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Inequality decomposition values: the trade-off between marginality and efficiency

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Abstract This paper presents a general procedure for decomposing income inequality measures by income sources. The methods of decomposition proposed are based on the Shapley value and extensions of the Shapley value of transferable utility cooperative games. In particular, we find that Owen's value can find an interesting application in this context. We show that the axiomatization by the potential of Hart and Mas-Colell remains valid in the presence of the domain restriction of inequality indices. We also examine the properties of these decomposition rules and perform a comparison with Shorrocks' decomposition rule properties.

Keywords Inequality · Decomposition rule · Income sources · Shapley value

1 Introduction

The idea of marginal contribution is fundamental in much economic analysis. The Shapley value [13] of transferable utility cooperative games illustrates the strength of this idea. Indeed, for every coalitional game the Shapley value of a player is intuitively the average of all marginal contributions that this individual can make to all coalitions. In view of both its intuitive appeal and mathematical tractability, the Shapley value has been the focus of much research and applications (see for example the surveys of [1, 5, 12]). These applications concern fields as different as cost allocation, surplus sharing, models of taxation, market allocations and political

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power allocation ... Is it reasonable or not to add to this list inequality measurement? The goal of this paper is to give an appropriate answer to this question.

A natural field of application of the Shapley value seems to be what is called the decomposition of inequality indices, namely, to decompose the aggregate inequality value into some relevant component contributions. The issues to which this kind of analysis has been applied fall into two broad categories. The first category deals with the influence of population subgroups such as those defined by age, sex or race (see for example [14, 16]). We consider an application of the Shapley value to the second category of decomposition problem, which covers situations within which different components of total income are examined. If we disaggregate total income into several factor components, such as pay, private incomes and profits, we wish to evaluate the contribution of each income source to the aggregate inequality. Kolm [8, 9] and Theil [17] have tackled this question but we can give Shorrocks [15] credit by setting up the theoretic foundations of the decomposition by income components. He discusses six general properties that one might wish to be satisfied by a decomposition rule and shows that the natural decomposition of the variance is the only decomposition rule satisfying these six properties. Furthermore the proportion of inequality attributed to each factor is the proportion obtained in the natural decomposition of the variance. For each component of income, the assessment of its relative contribution to the total income inequality will be independent of the inequality measure chosen. The author adds "This is a particularly attractive feature for those involved in applied research on income distribution ([15], p 209)". May be, but from a theoretical point of view, the result is debatable and there are good reasons to think that the relative contribution of a component must be dependent on the inequality measure chosen, dependent on how this measure weights a progressive transfer or a regressive transfer, dependent on the index being relative or absolute.

In this respect, the Shapley inequality decomposition by sources offers some advantages. It is sensitive to the choice of the inequality index, for example the contribution of an equally distributed factor component to inequality is zero if the index is absolute and negative if the index is relative. But this property does not preclude the Shapley value inequality decomposition by sources to correspond to the natural decomposition of the variance if the chosen inequality index is the variance. Moreover, the Shapley decomposition by income sources has a meaningful interpretation, inspired by the idea of marginal contribution. An approach would proceed by assigning to every income source its direct marginal contribution to the overall inequality, i.e., the difference between the overall inequality and the inequality if we dropped this income source or at least if we dropped inequality from this source. It is obvious however that it is not possible, in general, to solve the problem in this way. This is simply because the marginal contributions defined in that way may not add up to the amount of total inequality that needs to be explained. In most cases the marginalist view of inequality decomposition is not efficient. The key idea behind the Shapley value is to conciliate the marginalist interpretation with the efficiency requirement.

Inspired by Hart and Mas-Colell [6, 7], we keep the idea that the contribution could be regarded as marginal but due to the overall inequality constraint, marginal contributions have to be computed according to some function that differs from the inequality index. This idea leads to the following desirable property. There must exist some function related to the primitive of the models, the set of income sources and the inequality index, such that the contribution of any factor component is just equal to the marginal contribution according to this *new* function, the sum of these marginal contributions over the set of income sources being given equal to the amount of total inequality. This property would be meaningless if such a function did not exist for some inequality index or at contrary if we could build an infinity of such functions. Fortunately, we prove that such a function exists and is unique. Furthermore, we prove that the only decomposition rule satisfying this property is the Shapley decomposition rule. In other words, we only have one way to extend the problem of inequality decomposition by income sources such that the marginalist interpretation remains valid under the efficiency constraint.

The remainder of the paper deals with a difficulty of the Shapley decomposition rule. It does not respect a somewhat natural axiom of independence introduced by Shorrocks [15]. The contribution of an income source must be independent of the number of income sources considered. A partial answer can be found in extensions of the Shapley value promoted by Owen [11] and Winter [18]. This value leads to a decomposition of inequality indices, which satisfies a milder request of independence, once a more general framework consisting in some partition of the set of income sources has been introduced. For instance, some income sources can be labelled as market incomes, while others can be considered as transfers. With the Owen decomposition rule, the contribution of, for example, labour income would be independent of the number of sources gathered under the label of transfers.

We begin in Section 2 with an introductory discussion of the issues involved in factor decomposition problems. Section 3 introduces what we call the inequality game. Section 4 states the main theorem about the Shapley decomposition. Section 5 extends the Shapley decomposition to inequality games within which the set of sources is a priori decomposed into a partition of subgroups of sources. Section 6 summarizes the results and adds a few concluding remarks and extensions.

2 Inequality decomposition by factor components

Let X_i^j denote the income of individual i, $i \in N = \{1,\ldots, n\}$ from source j, $j \in K = \{1,\ldots, k\}$ and let $X = (X_1,\ldots, X_n)$ represent the distribution of total incomes among individuals, and $X^j = \left(X_1^j, ..., X_n^j\right)$ represent the distribution of income source j among individuals. We suppose that inequality is measured by a function I which is continuous, symmetric and Shur concave and such that $I(X) = 0$ if and only if $X = \mu e$, with $e = (1, ..., 1, ..., 1)$ and $\mu \in R_+$.[1] If we consider that the sources of income are disjoint and exhaustive, the contribution of source j to the total income inequality is represented by $\phi_j(K; X)$. We recall here the six properties introduced by Shorroks [15].

- **Axiom 1** Consistent decomposition
  The sum of the contributions of all sources equals the overall amount of inequality.

[1] In the following we will assume, without loss of generality, that the value of inequality is contained in the interval [0,1].

- **Axiom 2** (a) Continuity, (b) Symmetric treatment of factors
  The contribution of source j is continuous in X^j and no significance is attached to the labelling of the different sources of income.

- **Axiom 3** Population symmetry
  The contribution of sources does not depend on the labelling of individuals.

- **Axiom 4** Two factor symmetry
  Two sources, say 1 and 2, have the same contribution to the total inequality if the distribution of income from source 1, X^1, is a permutation of the distribution of income source 2, X^2.

- **Axiom 5** The contribution of a source is zero if all individuals receive the same amount of income from that source.

- **Axiom 6** The contribution of any one source of income does not depend on how many other types of income sources are distinguished.

Shorrocks shows that Axioms 1–6 are satisfied if and only if $\varphi_j(X^1, ..., X^k)/I(X) = \mathrm{cov}(X^j, X)/\sigma^2(X)$. In other words, assumptions 1–6 are satisfied if and only if the relative contributions of each source of income correspond to the relative contributions of the natural decomposition of the variance. Taken separately, the six axioms are clearly attractive but the joint result of requiring all of them, is a bit disturbing. The decomposition by factor components in relative terms must be independent of the choice of the inequality index.

We dispute the validity of the solution proposed by Shorrocks from two perspectives (see for example, [10], for a discussion of Shorrocks' approach). First, inequality indices are different according to their sensibility to Pigou-Dalton transfers. Some of them are very sensitive to transfers performed at the top of the distribution (Theil's index), while other are more sensitive to transfers performed at the bottom of the distribution (Atkinson index for some values of the parameter).

Example 1 Suppose we have 10 individuals who obtain income from two sources, such that the average incomes are the same for each source, i.e. $\mu^1 = \mu^2$. The distributions of incomes are the following:

$$X^1 = (1, 1, 1, 1, 1, 1, 1, 1, 100, 892)$$
$$X^2 = (1, 1, 1, 1, 1, 1, 1, 1, 484, 508)$$
$$X = (2, 2, 2, 2, 2, 2, 2, 2, 584, 1400)$$

These distributions are such that X^2 can be deduced from X^1 by a Pigou-Dalton transfer of an amount of 384 between the two richest individuals. It is well known that the variance is more sensitive to income transfer at the top than the Gini index. Actually, the natural relative contributions for the variance are:

$$\varphi_1(X^1, X)/I(X) = \mathrm{cov}(X^j, X)/\sigma^2(X) = 59{,}9\% \text{ and } \varphi_2(X^2, X)/I(X) = 40{,}1\%$$

Whereas with Gini's index[2] they are:

$$\varphi_1\left(X^1, X\right)/I\left(X\right) = \mu^1 G\left(X^1\right)/\mu G\left(X\right) = 52{,}3\% \text{ and } \varphi_2\left(X^2, X\right)/I\left(X\right) = 47{,}7\%$$

The difference between the relative contributions has an economic interpretation. The distribution for the second source is deduced from the distribution of the first source by a transfer between two rich individuals. Since the variance is more sensitive than Gini's index to transfer in the top of the distribution, the difference of inequality between sources 1 and 2 is larger with the variance than with the Gini index. Hence, it is natural that the relative contribution of the second source is weaker with the variance than with the Gini index.

A second difference between indices concerns the relative or absolute character of the indices. The following example illustrates that the contribution of an equally distributed source to overall inequality cannot be the same for relative and absolute inequality indices. Suppose that the distribution of total incomes is $X = X_0 + \lambda e$, with $e = (1,1,...,1)$ and consider the two sources $X^1 = X_0$ and $X^2 = \lambda e$. With an absolute index, we obtain $I(X) - I\left(X - X^2\right) = 0$, while with a relative index, we have $I(X) - I\left(X - X^2\right) < 0$. Hence, inequality remains constant, if we drop the equally distributed source, when the inequality index is an absolute one while inequality increases with a relative index. This observation raises the opportunity of questionning the fifth axiom Shorrocks proposed, namely that the contribution to the total inequality of an equally distributed factor is zero for all inequality indices. Although this axiom seems to be reasonable for an absolute index, relaxing this assumption seems more appropriate for a relative one.

These two comments found our interest for the characterization of decomposition rule by factor components inspired by alternate properties.

3 The inequality game

Let us recall that the fundamental idea of cooperative games is that we must take into account the possibility that some subsets of players might form a cooperative coalition without the other players. The assumption of transferable utility, in an n-player game, allows us to describe the cooperative possibilities of a game by mean of a *characteristic function* V that assigns a number $V(S)$ to every coalition S, an element of the power set of N. By convention, we always let $V(\emptyset) = 0$, where $\emptyset$ denotes the empty set. We say that a characteristic function V is superadditive if and only if, for every pair of coalitions S and T, if $S \cap T = \emptyset$, then $V(S \cup T) \geq V(S) + V(T)$. Roughly speaking, a game in coalitional form with transferable utility is given by two ingredients:

- a set of players;
- a characteristic function which assigns a real number to every subset of players.

An easy task is to define the set of players. The set of income sources would be an obvious candidate to "play" this role. A coalition of players would be simply

[2] We use here the natural decomposition of Gini (see for example [17] or [4]) which is equal to $\frac{\mu^j}{\mu}G(X^j)$, $j = 1{,}2$ since the ranking of individuals are the same for each source and the total income distribution.

a subgroup of income sources and the set of coalitions is the power set 2^K. More difficult is to accept the consequences of transferable utility in this context. Suppose that we have made the choice of some inequality index. We must agree not only with the ordinal meaning of this index but also with its cardinal meaning. A value of 0.365 for the Gini index must have true cardinal meaning. Maybe surprisingly we must admit that many experts in applied studies find some relevance in the cardinal value of the inequality indices. We will stick to this idea, which opens the possibility to debate on the computation of the characteristic function. At least, two computations seem reasonable. They differ in the treatment of sources not included in the particular subset of sources considered. In the first one, which will help us to define zero income inequality games, the value of the characteristic function for some subset of sources S is simply the value of the inequality index if the individuals receive nothing from sources not included in S. In the second computation, which leads to the definition of equalized inequality games, the value of the characteristic function for some subset S is given by the value of the inequality index when we have equalized the income for all sources not in S. Probably the second computation would be more in the spirit of inequality measurement.

Formally, the parameters of the model are a particular inequality index I: $R^n \to [0,1]$, a set of sources K, a power set of sources 2^K with S an element of 2^K and a distribution of income by sources X: $K \to R^n$. The distribution of income by sources helps us to build a distribution of income among subsets of sources, namely an application

$$Y\colon 2^K \to R^n, \text{such that } Y(\emptyset) = 0 \text{ and for all } S \in 2^K,\ Y(S) = \left(\sum_{j\in S} X_1^j, \ldots, \sum_{j\in S} X_n^j\right).$$

The first computation of the characteristic function V, with respect to the inequality index I, would a function V_I: $2^K \to [0,1]$, such that $V_I = I \circ Y$, with $V_I(\emptyset) = 0$, $V_I(K) = I(X)$ and $V_I(S) = I(Y(S))$, for all S.

Let $\boldsymbol{V}_I = \{V_I : \exists Y : 2^K \to R^n \mid V_I = I \circ Y\}$ denote the set of characteristic functions. The following definition summarizes the above discussion.

Definition 1 A zero income inequality game is a pair (K, V_I), where $K = \{1,\ldots k\}$ is the set of players and V_I is a function defined on all subsets $S \in 2^K$ such that $V_I \in \boldsymbol{V}_I$.

The definition of an equalized income inequality game is the same except that now the distribution of income among subsets of sources is obtained equalizing complementary sources, i.e. we define the application $\tilde{Y} : 2^K \to R^n$, such that $\tilde{Y}(\emptyset) = 0$ and for all $S \in 2^K$, $\tilde{Y}(S) = \left(\sum_{j\in S} X_1^j + \mu_X - \mu_{Y(S)}, \ldots, \sum_{j\in S} X_n^j + \mu_X - \mu_{Y(S)}\right)$, where μ_X and $\mu_{Y(S)}$ are the arithmetic means of the vector X and $Y(S)$ respectively. With $\tilde{V}_I(\emptyset) = 0$, $\tilde{V}_I(K) = I(X)$, $\tilde{V}_I(S) = I(\tilde{Y}(S))$ for all S and $\tilde{V}_I = \{\tilde{\boldsymbol{V}}_I : \exists \tilde{Y} : 2^K \to R^n \mid \tilde{V}_I = I \circ \tilde{Y}\}$, we state:

Definition 2 An equalized income inequality game is a pair $(K, \tilde{V}_I)$, where $K = \{1,\ldots k\}$ is the set of players and $\tilde{V}_I$ is a function defined on all subsets $S \in 2^K$ such that $\tilde{V}_I \in \tilde{\boldsymbol{V}}_I$.

Let us remark that in each case, the characteristic function is not supposed to be superadditive. The following example illustrates that it is better to not impose this restriction.

Example 2 Let X be a distribution of three income sources, wages (W), capital (C) and social allowances (SA) among three individuals

	W	**C**	**SA**
1	3	9	2
2	45	17	1
3	7	25	3

Suppose now that inequality is measured by Gini's index. Then, the characteristic function of the zero income inequality game $(K = \{W, C, SA\}, V_I)$ is:

$$V_I(\emptyset) = 0 \quad V_I(\{W\}) = \frac{8}{45} \quad V_I(\{W, SA\}) = \frac{10}{63} \quad V_I(\{W, C\}) = \frac{20}{99}$$
$$V_I(\{C\}) = \frac{32}{153} \quad V_I(\{SA\}) = \frac{2}{9} \quad V_I(\{C, SA\}) = \frac{34}{171} \quad V_I(\{W, C, SA\}) = \frac{42}{207}$$

While the characteristic function of the equalized income inequality game $\left(K = \{W, C, SA\}, \tilde{V}_I\right)$ is:

$$V_I(\emptyset) = 0 \quad V_I(\{W\}) = \frac{1}{27} \quad V_I(\{W, SA\}) = \frac{5}{108} \quad V_I(\{W, C\}) = \frac{5}{27}$$
$$V_I(\{C\}) = \frac{21}{108} \quad V_I(\{SA\}) = \frac{4}{207} \quad V_I(\{C, SA\}) = \frac{17}{108} \quad V_I(\{W, C, SA\}) = \frac{42}{207}$$

It is easy to see that V_I and $\tilde{V}_I$ are not superadditive, since $V_I(\{W, C, SA\}) < V_I(\{SA\})$ and $\tilde{V}_I(\{W, C\}) < \tilde{V}_I(\{C\})$. The absence of superadditivity is a specific feature of the inequality games and it has some severe consequences. Mostly axiomatizations of the Shapley value assume the superadditivity of the characteristic function, and then cannot be supposed to be true when we consider the restricted domain of inequality games.

Definition 3 Given an inequality index I, a decomposition rule is a rule that assigns to every inequality game (K, V_I) a function $\phi(K, V_I) \in R^K$ such that:

$$\sum_{j=1}^{n} \varphi_j(K, V_I) = V_I(K) \leq I(X) \tag{1}$$

where $\phi_j(K, V_I)$ represents the contribution of the income source j to the total income inequality.

In our framework, we state as a definition what Shorrocks labels as his consistency axiom (Axiom 1).

Definition 4 Given an inequality index I, a consistent decomposition rule is a rule that assigns to every inequality game (K, V_I) a function $\phi(K, V_I) \in R^K$ such that:[3]

$$\sum_{j=1}^{n} \varphi_j(K, V_I) = V_I(K) = I(X) \tag{2}$$

[3]We have a similar definition for every equalized income inequality game. Note also that we do not restrict the contribution of a given income source to be positive.

4 The Shapley decomposition by factor components

The only property we might wish to be satisfied by the decomposition rule is the property of marginality, inspired by the work of Hart and Mas-Colell [7]. We keep the idea that the contribution could be regarded as marginal but due to the overall inequality constraint, the marginal contribution has to be computed according to some function that differs from the inequality index. This idea leads to the following desirable property. There must exist some function related to the primitive of the model, the set of sources and the inequality index, such that the contribution of any factor component is just equal to the marginal contribution according to this *new* function, the sum of the marginal contributions over the set of income sources being equal to the amount of total inequality.

Marginality property The decomposition rule satisfies the marginality property if there exists some function $P : N \times V_I \rightarrow [0, 1]$ with $P(\emptyset, V_I) = 0$ and such that

$$\forall j \in K, \varphi, (K, V_I) = P(K, V_I) - P(K - \{j\}, V_I) \tag{3}$$

The function P called the potential by Hart and Mas-Colell [7] can be any function which assigns a number $P(K, V_I)$ in [0,1] to every inequality game (K, V_I). The marginality property requires that the contribution of any source is given by the difference of the value of the function for the set of sources and the value of the function for the set of sources minus this specific source. This property would be meaningless if such a function did not exist for some inequality index or at the contrary if we could build an infinity of such functions. Fortunately, we prove that such a function exists and is unique. Furthermore, we prove that the only consistent decomposition rule satisfying this property is the Shapley decomposition.

Proposition 1 *Let K a set of income sources and I any inequality index. A consistent decomposition rule satisfies the marginality property with respect to some function P if and only if it is given by the Shapley formula, namely:*

$$\varphi_j(K, V_I) = Sh_j(K, V_I) = \sum_{\substack{S \subseteq K \\ j \in S}} \frac{(s-1)!\,(k-s)!}{k!} \left[V_I(S) - V_I(S - \{j\})\right] \tag{4}$$

Proof Let

$$P(K, V_I) = \sum_{S \subseteq K} \frac{(s-1)!\,(k-s)!}{k!} V_I(S)$$

We thus have for every inequality game (K, V_I) and each income source $j \in K$

$$Sh_j(K, V_I) = P(K, V_I) - P(K - \{j\}, V_I)$$

Since we are interested in a consistent decomposition rule, we deduce from Eqs. 2 and 3 that

$$\sum_{j=1}^{k} \varphi_j(K, V_I) = V_I(K) = \sum_{j=1}^{k} \left[P(K, V_I) - P(K - \{j\}, V_I)\right] \tag{5}$$

which gives

$$P(K, V_I) = \frac{1}{k}\left[V_I(K) + \sum_{j=1}^{k} P(K - \{j\}, V_I)\right] \tag{6}$$

This equation proves the existence of P and recalling that $P(\emptyset, V_I) = 0$, we deduce the uniqueness of P by induction. □

Equation 4 gives what we label the Shapley inequality decomposition. It is easy to check that this decomposition rule satisfies Axioms 1–4. For the treatment of an equally distributed factor component we obtain the following results.

Proposition 2

a) *If one decomposes income inequality according to the Shapley decomposition and if inequality is measured by an absolute index, then the contribution of an equally distributed source to total inequality is null.*
b) *If one decomposes income inequality according to the Shapley decomposition and if inequality is measured by a relative index, then the contribution of an equally distributed source to total inequality is negative.*

Proof Let X be any distribution of total incomes from a set of sources $K = \{1,..., k\}$, such that one source, say j, is equally distributed.

a) If inequality is measured by an absolute index, we have $V_I(S) - V_I(S - \{j\}) = 0$ for all $S \in 2^K$.
b) If inequality is measured by a relative index, we have $V_I(S) - V_I(S - \{j\}) < 0$ for all $S \in 2^K$, $S \neq \{j\}$ and for $S = \{j\}$, we have $V_I(\{j\}) - V_I(\emptyset) = 0$. □

For some specific inequality indices we can obtain an easy computable formulation for the Shapley inequality decomposition.

Proposition 3 (Auvray and Trannoy [2]) *If inequality is measured with the variance, then the natural decomposition of the variance coincides with the Shapley decomposition.*

Proof If inequality is measured with the variance, then the computation of the characteristic function gives for $S \subseteq K$:

$$V_{\sigma^2}(S) = \sum_{k\in S} \sigma^2(k) + \sum_{k\in S} \sum_{\substack{k' \in S \\ k' \neq k}} \rho_{kk'}\sigma(k)\sigma(k')$$

and

$$V_{\sigma^2}(S - \{j\}) = \sum_{\substack{k \in S \\ k \neq j}} \sigma^2(k) + \sum_{\substack{k \in S \\ k \neq j}} \sum_{\substack{k' \in S \\ k' \neq k, k' \neq j}} \rho_{kk'}\sigma(k)\sigma(k')$$

thus:

$$V_{\sigma^2}(S) - V_{\sigma^2}(S - \{j\}) = \sigma^2(j) + 2\sum_{k' \neq j} \rho_{jk'}\sigma(j)\,\sigma(k')$$

Then, the contribution of source j to the overall inequality is:

$$Sh_j(K, V_I) = \sum_{\substack{S \subseteq K \\ j \in S}} \frac{(s-1)!\,(k-s)!}{k!}\left[\sigma^2(j) + 2\sum_{k' \neq j} \rho_{jk'}\sigma(j)\,\sigma(k')\right]$$

since

$$\sum_{\substack{S \subseteq K \\ j \in S}} \frac{(s-1)!\,(k-s)!}{k!} = 1$$

and

$$\sum_{\substack{S \subseteq K \\ j \in S,\, k \neq j}} \frac{(s-1)!\,(k-s)!}{k!} = 0.5$$

We conclude that

$$Sh_j(K, V_I) = \sigma^2(j) + \sum_{k' \neq j} \rho_{jk'}\sigma(j)\,\sigma(k') = \mathrm{Cov}(X_j, X).$$

□

Obviously, all the properties[4] satisfied by the Shapley inequality decomposition are obtained at some price. Indeed the Shapley decomposition does not satisfy the property of independence of the level of disaggregation (Axiom 6). The following example illustrates this point.

Example 3 Consider the distribution X of Example 2 and suppose that SA and C are combined into a unique income source Z. Then we have a new distribution X' comprising two factors.

	W	**Z**
1	3	11
2	45	18
3	7	28

If inequality is measured with Gini's index, then we have the following characteristic function

$$V_I(\emptyset) = 0 \;\; V_I(\{W\}) = \frac{8}{45} \;\; V_I(\{Z\}) = \frac{34}{171} \;\; V_I(\{W, Z\}) = \frac{42}{207}$$

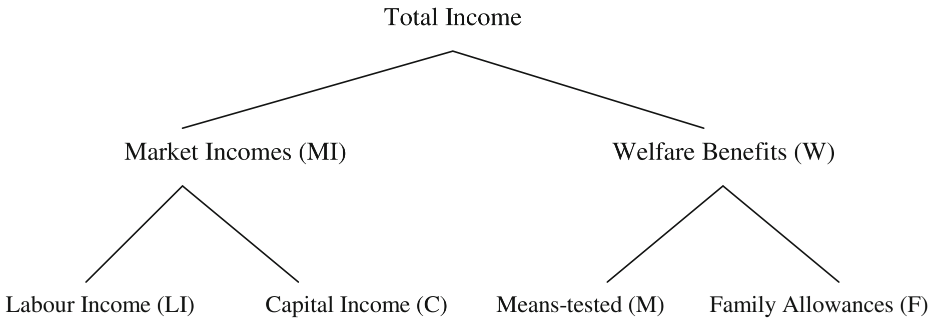
It is easy to check that the contribution of income source W to the total inequality will differ, in this inequality game, from its contribution in the inequality game obtained

[4]The reader can easily check that these propositions hold for equalized inequality games.

in Example 2.[5] In the following section, we develop a second application of values of cooperative games, which improve the Shapley inequality decomposition rule with respect to the request of independence.

5 The Owen decomposition by factor components

In this section we consider a framework in which the different components of total income are decomposed into several subgroups of sources. For the sake of illustration consider the following example: we can suppose that individuals receive income from two sources, say Market Incomes and Welfare Benefits. Furthermore, suppose that the source Market Incomes is decomposed into several sub-sources such as Labour Income and Capital Income, while the source Welfare benefits is decomposed into Means-tested and Family Allowances as illustrated by the following income sources tree.



Consider, for example, the contribution of a source, say Labour Income. Decomposing income inequality using the Shapley value decomposition, the contribution of Labour Income will depend on the number of sub-groups considered in the disaggregation of both Factor Income and Welfare Benefits.

A milder requirement is that the contribution of the source Labour Income would be at least independent of the disaggregation of the source Welfare Benefits. It turns out that this milder requirement of independence of the level of disaggregation leads to the choice of another value of cooperative games proposed by Owen [11].

Some further notations are needed to be introduced. A partition of the different components of total income is the set

$$P_K = \{S_1, \ldots, S_l, \ldots, S_m\} \text{ such that for all } S_h, S_l \in P_K, S_h \cap S_l = \oslash \text{ and } \bigcup_{h=1}^{m} S_h = K$$

Definition 5 Given a zero income inequality game,[6] a two-stage inequality game is a triple (K, P_K, V_I).

[5] In Example 2, the contribution of W is 0.0488, while it is 0.0909 in Example 3.

[6] We have a similar definition for an equalized income game.

Let ϕ_j (K, P_K, V_I) denoting the contribution of source j to the total income inequality. We need to define the inequality game between subsets of sources, namely the one stage inequality game (P_K, V_I). ϕ_l (P_K, V_I) will denote the contribution of the subset of sources S_l to the total income inequality in this specific game. We strengthen here the efficiency property in requiring that it should be satisfied at each stage of the income-source tree. As previously, we require that the sum of the contributions adds up to the amount of total inequality.

Definition 6 Given an inequality index I, a consistent decomposition rule assigns to every two-stage inequality game (K, P_K, V_I) a function $\phi_l(K, P_K, V_I)$, such that

$$\sum_{j=1}^{k} \varphi_j\big(K, P_K, V_I\big) = V_I(K) = I(X) \tag{7}$$

Moreover, we require that the sum of contributions of income sources belonging to the same subset of income sources adds up to the contribution of this specific subset of income sources to the total income inequality in the one stage inequality game.

Definition 7 The decomposition rule satisfies the property of efficiency among subsets of income sources if for all subsets $S_l \in P_K$

$$\sum_{j \in S_1} \varphi_j\big(K, P_K, V_I\big) = \varphi_1\big(P_K, V_I\big)$$

Within the context of two-stage inequality games, the property of marginality required is directly inspired by the one considered previously.

Marginality property The decomposition rule satisfies the marginality property if there exists some function $P : N \times \mathcal{P} \times V_I \to [0, 1]$, with $P(\emptyset,\emptyset,V_I) = 0$ and such that

$$\forall j \in K, \varphi_j\big(K, P_K, V_I\big) = P\big(K, P_K, V_I\big) - P\big(K - \{j\}, P_{K-\{j\}}, V_I\big) \tag{8}$$

Proposition 4 *Let K be a set of income sources, P a partition of K and I an inequality index. A consistent decomposition rule satisfies the property of efficiency among subgroups of sources and the marginality property with respect to some function P if and only if it is given by the Owen formula, namely:*

$$\varphi_j\big(K, P_K, V_I\big) = Ow_j\big(K, P_K, V_I\big)$$

with

$$Ow_j\big(K, P_K, V_I\big) = \sum_{\substack{C \in 2^P \\ S_l \notin C}} \sum_{\substack{S \in S_l \\ j \notin S}} \frac{c!(m-c-1)!s!(s_l-s-1)!}{m!s!} \times \big[V_I(C \cup S \cup \{j\}) - V_I(C \cup S)\big] \tag{9}$$

Proof Let

$$P\big(K,P_K,V_I\big)=\sum_{\substack{C\in 2^P\\ S_l\notin C}}\sum_{\substack{S\in S_l\\ j\notin S}}\frac{c!(m-c-1)!s!(s_l-s-1)!}{m!s!}\big[V_I(C\cup S\cup\{j\})\big]$$

Thus for every inequality game with a partition structure (K, P_K, V_I) and each income source $j \in K$ we have:

$$Ow_j\big(K,P_K,V_I\big)=P\big(K,P_K,V_I\big)-P\big(K-\{j\},P_{K-\{j\}},V_I\big)$$

Furthermore, Owen's value satisfies the property of efficiency among subgroups of sources,[7] thus

$$\varphi_I\big(P_K,V_I\big)=Ow_1\big(P_K,V_I\big)=\sum_{j\in S_1}Ow_j\big(K,P_K,V_I\big)=\sum_{j\in S_1}\varphi_j\big(P_K,V_I\big)$$

In order to prove the uniqueness, note that we are interested in consistent decomposition rule. From Eqs. 4 and 6, we deduce that

$$\sum_{j=1}^{k}\varphi_j\big(P_K,V_I\big)=V_I(K)=\sum_{j=1}^{k}\big[P\big(K,P_K,V_I\big)-P\big(K-\{j\},P_{K-\{j\}},V_I\big)\big]$$

which gives

$$P\big(K,P_K,V_I\big)=\frac{1}{k}\left[V_I(K)+\sum_{j=1}^{k}P\big(K-\{j\},P_{K-\{j\}},V_I\big)\right]$$

This equation proves the existence of P and recalling that $P(\emptyset,\emptyset,V_I)=0$, we deduce the uniqueness of P by induction. □

The reader can easily check that the marginality property is satisfied at each stage of the game. Owen's value assigns to each source $j \in K$ its marginal contribution with respect to a uniform distribution over all orders that are admissible with respect to the considered partition of subgroups of sources.

To understand the difference between the Shapley decomposition and the Owen decomposition, let us consider the previous income-source tree. Using the Shapley value decomposition, we do not take into account the fact that the set of income sources is decomposed into a partition of subgroups of income sources. Then, the contribution of a source, say Labour Income, depends of the level of disaggregation of the remaining part of the total income. Computing the Shapley value of Labour Income, the available coalitions are $\{L, C\}$, $\{L, M\}$, $\{L, F\}$, $\{L, C, M\}$, $\{L, C, F\}$, $\{L, M, F\}$ and $\{L, C, M, F\}$. The Owen value takes into account additional information in the characteristic function of the inequality game in order to compute the contributions of income sources. To compute the Owen value of the income source Labour Income, the only available coalitions are $\{L, C\}$, $\{L, W\}$ and $\{L, C, W\}$. The economic meaning behind the Owen decomposition rule is that some

[7] This property is easily proved by formula (9). For further technical discussion on the Owen's value the reader is referred to Owen [11], Winter [18].

coalitions of sources need not be considered. To illustrate this point, consider the income distribution X associated to the income sources tree in the case of three individuals.

	L	***C***	***M***	***F***
1	3	2	5	3
2	4	3	2	1
3	7	4	1	0

Thus, we have the following distribution for the two subsets of sources (*MI* and W).

	MI	***W***
1	5	8
2	7	3
3	11	1

Assuming that the income inequality is measured by the Gini index, the characteristic function of the zero income inequality game is then

$$V_I(\emptyset) = 0 \quad V_I(\{F\}) = \frac{1}{2} \quad V_I(\{C, M\}) = \frac{4}{51} \quad V_I(\{L, C, F\}) = \frac{1}{27}$$
$$V_I(\{L\}) = \frac{4}{21} \quad V_I(\{L, C\}) = \frac{4}{23} \quad V_I(\{C, F\}) = \frac{2}{39} \quad V_I(\{L, M, F\}) = \frac{4}{39}$$
$$V_I(\{C\}) = \frac{4}{27} \quad V_I(\{L, M\}) = \frac{2}{33} \quad V_I(\{M, F\}) = \frac{7}{18} \quad V_I(\{C, M, F\}) = \frac{10}{63}$$
$$V_I(\{M\}) = \frac{1}{3} \quad V_I(\{L, F\}) = \frac{2}{27} \quad V_I(\{L, C, M\}) = \frac{2}{31} \quad V_I(X) = I(X) = \frac{2}{35}$$

Using the Shapley decomposition rule and considering the zero inequality game (K, V_I) we obtain the following contributions: $Sh_L(K, V_I) = -0.056972$, $Sh_C(K, V_I) = -0.053188$, $Sh_M(K, V_I) = 0.055421$ and $Sh_F(K, V_I) = 0.111882$; while using the Owen decomposition rule and considering the zero inequality game (K, P_K, V_I), we obtain the following contributions: $Ow_L(K, P_K, V_I) = -0.042918$, $Ow_C(K, P_K, V_I) = -0.035999$, $Ow_M(K, P_K, V_I) = -0.023974$, and $Ow_F(K, P_K, V_I) = -0.112086$.

Now consider that for the next year the data set does not allow to make the distinction between M and F, but that the income distribution remains identical to that one of the previous year restricted to the three income sources L, C and W. Then we have a new income distribution X' comprising three factor components:

	L	***C***	***W***
1	3	2	8
2	4	3	3
3	7	4	1

While the number of income sources decreases, the income distribution for the two subsets of sources (MI, W) remains invariant. In this case, the contribution of the income source L for the Shapley decomposition changes as we point out in Example 3. But, considering the two-stage inequality game and using the Owen decomposition rule we obtain the following decomposition results: $Ow_L(K, P_K, V_I) = -0.042918$, $Ow_C(K, P_K, V_I) = -0.035999$ and $Ow_W(K, P_K, V_I) = -0.136059$. Hence, accounting for the partition of income sources, the contributions of L, C and MI are independent of the number of income sources aggregated under the label Welfare Benefits, but depend on the number of income sources belonging to the subset of sources Market Income. This example illustrates that the independence property contained in the Owen decomposition is still weaker than the axiom of independence proposed by Shorrocks [15].

A final remark deals with the extension of the Owen decomposition rule by factor components. We only consider situations within which the set of income sources is decomposed into subgroups of income sources. Obviously, situations where the set of income of sources is decomposed into a level structure, that is, a sequence of partitions of subgroups of sources, can be considered. In such cases, the contribution of a given income source can be determined using a level structure value. Calvo et al. [3] characterize this value with three axioms: consistency, marginality and efficiency. The efficiency property is an extension of the one we use. The basic idea of this property is that the rule according to which every source's contributions are determined is implemented into a r-stage process, where r is the number of levels of the coalition level structure. At each stage, the blocs behave as players and get their contribution that they "share" in a lower stage among their members.

6 Conclusion

Papers in the field of income inequality have developed specific techniques to solve the problem of inequality decomposition by factor components. In this paper, we illustrate how general tools of cooperative game theory can solve this problem. We have showed that the problem is clearly specific, the characteristic function is not superadditive, there are different ways to define it and the more natural way is to consider that the other sources that source S are equally distributed. In particular, we demonstrate that the Shapley inequality decomposition by sources offers some advantages. It is sensitive to the choice of the inequality index, for example the contribution to inequality of an equally distributed factor component is zero when inequality is measured with an absolute index and negative if inequality is measured with a relative index. This property does not preclude the Shapley inequality decomposition to correspond to the natural decomposition of the variance, when inequality is measured with the variance. However, the Shapley inequality decomposition does not satisfy the axiom of independence of the level of disaggregation, but we show how to improve this decomposition rule with respect to the request of independence when the set of income sources is a priori decomposed into a partition of subgroups of sources. As mentioned above, the notion of independence used in this paper is different from that of Shorrocks, even though these two notions of independence tend to coincide as the decomposition of the set of sources becomes finer.

As we argue in the introduction, the natural field of application of the Shapley value and of its extensions seems to be what it is called the decomposition of inequality indices, namely, to decompose the aggregate inequality value into some relevant component contributions. The issues to which this kind of analysis has been applied fall into two broad categories. We have considered an application of the Shapley value to the category of decomposition problem which covers situations within which different components are examined. One possible extension of this paper is to consider an application of the Shapley value to the other category of decomposition problem, which deals with the influence of population subgroups on the total inequality.

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