U_i + (1 - P_i)u_i$. The second-best optimum entails,

$$(\mathbf{E}u_2 - c_2) - (\mathbf{E}u_1 - c_1) = (P_2 - P_1)K_1 + (c_1 - c_2) > 0,$$

since $c_2 \leq c_1$. The talented types are strictly better off ex ante. Both types obtain the same expected payment in the event of success as well as in the event of failure, but *(i)*, insurance is incomplete: $U_i - u_i = K_1$, and *(ii)*, the talented types have a higher probability of success, $P_2 > P_1$.

The optimal second-best allocation is trivially not first-best efficient, since first-best efficiency requires full insurance. The solution potentially entails a limited form of *exploitation of the talented*, by means of cross-subsidies between types, since the less-talented are also producing less surplus *per capita*. In the case described by Proposition 2, this subsidy from the talented is a price paid to solve the incentive problem. Indeed, both $\overline{IC}_i$ constraints are binding and incentive compatibility holds since $U_1 = U_2$. It is only when the social welfare function sufficiently favors the talented that the incentive problem is solved by means of *screening*, imposing a higher level of risk (and return) on the most productive agents (this is shown in Section 6 below).

As discussed in Section 2, we assumed that types k and i are independent, since the probability π of drawing the ex post type $k \neq q$ doesn't depend on the ex ante type i. The

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[23]This is because we must have $P_i(U_i - u_i) + u_i \geq P_i(U_j - u_j) + u_j$ for all $i \neq j$.

reader may wonder if the equal treatment result depends on this simplification. The answer is no. It can be shown that Proposition 2 is still true with a more general distribution of ex post types of the form

$$\Pr(k \neq q \mid q, \ i, \ \text{success}) = \pi_{iq}.$$

For instance we could have in mind that $\pi_{22} < \pi_{12}$. We could also assume that b varies with q. In such extensions of the base model, the interim expected utility of a student, given success, would more generally depend on i and q: we would need to handle interim utilities that depend separately on i and q, denoted U_{iq}, instead of simply U_q. But by definition, in this more general model, the *ex post* incentive constraints would not depend on π_{iq}. As a consequence, we would find that $U_{22} = U_{12}$ and $U_{11} = U_{21}$ when ICX_i constraints are binding. Hence, the analysis would entirely rest on two pairs of interim expected utilities (U_i, u_i) with $q = i$, where $U_{12} = U_{22} = U_2$, and $U_{21} = U_{11} = U_1$. For the same reason, the IC_i constraints, the MH_i constraints and the analysis of their interplay at the optimum would be the same as above.

Now suppose that there are three ex ante types $i = 1, 2, 3$, with the following natural ranking of key parameters,

$$P_1 < P_2 < P_3 \quad and \quad K_1 > K_2 > K_3.$$

If ex ante moral hazard constraints MH_i were all binding, we would have $U_1 - u_1 > U_2 - u_2 > U_3 - u_3$. On the other hand, $\overline{IC}_i$ constraints impose condition D, that is, $U_3 - u_3 \geq U_2 - u_2 \geq U_1 - u_1$. This suggests the existence, in parameter space, of a nonnegligible subset at least, over which the equal difference property will hold. In other words, the second best would be to set $U_3 - u_3 = U_2 - u_2 = U_1 - u_1$, and we know that equal differences imply equal treatment under IC constraints[24]. These remarks show that the interim equalization property is more robust than it seems.

To sum up, the optimal solution involves, *(i)*, ex ante inequality of ex ante types in expected utility terms, due to incomplete insurance; *(ii)*, equality of interim expected utilities conditional on success or failure (*i.e., equal treatment*), and *(iii)*, ex post inequality

---

[24]Of course, this will not always be true, particularly if there are many types that are very different from each other. Equal treatment may break down for some types, but will remain true for an interval of types.

in monetary terms, conditional on success, due to the operation of the labor market and constraints on income taxation.

We now list the *ex post* utilities associated with the solution, and derive the *optimal graduate tax schedule.* Given that $U_1 = U_2$, we have $V_1 + b = V_2$ and therefore $V_2 > V_1$ (recall that $b > 0$). Recall that ICX_i constraints impose,

$$v_1 = V_1 + b, \qquad \text{and} \qquad v_2 = V_2 - b.$$

We can therefore state the following result.

Proposition 3. *In the standard utilitarian case, the second-best optimal allocation has the following properties,*

$$
\begin{aligned}
u_1 &= u_2 = \mathbf{u}, \\
v_2 &= V_1 = \mathbf{u} + K_1, \\
v_1 &= V_2 = \mathbf{u} + K_1 + b,
\end{aligned}
$$

It follows that,

    (a) $v_1 = V_2 > V_1 = v_2$;

    (b) $V_1 > u_1$.

The proof of Proposition 3 is very easy, given Proposition 2, and the relationships between U_i, V_i and v_i that it implies. In practice, it follows from this result that the successful "self-made man", *i.e.*, type $(1, 2)$, earns the same after-tax income than the successful graduate from a top school with the necessary job-market skills, *i.e.*, type $(2, 2)$, since $v_1 = V_2$. A top-school student lacking the necessary job-market skills, *i.e.*, type $(2, 1)$, has the same after tax income than the plain student, *i.e.*, type $(1, 1)$, since $V_1 = v_2$. These consequences of the optimal graduate tax become explicit when we derive the implications of Propositions 2 and 3 in terms of income-contingent repayment (or education-contingent income tax).

The next result shows that the equal treatment property is achieved by means of an unequal *ex post* treatment of graduates.

Corollary 2. (Optimal graduate tax, repayment schedule)

The optimal graduate tax has the following properties,

      (a) $u_1 = u_2$ implies $r_1 = r_2$ (equal treatment in case of failure);

      (b) $V_2 > V_1$ implies $W_2 - w_1 > R_{22} - R_{11}$ (the talented are not fully exploited);

      (c) $v_1 = V_2$ implies $R_{22} > R_{12}$ (self-made (wo)men repay less);

      (d) $V_1 = v_2$ implies $R_{21} > R_{11}$ (top-school students lacking the job-market skills

repay more);

In the statement of Corollary 2, points *(c)* and *(d)* show that successful individuals who studied longer repay more. Thus, in case of success, past education choices matter: q_i contributes to the determination of repayments. Point *(c)* is a consequence of $v_1 = u(W_1 - R_{12}) = u(W_2 - R_{22}) = V_2$ implying $R_{22} - R_{12} = W_2 - W_1 > 0$. Similarly, to prove point *(d)*, we have, $V_1 = u(w_1 - R_{11}) = u(w_2 - R_{21}) = v_2$ implying $R_{21} - R_{11} = w_2 - w_1 > 0$.

In case of failure, point *(a)* shows that the lender "forgets" the choices of q_1 and q_2: this is mainly due to the fact that ex post effort does not matter for the students who failed. Nobody can falsely report failure, and the base wage $\underline{w}$ does not depend on effort or type. In a more general model, (r_1, r_2) would be shaped by the tradeoff between incentives and insurance, and the planner might be willing to set different values for r_1 and r_2. Point *(b)* shows that the talented cannot be fully exploited (the marginal income tax rate is smaller than 100%): this is due to the presence of an ex post moral hazard problem, for otherwise, the planner would be tempted to tax the $W_2 - w_1$ increment completely, in order to achieve greater equality.

In the first-best world, full insurance can be implemented with the help of a lump-sum tax: transfers would depend on type, success, and therefore, would also depend on education, insofar as ex ante types determine education. When negative repayments are permitted, it follows that income-contingent loans would be useless. Indeed, absent informational problems, the private lending sector could distribute standard loans efficiently, while the government would protect individuals against risk and redistribute the income. In contrast, in the second-best economy, income-transfer instruments are constrained by limited observation and incentives. The income tax is typically a function of observable

income; standard loan repayments are proportional to years of education, with the addition of a bankruptcy clause that provides some form of insurance. These standard tools are not enough to achieve the second-best optimum. In particular, the loan repayments of successful individuals have no reason to be simply expressed as the product of an interest rate ρ times the cost of education, that is, $(1 + \rho)\gamma_q$.[25] More flexible instruments are needed: appropriately designed income-contingent loans and (or) a graduate tax (with 100% subsidization of tuition fees) can do the required job.

In general, the optimal graduate-tax repayment schedule cannot be implemented by means of the addition of an ordinary loan repayment (that would depend only on education q_i), and an income tax, (that would not depend on education).

To be more precise, we would like to find an income tax schedule T_i and a loan repayment schedule L_i, such that for any given 4 numbers $(R_{11}, R_{22}, R_{12}, R_{21})$ representing optimal repayments, we have,

$$
\begin{aligned}
R_{11} &= T_1 + L_1, \\
R_{22} &= T_2 + L_2, \\
R_{12} &= T_2 + L_1, \\
R_{21} &= T_1 + L_2.
\end{aligned}
\tag{14}
$$

This implies

$$
\begin{aligned}
R_{22} - R_{12} &= L_2 - L_1 = R_{21} - R_{11}, \\
R_{22} - R_{21} &= T_2 - T_1 = R_{12} - R_{11},
\end{aligned}
$$

which happens to be typically impossible. To see this, define the quantity,

$$
\boldsymbol{\Delta} = (R_{22} - R_{21}) - (R_{12} - R_{11}).
\tag{15}
$$

The linear system stated above (*i.e.*, (14)) implies that if the second-best repayments are decomposable, then $\boldsymbol{\Delta} = 0$. For a generic $(R_{11}, R_{22}, R_{12}, R_{21})$, these relations will not be true and $\boldsymbol{\Delta} \neq 0$. Using the definitions of V_i and v_i, we find

$$
\boldsymbol{\Delta} = z(v_1) - z(V_1) + z(v_2) - z(V_2) + W_2 - W_1 + w_1 - w_2.
$$

---

[25] But γ_q itself may be a function of the interest rate on government bonds, and the latter rate would thus of course play a role in the resource constraint.

Using the properties $V_1 = v_2$ and $V_2 = v_1$ we find,

$$\boldsymbol{\Delta} = (W_2 - W_1) - (w_2 - w_1). \tag{16}$$

By inequality (1), we have $\boldsymbol{\Delta} \geq \boldsymbol{0}$. We can have $\boldsymbol{\Delta} = \boldsymbol{0}$ only in a special case. Generically, we will have $\boldsymbol{\Delta} > \boldsymbol{0}$. The required decomposition is impossible because wages typically depend on effort and education in a nonseparable way.

We summarize the above remark as follows.

Proposition 4. (Nondecomposition Property) *Under Assumptions 1-5, it is generically impossible to decompose the second-best optimal transfers R_{ik} and r_i as the sum of student-loan repayments, depending only on education, and an income tax, not depending on education. It must be that, either the student-loan repayments are income-contingent, or the income tax is education-contingent.*

5   Application of Occam's Razor

In the above theory, we have introduced 4 important ingredients: *(i)*, wage risk and risk aversion; *(ii)*, unobservable *ex ante types* leading to a *screening,* or *self-selection* problem; *(iii)*, unobservable study effort generating *ex ante moral hazard*; *(iv)*, unobservable *ex post types* and effort causing an *ex post moral hazard* problem. Can we remove each of these ingredients from the model, to simplify the theory, and still obtain the key results? What do we find if we remove each of the ingredients in turn?

Without risk or risk aversion

If there is no wage risk, the problem posed above becomes rather trivial because, as soon as they would know their ex ante types, students could perfectly predict their future payoffs. There are two sources of risk in the model: random success and ex post type draws. In a riskless world, study-effort choices would have deterministic effects and ex post types would play no role.

If, on the other hand, we impose risk neutrality, the model becomes quite degenerate, because there is no private demand for insurance and no social motive for redistribution (*i.e.,*

no demand for insurance against the lack of talents, behind the veil of ignorance). Insurance is an essential element of student loan design. It follows that we need some risk and some degree of risk aversion.

Without ex ante moral hazard and ex ante screening

Before discussing what happens in the absence of moral hazard or of a screening problem ex ante, we briefly consider the situation in which ex post moral hazard is the only problem, *i.e.*, the model in which ICX_i inequalities are the only constraints. This can be called the "pure income taxation problem".

If we maximize $\Sigma_i \lambda_i (P_i U_i + (1 - P_i) u_i)$ subject to ICX_i and $\overline{RC}$, we find the following result.

Proposition 5. (Second best with observable ex ante types and observable study effort.) *When the only constraints are* $\overline{RC}$, *and* ICX_i, $i = 1, 2$, *then, in the standard utilitarian case, the second-best optimal solution has the following properties:*

$$u_1 = u_2 = \mathbf{u},$$

$$Ez_i' = z'(\mathbf{u}),$$

for all i, where by definition, $Ez_i' = \pi z'(v_i) + (1 - \pi)z'(V_i)$. *The constraints* $\overline{RC}$, ICX_1 *and* ICX_2 *are all binding.*

Proposition 5 is a relatively direct consequence of the first-order conditions for optimality. The solution requires *efficient interim insurance* given the remaining *ex post* moral hazard problem[26]. To achieve this goal, the marginal social cost of providing the basic welfare level $\mathbf{u}$, *i.e.*, $z'(\mathbf{u})$, must be equated with Ez_i', the expected marginal cost of providing the interim expected welfare level $U_i = (1 - \pi)V_i + \pi v_i$. As a consequence, we have $z'(v_1) > z'(u_1) > z'(V_1)$ and $z'(v_2) < z'(u_2) < z'(V_2)$ or equivalently, $v_1 > u_1 > V_1$ and $V_2 > u_2 > v_2$. These properties are unpalatable, because a student who failed ends up

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[26]Remark that when $\pi = 0$, the marginal social cost boils down to $Ez_i' = z'(U_i)$ and the condition $Ez_i' = z'(\mathbf{u})$ boils down to $z'(U_i) = z'(\mathbf{u})$, or $U_i = \mathbf{u}$.

being better off than an academically successful student with an unlucky draw of job market opportunities.

Without a screening problem

If there are no ex ante types, there is no ex ante screening problem; students do not self-select by choosing their level (or quality) of education. In a simple version of the model, there would be only one level of education and all students would face the same probability of success, conditional on study effort. Student loans would then be designed so as to induce effort, that is, to solve the moral hazard problems, both ex ante and ex post, but the graduate tax decomposition problem would be vacuous, because we need at least two types (and two levels of education) to pose this problem in a meaningful way.

Yet, there is another possibility, which is to study a model with two fully observable ex ante types: each type of student would then be assigned to an education level by the authoritarian social planner. To study this case, it is enough to compute second-best optima while ignoring all self-selection constraints, that is, both $\overline{IC}_i$ and $\underline{IC}_i$ constraints. The only constraints taken into consideration are $\overline{RC}$, MH_i and ICX_i. The results are quite different. When ex ante types are public information, we find the following result.

Proposition 6. (Second best with observable ex ante types and moral hazard.) *When the constraints are* $\overline{RC}$, MH_i *and* ICX_i, $i = 1, 2$, *then, in the standard utilitarian case, the second-best optimal solution has the following properties:*

$$U_i = u_i + K_i, \quad \text{for all } i = 1, 2.$$

$\overline{RC}$, MH_1, MH_2, ICX_1 *and* ICX_2 *are all binding, and*

$$\mathbf{E}z_1' = \mathbf{E}z_2',$$

where by definition, $\mathbf{E}z_i' = P_i[\pi z'(v_i) + (1 - \pi)z'(V_i)] + (1 - P_i)z'(u_i)$.

We skip the proof of Proposition 6. This proof relies on a discussion of the Kuhn and Tucker conditions for optimality.

In the above statement, it is striking that the *equal treatment* property is no longer required. To see this, note that MH_1 and MH_2 are simultaneously binding. It follows that, if $K_1 \neq K_2$, we cannot have $U_1 = U_2$ and $u_1 = u_2$. In the "normal" case, that is, if $K_1 > K_2$, we find that $\overline{IC}$ constraints must be violated. Indeed, condition D (in Lemma 3 above) implies $U_2 - u_2 \geq U_1 - u_1$, or equivalently, $K_2 \geq K_1$, a contradiction.

If $K_1 > K_2$, since the MH_i are binding, we must have $u_1 - u_2 < U_1 - U_2$. It follows that $U_1 > U_2$ is now a possibility, in particular if u_1 is close to u_2. Thus, $\overline{IC}_2$ may be violated. In other words, better insurance against a lack of academic talents, in case of failure, would imply that high quality studies are not chosen, because type $i = 2$ students would all prefer q_1. As a result, if ex ante types are assumed observable, the social planner would have to force the talented students to study longer.

The solution is indeed strikingly different from that of Proposition 2 above. Proposition 6 shows that *equal treatment* is a consequence of $\overline{IC}$ constraints. Interim equalization is not implied by the sole redistribution motive (risk aversion), it is a way of solving the self-selection problem. Absent self-selection constraints, Pareto-optimality does not impose equal treatment or full equality[27].

We conclude that with observable ex ante types, we would lose an important and realistic feature of our model, namely, the fact that education is an object of individual choice, which is also used to screen individuals[28]. In addition, the ex ante selection problem has a nontrivial influence on the shape of optimal repayment schedules.

With an ex ante screening problem but without ex ante moral hazard

If we remove study effort from our theory, and if ex ante types are not publicly observable, the only constraints are $\overline{RC}$, ICX_i and $\overline{IC}_i$, $i = 1, 2$. In this case, we know that the best we

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[27]This point is also shown by the proof of Proposition 1, in the Appendix: full equality is not a consequence of first-best optimality, whereas *full insurance* is implied by first-best efficiency. The two things are different here. In the next subsection, Proposition 2' shows that equal treatment is a second-best device: it is imposed to take care of ex ante self-selection constraints, even when it is not a requirement of first-best optimality.

[28]This has been discussed by Spence (1973), although in the present paper, we model screening by means of self-selection in a menu of contracts as in Rothschild and Stiglitz (1976).

could possibly achieve is *equal treatment in case of failure* with *efficient interim insurance*, that is, $u_1 = u_2 = \mathbf{u}$ and $Ez'_i = z'(\mathbf{u})$ for all i (as in Proposition 5). But the self-selection constraints $\overline{IC}_i$ would now impose $U_1 = U_2 = \mathbf{U}$ in addition (by Lemma 3c). Given $\overline{RC}$, we would then have three equations to determine two variables, namely $(\mathbf{U}, \mathbf{u})$. This is typically impossible. It follows that the solution would not just be the allocation of Proposition 5, with the addition of $\mathbf{U} = \mathbf{u}$: we would find $Ez'_i \neq z'(\mathbf{u})$ for at least one i.

Another way of looking at the same difficulty is to take the equal treatment solution of Proposition 2 and let the cost of study effort go to zero, *i.e.*, $c_i \to 0$ for all i, so that, in the limit, we obtain $K_1 = 0$, and ex ante moral hazard disappears as an economic problem. By continuity, the solution is now such that $\mathbf{U} = \mathbf{u}$, but we have imposed the constraint $U_1 - u_1 \geq 0$, and this constraint is not a consequence of $\overline{IC}$ alone. In fact, we need a specific treatment of this case. The results are disappointing: absent MH_i constraints, it can be shown that the structure of the optimal solution depends on the shape of inverse marginal utility $z'(.)$.

Proposition 7. (Second best with unobservable ex ante types and observable study effort.) *When the only constraints are $\overline{RC}$, $\overline{IC}_i$ and ICX_i, $i = 1, 2$, then, in the standard utilitarian case, the second-best optimal solution may be an* equal treatment *or a* separating *allocation. If π is sufficiently small,*

(a) if $z'(.)$ is concave, equal treatment is optimal:

$$u_1 = u_2 = \mathbf{u}, \qquad U_1 = U_2 = \mathbf{U},$$

$$Ez'_2 \geq z'(\mathbf{u}) \geq Ez'_1,$$

$z'(\mathbf{u})$ is a weighted average of Ez'_2 and Ez'_1,

$$z'(\mathbf{u}) = \frac{\Sigma_i \lambda_i P_i Ez'_i}{\Sigma_i \lambda_i P_i},$$

where, by definition, $Ez'_i = \pi z'(v_i) + (1 - \pi) z'(V_i)$;

(b) if $z'(.)$ is strictly convex, $\overline{IC}_1$ is binding and $\overline{IC}_2$ is slack. We have $U_2 > U_1$, $u_1 > u_2$, and

$$z'(u_1) = Ez'_1 > Ez'_2 > z'(u_2).$$

(c) There is no solution such that $\overline{IC}_2$ is binding and $\overline{IC}_1$ is slack. The constraints $\overline{RC}$ and ICX_i are always binding.

We skip the proof of this Proposition[29]. The result shows that the second-best optimal allocation depends on specific properties of the inverse marginal utility function $z' = 1/u'$. The solution is less clear-cut than when ex ante moral hazard is taken into account. In fact, adding the MH_i constraints simplifies the solution and the proof of the optimum characterization result: the solution is easier to find under moral hazard, combined with adverse selection, than under adverse selection alone. The equal treatment solution described by Proposition 2 is also more robust in many ways. It is robust to changes in the values of key parameters and in the shape of the utility function, provided that the latter is strictly concave; it is also robust to changes in the weights of types in the social welfare objective, as shown by Proposition 2' below. Finally, in the student loan problem, it is not reasonable to assume that study effort can be perfectly monitored, or that study effort does not affect the probability of success. We conclude that ignoring ex ante moral hazard does not yield an equivalent, but more parsimonious theory.

Without ex post moral hazard

The study of this case is straightforward. The solution can be obtained by letting $\pi \to 0$: the ex post type $k \neq i$ is drawn with probability zero. Proposition 2 is still valid when ex post moral hazard does not play a role: the optimum exhibits equal treatment and incomplete insurance. The important difference with the general case is now that $V_1 = V_2$. These relations indicate that the *talented are completely exploited*. We have $R_{22} - R_{11} = W_2 - w_1$. In other words, the marginal income-tax rate is 100% for earnings above w_1. The student loan problem is now combined with a first-best taxation problem. This solution is clearly unrealistic. The general conclusion of this discussion is that none of the "ingredients" of our model can be eliminated without losing an important feature of the solution.

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[29]The proof is available upon request.

6   Complete Characterization of Second-best Optima

Our goal is now to characterize all second-best Pareto optima in our model. To this end, we will consider weights for ex ante types that may differ from their frequencies in the population. Let α_1 and α_2 be the weights of type $i = 1$ and type $i = 2$ in the welfare function. We assume $\alpha_1 + \alpha_2 = 1$ without any loss of generality, and $\alpha_i > 0$ for all i. A first-best optimum can be obtained as a solution of the following problem,

$$\text{Maximize} \quad \sum_i \alpha_i[\mathbf{p}_i(e_i)U_i + (1 - \mathbf{p}_i(e_i))u_i - c_i] \tag{17}$$

with respect to $\{(R_i, R'_i, r_i, e_i, \varepsilon_{ik})\}_{i,k=1,2}$, subject to the resource constraint RC, and e_i, ε_{i1}, ε_{i2} are equal to 0 or 1.

Under the assumptions made above, using the inverse utility function $z(x) = u^{-1}(x)$ and eliminating R_{ik} and r_i from the objective and the resource constraint RC, the second-best optimality problem can be rewritten as follows:

$$\text{Maximize} \quad \sum_i \alpha_i[P_iU_i + (1 - P_i)u_i - c_i] \tag{18}$$

with respect to $(V_i, v_i, u_i)_{i=1,2}$, subject to the resource constraint $\overline{RC}$ and all incentive and moral-hazard constraints, exactly as defined in Section 3 above. The statement of Proposition 2 is in fact a particular case of a more general result, called Proposition 8, that we now state. The proof of this proposition is provided in the appendix.

Proposition 8. (Equal treatment as a second best under adverse selection, ex ante and ex post moral hazard.) *Under Assumptions 1-5, for all $\pi < 1/2$, there exists an open interval $(\underline{\lambda}_2, \overline{\lambda}_2)$, including λ_2, such that if $\alpha_2 \in (\underline{\lambda}_2, \overline{\lambda}_2)$, then, the second-best optimal solution has the following properties:*

$$U_1 = U_2 = \mathbf{U}, \qquad u_1 = u_2 = \mathbf{u} \quad (\text{equal} \quad \text{treatment}),$$

$$\mathbf{U} = \mathbf{u} + K_1 \quad (\text{incomplete} \quad \text{insurance}),$$

where $\mathbf{U}$, $\mathbf{u}$ are functions of π. Constraints $\overline{RC}$, MH_1, $\overline{IC}_1$, $\overline{IC}_2$, ICX_1 and ICX_2 are all binding. If, in addition, $K_1 > K_2$, then, MH_2, $\underline{IC}_1$ and $\underline{IC}_2$ hold as strict inequalities.

Proposition 8 shows that the result of Proposition 2 can be extended: the equal treatment optima are obtained for all weights α in a neighborhood of type frequencies λ.

At this point, several remarks can be made. We first find a local invariance property: when α is close to λ, the optimal solution does not depend on α.

Corollary 3. *The second-best optimal solution of Proposition 8 is independent of* α, *when* α_2 *is small enough, i.e., if* $\alpha_2 < \overline{L}_2$, *we have,*

$$\frac{\partial}{\partial \alpha_2}(U_2, u_2, U_1, u_1) = 0.$$

Proof of Corollary 3: The second-best allocation is the solution of a system of four equations with four unknowns: *(i)*, $U_1 = U_2$; *(ii)*, $u_1 = u_2$; *(iii)*, $U_1 = u_1 + K_1$; and *(iv)*, given these constraints, $\overline{RC}$ pins down $u_1 = \mathbf{u}$. None of these equations involve α. *Q.E.D.*

For the sake of completeness, the remaining question is to find the second-best optimal solutions when $\alpha_2 > \overline{\lambda}_2$. We look for a second-best allocation in which a single $\overline{IC}$ constraint is binding. By Lemma 5, proved in the appendix, we know that $\overline{IC}_1$ must be the binding constraint, and we can show that this happens *only if* $\alpha_2 > \lambda_2$.

Proposition 9. (Separating Second-best Optima) *Under Assumptions 1-5, for all* $\pi < 1/2$, *if a second-best optimum has only one binding* $\overline{IC}$ *constraint, then,*

   (a) $\overline{IC}_1$ *is the binding constraint;* $\overline{IC}_2$ *is slack;*
   (b) *constraints* MH_1, $\overline{RC}$, ICX_1 *and* ICX_2 *are binding;*
   (c) *we have* $U_2 > U_1 > u_1 > u_2$ *and, necessarily,* $\alpha_2 > \lambda_2$.
   (d) *The second-best solution is fully determined by the following 4 equations:* $\overline{IC}_1$, MH_1 *and* $\overline{RC}$, *expressed as equalities, and the condition,*

$$\frac{\lambda_2}{\lambda_1 \alpha_2} \frac{[P_2(1 - P_\alpha)Ez_2' - P_\alpha(1 - P_2)z'(u_2)]}{(P_2 - P_1)} = P_1 Ez_1' + (1 - P_1)z'(u_1), \qquad (K)$$

where $P_\alpha = \alpha_1 P_1 + \alpha_2 P_2$ *and* $Ez_i' = (1 - \pi)z'(V_i) + \pi z'(v_i)$.

We have completely characterized the second-best optima under adverse selection, *ex ante* and *ex post* moral hazard, in the optimal graduate tax problem. Again, since the Kuhn-Tucker conditions are necessary and sufficient, we have shown that when $\alpha_2 > \overline{\lambda}_2$, the solution is that described by Proposition 9.

Proposition 9 shows that the second-best optimum is a *separating* allocation à la Rothschild-Stiglitz, *i.e.*, $U_2 > U_1 > u_1 > u_2$, when the weight of the talented types is sufficiently higher than their frequency in the population. In other words, to get a *separating optimum*, the social planner must be willing to markedly favor the talented ex ante types. It is still true that types are *ex ante* unequal since, using $\overline{IC_i}$ and MH_i constraints, we find that

$$Eu_2 = u_2 + P_2(U_2 - u_2) > u_2 + P_1(U_2 - u_2) = u_1 + P_1(U_1 - u_1) = Eu_1.$$

But the talented types are now less well insured in case of failure, since $u_1 > u_2$.

7 Conclusion

We have studied optimal student-loan contracts in a simple private information economy with unobservable student types. Types differ in the probability distributions of individual labour-market outcomes. Future earnings are risky. Students are risk-averse and choose an *ex ante* effort variable, that is not observed by the lender but affects the probabilities of academic success. This poses an optimal insurance problem. Students can also reduce their effort *ex post* and thus reduce their earnings below potential. This poses an additional income taxation problem à la Mirrlees. We completely describe the set of second-best optimal (or interim efficient) incentive-compatible menus of loan contracts. There are two types of optima: the *separating* and *equal treatment* allocations. Equal treatment arises when the social weights of types are in the neighborhood of their frequencies in the student population. In this case, the *interim* expected utilities of students of different types are equalized, conditional on the student's observable success or failure. However, students are *ex ante* unequal since they differ in their probability of success. In addition, students are *ex post* unequal since the second-best allocation trades off incentives and insurance-redistribution motives. The optimal menus of contracts exhibit incomplete insurance, as a consequence of moral hazard; they typically involve cross-subsidies in favor of the less-talented; the latter obtain the maximal amount of insurance, compatible with effort incentives; the talented types bear too much risk, due to the interplay of moral hazard and adverse selection. Optimal student loans are always income-contingent, even in the presence of an income tax. In other words, the second-best transfers

cannot be decomposed as the sum of a loan repayment, depending only on education, and an income tax, not depending on education. It must be that the optimal loan-repayments are income contingent, or that the income tax is itself education-contingent. The student-loan contracts can thus be interpreted as a form of *graduate tax*.

To a certain extent, these results provide a justification for the policies implemented in Australia and the UK. Recent developments in Public Finance, including the present contribution, consider the implementation of second-best allocations by means of a tax system that is not only income-contingent, but also education-dependent and even age-dependent. It is striking that student loans belong to a hybrid category, lying somewhere in between income taxation and a structured loan contract. The theory, in its present state, views a graduate tax as equivalent to an income-contingent college loan. A compulsory payment imposed by the State is treated as equivalent to a contract with a publicly regulated lending institution. This equivalence rests on some implicit assumptions. In the real world, it is likely to break down. A discussion of the reasons for which a loan contract and a graduate tax are in fact different opens new avenues for research. An important problem is of course the commitment of fiscal authorities over long horizons, sometimes called the hold-up problem. It appears as soon as it is recognized that there is no benevolent planner. The State has an obvious advantage over private lenders when it comes to enforce repayments, but governments may decide to change rules and may exploit some categories of students by means of unilateral changes in graduate-tax provisions. In contrast, an income-contingent loan, signed with a lending institution, may protect former students against discretionary power.

Post-graduation student migrations pose a well-known problem, given the absence of tax agreements and harmonization. It is not clear which of the private banker or the public lender has an advantage in solving this problem, since non-residents can sometimes be taxed, and it is typically more difficult to pursue debts overseas.

Another long-recognized difficulty is the cream-skimming problem. Most discussions at least implicitly assume the existence of a benevolent, or publicly monitored lender, endowed with monopoly power. All students must choose a loan contract in the publicly designed menu. But in a free-market economy, it may be hard to force the most profitable

student-borrowers to stay within the public system. A sub-population of students would opt out and prefer a standard (*i.e.*, non-contingent) bank loan, in combination or not with parental transfers, instead of the publicly regulated loan. This fact in itself does not invalidate the analysis, but deserves a specific study. Cream-skimming will of course have the likely effect of putting further limits on the amount of redistribution that is allowed by the public lending system.

Finally, the student-loan system may be more useful as a way of increasing access, and thus the overall investment in higher education, than as a redistribution instrument. This consideration would lead to the study of inter-generational transfers (parental support) and borrowing constraints as an additional dimension in the design of a lending system.

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Appendix: Proofs

Maximizing a weighted sum of expected utilities.

Our goal is to characterize all first-best and second-best Pareto optima in our model. To this end we consider weights for ex ante types that may differ from their frequencies in the population. Let α_1 and α_2 be the weights of type $i = 1$ and type $i = 2$ in the welfare function. We assume $\alpha_1 + \alpha_2 = 1$ without loss of generality, and $\alpha_i > 0$ for all i.

Under the assumptions made above, the first-best optimality problem can be rewritten as follows: maximize $\Sigma_i \alpha_i [P_i U_i + (1 - P_i) u_i]$ with respect to $(V_i, v_i, u_i)_{i=1,2}$, subject to the resource constraint $\overline{RC}$, exactly as defined in the text above. Second-best optima are obtained while maximizing the same objective function, subject to the resource constraint, plus all incentives and moral hazard constraints.

First-best efficiency

Proof of Proposition 1.

The first-best optimality problem is a convex programming problem, since $z(.)$ is a convex function and the objective is a linear function of utility levels V_i, v_i, and u_i. To write the first-order necessary conditions for optimality, let κ denote the Lagrange multiplier of the resource constraint. We find, for $i = 1, 2$,

$$\alpha_i = \kappa \lambda_i z'(V_i) = \kappa \lambda_i z'(v_i) = \kappa \lambda_i z'(u_i)$$

for all i. This immediately yields,

$$\frac{\alpha_i}{\kappa^* \lambda_i} = z'(V_i^*) = z'(v_i^*) = z'(u_i^*), \tag{19}$$

$$\frac{z'(V_1^*)}{z'(V_2^*)} = \frac{\alpha_1/\lambda_1}{\alpha_2/\lambda_2} = \frac{z'(v_1^*)}{z'(v_2^*)} = \frac{z'(u_1^*)}{z'(u_2^*)}, \tag{20}$$

$$\kappa^* = \frac{1}{\Sigma_i \lambda_i z'(u_i^*)} > 0. \tag{21}$$

It follows that first-best optimality implies *full insurance*, that is, for all i,

$$V_i^* = v_i^* = u_i^*, \tag{22}$$

and the resource constraint must be binding. If, in addition, $\alpha_i = \lambda_i$, we get *full equality*, i.e., $V_1^* = V_2^*$, $v_1^* = v_2^*$ and $u_1^* = u_2^*$. *Q.E.D.*

Proof of Corollary 1

(i) There does not exist an unconstrained optimum with $r_i^* \geq 0$ for all i. If such an optimum did exist, then, because of full insurance, we would have $\boldsymbol{\omega}(q_i, \varepsilon_{ii}^*) - R_{ii}^* = \boldsymbol{\omega}(q_i, \varepsilon_{ij}^*) - R_{ij}^* = \underline{w} - r_i^* \leq \underline{w}$ and therefore, $\Sigma_i \lambda_i \mathbf{E} z_i^* \leq \underline{w} < \Sigma_i \lambda_i S_i$, a contradiction, since resources would then be wasted.

(ii) First-best efficiency implies full insurance and full insurance in turn implies $\boldsymbol{\omega}(q_i, \varepsilon_{ii}^*) - R_{ii}^* = \underline{w} - r_i^*$. This yields $\boldsymbol{\omega}(q_i, \varepsilon_{ii}^*) - R_{ii}^* - \underline{w} = -r_i^*$. So, if we require $\boldsymbol{\omega}(q_i, \varepsilon_{ii}^*) - R_{ii}^* \geq \underline{w}$, we must have $r_i^* \leq 0$. *Q.E.D.*

Incentives. Proofs of Lemmas

Proof of Lemma 2:

Adding up the $\overline{IC}_i$ constraints immediately yields

$$(P_2 - P_1)(U_2 - U_1) \geq (P_2 - P_1)(u_2 - u_1)$$

and $P_2 > P_1$ in turn implies $U_2 - U_1 \geq u_2 - u_1$. With MH_1, we then obtain the following string of inequalities:

$$U_2 - u_2 \geq U_1 - u_1 \geq K_1 \geq K_2.$$

It follows that MH_2 is satisfied.

Q.E.D.

Proof of Lemma 3:

As in the proof of Lemma 2, adding up the $\overline{IC}_i$ constraints and $P_2 > P_1$ immediately yields $U_2 - U_1 \geq u_2 - u_1$: this is property D. Given $\overline{IC}$, since $P_2 > P_1$, if $U_1 > U_2$ we find a contradiction. $\overline{IC}$ above shows that $U_2 \geq U_1$ implies $u_1 - u_2 \geq 0$. This yields D'.

The proof of Lemma 3a is trivial, since $P_2 > P_1$. Lemma 3b follows from the fact that $u_1 = u_2$ and $\overline{IC}$ imply $U_2 - U_1 \geq 0 \geq U_2 - U_1$ and therefore $U_2 = U_1$. But under $\overline{IC}$, we also have that $U_2 = U_1$ implies $u_1 = u_2$. *Q.E.D.*

Proof of Lemma 4:

(a) If $\overline{IC}_1$ holds, then,

$$(1 - P_1)(u_1 - u_2) \geq P_1(U_2 - U_1),$$

and since under IC, $U_2 - U_1 \geq 0$, and we assumed $P_1 > p_1$, we also have $(1 - p_1)(u_1 - u_2) \geq p_1(U_2 - U_1)$. But MH_1 implies $c_1 - (P_1 - p_1)(U_1 - u_1) \leq 0$. This trivially implies

$$(1 - p_1)(u_1 - u_2) \geq p_1(U_2 - U_1) + c_1 - (P_1 - p_1)(U_1 - u_1), \tag{23}$$

and rearranging terms we get the equivalent inequality,

$$P_1 U_1 + (1 - P_1)u_1 - c_1 \geq p_1 U_2 + (1 - p_1)u_2,$$

but this is exactly $\underline{IC}_1$.

(b) Given that MH_1 is binding, $\underline{IC}_2$ can be expressed as follows,

$$P_2 U_2 + (1 - P_2)u_2 - c_2 \geq p_2(u_1 + K_1) + (1 - p_2)u_1 = u_1 + p_2 K_1. \qquad (\underline{IC}_2 + MH_1)$$

Combining $\overline{IC}_1$ and MH_1, holding as equalities, we easily obtain,

$$u_1 + P_1 K_1 = P_1 U_2 + (1 - P_1)u_2. \qquad (\overline{IC}_1 + MH_1)$$

Substituting the value of u_1 derived from $(\overline{IC}_1 + MH_1)$ in $(\underline{IC}_2 + MH_1)$ yields, after some rearrangement of terms,

$$(P_2 - P_1)(U_2 - u_2) \geq c_2 + (p_2 - P_1)K_1.$$

Dividing both sides by $(P_2 - p_2) > 0$ and rearranging terms, we obtain,

$$(U_2 - u_2)\left[\frac{P_2 - P_1}{P_2 - p_2}\right] \geq K_2 + \frac{p_2 - P_1}{P_2 - p_2}K_1. \tag{Z}$$

From condition D and MH_1 we know that $U_2 - u_2 \geq U_1 - u_1 = K_1$. In addition, $(P_2 - P_1)/(P_2 - p_2) = 1 + (p_2 - P_1)/(P_2 - p_2) > 0$. Hence, the following string of inequalities:

$$(U_2 - u_2)\left[\frac{P_2 - P_1}{P_2 - p_2}\right] \geq K_1\left[1 + \frac{p_2 - P_1}{P_2 - p_2}\right] \geq K_2 + \frac{p_2 - P_1}{P_2 - p_2}K_1,$$

since, by Assumption 5, $K_1 \geq K_2$. This shows that condition Z is satisfied, hence, $\underline{IC}_2$ is satisfied when $\overline{IC}_2$ holds and when $\overline{IC}_1$ and MH_1 are binding.

Q.E.D.

Optimal graduate tax

Before the proof of Propositions 2 an 8, we state a useful preliminary result. Lemma 5 gives indications on the constraints that must be binding at a second-best optimum. There is an asymmetry between the two $\overline{IC}_i$ constraints. Lemma 5 shows that, if a single $\overline{IC}$ constraint is binding at the optimum, this constraint must be $\overline{IC}_1$.

Lemma 5. *At a second-best optimum,*

      (a) $\overline{RC}$ is binding;

      (b) if $\overline{IC}_1$ and $\overline{IC}_2$ are binding, then, MH_1 must be binding;

      (c) if $\overline{IC}_2$ is binding, then, $\overline{IC}_1$ must be binding too.

Proof of Lemma 5:

Using Lemmata 1-4, the second-best optimality problem can be further simplified. The benevolent public banker should maximize $\Sigma_i \alpha_i (P_i U_i + (1 - P_i) u_i)$ with respect to (U_i, u_i), subject to $\overline{IC}_i$, MH_1, $\underline{IC}_2$ and $\overline{RC}$, $i = 1, 2$. To study this problem, we will also temporarily ignore (*i.e.*, relax) constraint $\underline{IC}_2$ and check at the end that it is indeed satisfied. Let κ, δ, μ_1 and μ_2 be the nonnegative Lagrange multipliers of, respectively, constraints $\overline{RC}$, MH_1, $\overline{IC}_1$ and $\overline{IC}_2$. The first-order conditions (*i.e.*, Kuhn-Tucker conditions) for the second-best optimality problem are the following. For $i, j = 1, 2$,

$$\alpha_1 P_1 + \mu_1 P_1 - \mu_2 P_2 + \delta \;=\; \kappa \lambda_1 P_1 E z_1'; \qquad (FOC1)$$

$$\alpha_2 P_2 + \mu_2 P_2 - \mu_1 P_1 \;=\; \kappa \lambda_2 P_2 E z_2'; \qquad (FOC2)$$

$$\alpha_1 (1 - P_1) + \mu_1 (1 - P_1) - \mu_2 (1 - P_2) - \delta \;=\; \kappa \lambda_1 (1 - P_1) z'(u_1); \qquad (FOC3)$$

$$\alpha_2 (1 - P_2) + \mu_2 (1 - P_2) - \mu_1 (1 - P_1) \;=\; \kappa \lambda_2 (1 - P_2) z'(u_2); \qquad (FOC4)$$

with, by definition,

$$Ez_1' \;=\; (1 - \pi) z'[U_1] + \pi z'[U_1 + b], \qquad (24)$$

$$Ez_2' \;=\; (1 - \pi) z'[U_2 + b] + \pi z'[U_2],$$

and with the complementary slackness conditions, *i.e.*,

$$\delta(U_1 - u_1 - K_1) = 0, \qquad\qquad (CS_1)$$

$$\kappa\{\Sigma_i\lambda_i[S_i - \mathbf{E}z_i]\} = 0, \qquad\qquad (CS_2)$$

$$\mu_i\{P_iU_i + (1 - P_i)u_i - P_iU_j - (1 - P_i)u_j\} = 0, \qquad\qquad (CS_{3i})$$

where $j \neq i$. These conditions are necessary and sufficient for an optimum, because as noted above, the problem is convex. It follows from this that, if we find a solution in which all multipliers are nonnegative, we have found the solution.

Proof of Lemma 5(a) Adding equations FOC1 to FOC4, we easily find,

$$1/\kappa = \Sigma_i\lambda_i\{P_iEz_i' + (1 - P_i)z'(u_i)\} > 0,$$

so that $\kappa > 0$, since $z'(.) > 0$. Hence, by CS_2, it follows that $\overline{RC}$ is binding.

Proof of Lemma 5(b) If $\overline{IC}_1$ and $\overline{IC}_2$ are binding, then $U_1 = U_2 = U$ and $u_1 = u_2 = u$. Suppose that MH_1 is slack, *i.e.*, $U_1 > u_1 + K_1$, at the second-best optimum, then by CS_1, we have $\delta = 0$. With $\delta = 0$, $FOC1$ and $FOC3$ form a linear system in (μ_1, μ_2); that is,

$$\mu_1P_1 - \mu_2P_2 = P_1[\kappa\lambda_1Ez_1' - \alpha_1];$$

$$\mu_1(1 - P_1) - \mu_2(1 - P_2) = (1 - P_1)[\kappa\lambda_1z'(u) - \alpha_1];$$

This system has a nonzero determinant, equal to $P_2 - P_1 > 0$, and a unique solution (μ_1^*, μ_2^*). It is easy to check that,

$$\mu_2^* = \frac{P_1(1 - P_1)}{P_2 - P_1}\kappa\lambda_1\left(z'(u) - Ez_1'\right).$$

But,

$$Ez_1' = (1 - \pi)z'[U] + \pi z'[U + b].$$

Since $b > 0$ and since MH_1 implies $U > u + K_1$, it follows that $Ez_1' > z'(U)$ and thus, $z'(u) - Ez_1' < 0$. We conclude that $\mu_2^* < 0$. This is a violation of Kuhn-Tucker conditions, since all multipliers must be non-negative. We have found a contradiction. This proves 5b.

Proof of Lemma 5(c): If $\overline{IC}_2$ is binding, and $\overline{IC}_1$ is slack, then, by CS_{31}, we have $\mu_1 = 0$. Using $FOC2$ and $FOC4$, we easily obtain, $\kappa\lambda_2 z'(u_2) = \alpha_2 + \mu_2 = \kappa\lambda_2 Ez'_2$ and therefore, $z'(u_2) = Ez'_2$. Using the definition of Ez'_i given above, $Ez'_2 = (1-\pi)z'[U_2 + b] + \pi z'[U_2]$, we find $Ez'_2 > z'(U_2)$. But $\overline{IC}$ and MH_1 together imply $U_2 \geq U_1 \geq u_1 + K_1 > u_1 \geq u_2$, so that $U_2 - u_2 \geq K_1 > 0$. It follows that we have a contradiction since $U_2 > u_2$ and hence $Ez'_2 > z'(u_2)$.

Q.E.D.

Proof of Propositions 2 and 8:

The FOCs can be rewritten,

$$\mu_1 P_1 - \mu_2 P_2 + \delta = P_1[\kappa\lambda_1 Ez'_1 - \alpha_1]; \qquad (FOC1b)$$

$$\mu_2 P_2 - \mu_1 P_1 = P_2[\kappa\lambda_2 Ez'_2 - \alpha_2]; \qquad (FOC2b)$$

$$\mu_1(1 - P_1) - \mu_2(1 - P_2) - \delta = (1 - P_1)[\kappa\lambda_1 z'(u_1) - \alpha_1]; \qquad (FOC3b)$$

$$\mu_2(1 - P_2) - \mu_1(1 - P_1) = (1 - P_2)[\kappa\lambda_2 z'(u_2) - \alpha_2]. \qquad (FOC4b)$$

Our optimum candidate exhibits equal treatment, $U_1 = U_2 = U(\pi)$, $u_1 = u_2 = u(\pi)$, and both $\overline{IC}$ constraints are binding. By Lemma 5 above, MH_1 must then be binding too. This imposes $U(\pi) = u(\pi) + K_1$. Adding the four $FOCb$ equations easily yields, $\kappa = \kappa(\pi)$, satisfying the relation,

$$\kappa(\pi)[\Sigma_i P_i\lambda_i Ez'_i + (1 - P_\lambda)z'(u(\pi))] = 1, \qquad (\Sigma FOC)$$

where, by definition, $P_\lambda = P_1\lambda_1 + P_2\lambda_2$, and where,

$$Ez'_i = (1 - \pi)z'[U(\pi) + (i - 1)b] + \pi z'[U(\pi) + (2 - i)b], \qquad (25)$$

for $i = 1, 2$. We know that $\overline{RC}$ is binding and it follows that $(U(\pi), u(\pi))$ is fully determined by the intersection of $\overline{RC}$ and MH_1. It is easy to check that the solution $(U(\pi), u(\pi))$ is a continuous function of π for $\pi \geq 0$, and this solution does not depend on α_2.

We must check that the associated multipliers $\mu_i(\pi)$ are nonnegative. $FOC2b$ and $FOC4b$ provide us with a linear system of equations for (μ_1, μ_2). The determinant of this

system is $P_2 - P_1 > 0$, so that there is a unique solution $(\mu_1(\pi), \mu_2(\pi))$. We easily derive,

$$\mu_1(\pi) = \frac{P_2(1 - P_2)\kappa(\pi)\lambda_2}{(P_2 - P_1)}[Ez_2' - z'(u(\pi))], \tag{26}$$

$$\mu_2(\pi) = \frac{\lambda_2 P_2(1 - P_1)}{(P_2 - P_1)}\left[\kappa(\pi)Ez_2' - \frac{\alpha_2}{\lambda_2}\right] + \frac{\lambda_2 P_1(1 - P_2)}{(P_2 - P_1)}\left[\frac{\alpha_2}{\lambda_2} - \kappa(\pi)z'(u(\pi))\right]. \tag{27}$$

Remark first that $Ez_2' > Ez_1' > z'(U(\pi)) > z'(u(\pi))$, since $U(\pi) > u(\pi)$ and $\pi < 1/2$. Using these inequalities, expression ΣFOC above, and the fact that $z' > 0$, we derive

$$1 > \kappa(\pi)z'(u(\pi)), \qquad \text{and} \qquad 1 < \kappa(\pi)Ez_2'. \tag{28}$$

From these remarks, we immediately obtain $\mu_1(\pi) > 0$. Note that this property is true for all values of α_2 in $(0, 1)$ since $\mu_1(\pi)$ does not depend on α_2. For the same reasons, $\kappa(\pi)$, given by equation ΣFOC, does not depend on α_2, and thus, we always have $Ez_2' > z'(u(\pi))$ and $\kappa(\pi) > 0$. Given the inequalities derived above, for any $\alpha_2 \le \lambda_2$, we also have $\kappa(\pi)Ez_2' - (\alpha_2/\lambda_2) > 0$, and therefore, using $P_2 > P_1$, we find

$$\mu_2(\pi) > \frac{\lambda_2 P_1(1 - P_2)}{(P_2 - P_1)}\left[\kappa(\pi)Ez_2' - \frac{\alpha_2}{\lambda_2} + \frac{\alpha_2}{\lambda_2} - \kappa(\pi)z'(u(\pi))\right]$$

$$= \frac{\lambda_2\kappa(\pi)P_1(1 - P_2)}{(P_2 - P_1)}[Ez_2' - z'(u(\pi))] > 0. \tag{29}$$

In particular, if $\lambda_2 = \alpha_2$, then $\mu_2(\pi) > 0$. By continuity, there exists an interval of values of α_2, including λ_2, such that $\mu_2(\pi)$ is positive. By Lemma 4, we know that both $\underline{IC}_i$ constraints are satisfied. By Lemma 2, we know that MH_2 is also satisfied.

Finally, we must check that the multiplier $\delta = \delta(\pi) > 0$. Adding $FOC3b$ and $FOC4b$, we derive an expression for $\delta(\pi)$, that is,

$$\delta(\pi) = 1 - P_\alpha - \kappa(\pi)(1 - P_\lambda)z'(u(\pi)), \tag{30}$$

where by definition, $P_\alpha = \alpha_1 P_1 + \alpha_2 P_2$. This can be rewritten

$$\frac{\delta(\pi)}{(1 - P_\lambda)} = \frac{(1 - P_\alpha)}{(1 - P_\lambda)} - \kappa(\pi)z'(u(\pi)). \tag{31}$$

Since we have $1 > \kappa(\pi)z'(u(\pi))$, this means that there is an interval of values of α_2, around λ_2, such that $\delta(\pi) > 0$. In addition, $\delta(\pi) > 0$ for all values of $\alpha_2 \le \lambda_2$ since $\alpha_2 \to 0$ implies $P_\alpha \to P_1 < P_\lambda$. We conclude that there exists an interval $(\underline{\lambda}_2, \overline{\lambda}_2)$ of values of α_2, including

λ_2, such that all Lagrange multiplers are positive. Therefore, we have found the optimal solution for these values of α_2 and π.

Q.E.D.

Separating optima

Proof of Proposition 9:

If a second-best optimum has only one binding $\overline{IC}$ constraint, then, by Lemma 5, $\overline{IC}_1$ must be binding and $\overline{IC}_2$ is slack. Hence, $\mu_2 = 0$. By Lemmas 2 and 4, we can neglect MH_2 and $\underline{IC}_i$ constraints. By Lemma 1, ICX_i constraints are binding. Thus, using (13), define

$$Ez'_i = (1 - \pi)z'[U_i + (i - 1)b] + \pi z'[U_i + (2 - i)b].$$

Adding equations $FOC1$ to $FOC4$, we easily obtain, $1/\kappa = \Sigma_i \lambda_i (P_i Ez'_i + (1 - P_i)z'(u_i)) > 0$. Hence, $\overline{RC}$ is binding. Suppose now that MH_1 is slack. Then, $\delta = 0$. From $FOC1b$ and $FOC3b$, we easily derive,

$$\kappa \lambda_1 z'(u_1) = \alpha_1 + \mu_1 = \kappa \lambda_1 Ez'_1.$$

Hence, $z'(u_1) = Ez'_1$. But this leads to a contradiction, since MH_1 imposes $u_1 + K_1 < U_1$ and therefore,

$$Ez'_1 > z'(U_1) > z'(u_1 + K_1) > z'(u_1).$$

We conclude that MH_1 must be binding and $U_1 = u_1 + K_1$. From $FOC2b$, and the requirement that $\mu_1 \geq 0$, we derive

$$\mu_1 = \frac{P_2}{P_1}[\alpha_2 - \kappa \lambda_2 Ez'_2] \geq 0. \tag{A9}$$

From $FOC1b$, we derive,

$$\mu_1 + \frac{\delta}{P_1} = \kappa \lambda_1 Ez'_1 - \alpha_1 \geq 0. \tag{B9}$$

Combining $A9$ and $B9$, we obtain,

$$\frac{\alpha_2}{\lambda_2} \geq \kappa Ez'_2 \qquad \text{and} \qquad \kappa Ez'_1 \geq \frac{\alpha_1}{\lambda_1}. \tag{C9}$$

We must compare Ez'_2 and Ez'_1. We have $Ez'_2 > Ez'_1$ if and only if,

$$(1 - \pi)z'(U_2 + b) + \pi z'(U_2) > (1 - \pi)z'(U_1) + \pi z'(U_1 + b). \qquad (D9)$$

To prove that $D9$ is true, assume that $Ez_1' \geq Ez_2'$. Then, using $U_2 > U_1$, we must have,

$$(1 - \pi)z'(U_1) + \pi z'(U_1 + b) \geq (1 - \pi)z'(U_2 + b) + \pi z'(U_2) > (1 - \pi)z'(U_1 + b) + \pi z'(U_1),$$

but this leads to a contradiction since, obviously, $U_1 + b > U_1$ and $\pi < 1/2$. This proves $D9$. Now, it follows from $C9$ and $D9$ that we must have $\alpha_2(1 - \lambda_2) > (1 - \alpha_2)\lambda_2$, or $\alpha_2 > \lambda_2$.

Combining $A9$ and $B9$, and assuming that $\delta > 0$, we obtain

$$\delta = \lambda_1 P_1[\kappa Ez_1' - (\alpha_1/\lambda_1)] + P_2\lambda_2[\kappa Ez_2' - (\alpha_2/\lambda_2)] > 0,$$

which is equivalent to,

$$\Sigma_i \lambda_i P_i Ez_i' > \frac{P_\alpha}{\kappa}. \qquad (E9)$$

The allocation U_1, U_2, u_1, u_2 is determined by a system of four equations, the first three are obviously $\overline{IC}_1$, MH_1 and $\overline{RC}$, expressed as equalities. To find the fourth equation, we eliminate Lagrange multipliers from $FOC1b$, $FOC3b$ and $FOC4b$. More precisely, adding $FOC1b$ and $FOC3b$ yields

$$\mu_1 = \lambda_1 \kappa[P_1 Ez_1' + (1 - P_1)z'(u_1)] - 1 + \alpha_2. \qquad (F9)$$

On the other hand, substituting $A9$ yields,

$$\frac{P_2}{P_1}[\alpha_2 - \kappa\lambda_2 Ez_2'] = \lambda_1 \kappa[P_1 Ez_1' + (1 - P_1)z'(u_1)] - 1 + \alpha_2,$$

or equivalently,

$$\frac{P_\alpha}{\kappa} = \lambda_2 P_2 Ez_2' + \lambda_1 P_1[P_1 Ez_1' + (1 - P_1)z'(u_1)]. \qquad (G9)$$

Note that if $G9$ holds, since $Ez_1' > z'(u_1)$, then, necessarily, $E9$ holds: this confirms that $\delta > 0$. Substituting the expression for κ, derived above, in $G9$, and rearranging terms, yields the fourth equation that we need to solve the problem,

$$\frac{\lambda_2}{\lambda_1 \alpha_2} \frac{[P_2(1 - P_\alpha)Ez_2' - P_\alpha(1 - P_2)z'(u_2)]}{(P_2 - P_1)} = P_1 Ez_1' + (1 - P_1)z'(u_1) \qquad (K)$$

The second-best optimum (U_1, u_1, U_2, u_2) is fully determined by $\overline{RC}, MH_1, \overline{IC}_1$ expressed as equalities and condition K. The condition $\mu_1 \geq 0$ yields a lower bound on the values of α_2 that can be derived from $A9$, that is, equivalently, from $C9$, we have,

$$\frac{\alpha_2}{\lambda_2} \geq \kappa E z_2'. \tag{H9}$$

Summing the $FOCs$, we found $1/\kappa = \Sigma_i \lambda_i (P_i E z_i' + (1 - P_i) z'(u_i))$. Since MH_i implies $E z_i' > z'(u_i)$, and since by $D9$, $E z_2' > E z_1'$, we must have $1/\kappa < E z_2'$ or $\kappa E z_2' > 1$. We conclude from $H9$ that $\alpha_2 > \lambda_2$ is required for this type of solution to be optimal. $Q.E.D.$