Does less inequality among households mean less inequality among individuals?*

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Abstract

Consider an income distribution among households of the same size in which individuals, equally needy from the point of view of an ethical observer, are treated unfairly. Individuals are split into two types, these who receive more than one half of the family budget and those who receive less than one half. We look for conditions under which welfare and inequality quasi-orders established at the household level still hold at the individual one. A necessary and sufficient condition for the Generalized Lorenz test is that the income of dominated individuals is a concave function of the household income: individuals of poor households have to stand more together than individuals of rich households. This property also proves to be crucial for the preservation of the Relative and Absolute Lorenz criteria, when the more egalitarian distribution is the poorest. Extensions to individuals heterogeneous in needs and more than two types are also provided.

Key Words: Lorenz dominance, Intra-household inequality, Concavity, Sharing rule.

JEL Codes: D13, D63, D31.

1 Introduction

In modern western democracies, there is not much debate about the fact that the ultimate object of concern for economic policy is the well-being of individuals. When public authorities target social benefits to some specific group of individuals (e.g. children) or assess the impact of such a policy, their action is limited by asymmetries of information about the allocation of resources within the household. The household forms an informational screen between the government and the individuals, since intra-household allocation is considered as a domain of privacy and as such is protected by the law in many societies. A somewhat related informational problem is the fact that the elementary statistical unit remains the household in most data bases. There is at least one case where this veil of ignorance would be innocuous for the appraisal or the design of public policies. As pointed out by Bourguignon and Chiappori [2], if the household behavior is such that the intra-household distribution is optimal for the policy maker, then according to a decentralization device, it is sufficient for the state to ensure the resources allocation problem among families in order to achieve a grand optimum among individuals.

Yet, there is some empirical evidence that this rosy picture is out of this world and that the family, like other institutions, may be unfair in the sense that similar individuals from the policy's maker view point may be discriminated in the allocation of resources within a household. The origins of the literature on intra-household inequality are referred to in Sen [10]. He summarizes a number of studies which argue that girls are discriminated relatively to boys. The relevance of gender disparities has been recognized in The Human Development Report 1995, which introduces two new measures for ranking countries according to their performance in gender equality (see Anand and Sen [1] for more details). Hence, from the point of view of the decision maker, the most common background might be that some unknown intra-household inequality prevails.

Haddad and Kanbur ([5], [7]) provide a first theoretical demonstration of the relative importance of intra-household inequality with respect to inter-household inequality. In their first paper, they show that the neglect of intra-household inequality is likely to lead to an understatement of the *levels* of inequality up to 30%. They also find this problem 'not dramatic' for inequality *comparisons*. More precisely, when intra-household inequality in the two populations is 'sufficiently similar', ignoring intra-household behavior does not reverse the results of inequality comparisons based on a decomposable index of inequality.

Here we deal with the same kind of questions and we are concerned with statements about the evolution of inequality at the individual level which can be inferred from the knowledge of the evolution of inequality at the household stage. In other words, by taking into account the fact that intra-household inequality is unobservable but that some general pattern of discrimination may be postulated, we exhibit cases where knowing the pattern of inter-household discrimination may be sufficient to predict the evolution of overall inequality. A major difference with Haddad and Kanbur analysis is that we are interested in dominance tools like the Lorenz Curve, the Generalized Lorenz curve, with the advantages of robustness of conclusions associated to this approach.

Suppose that all individuals are homogeneous in the sense that either they are endowed with the same capacity of deriving welfare from income in a utilitarian perspective or their claim to obtain a share of the cake is considered to be identical from an ethical point of view. However, they are distinguished by some characteristics such as sex, age, which do not have to play a role in distribution issues. Despite the fact that the allocation within households ought to be equal, we suppose that the actual distribution of resources within households exhibits some inequality. Why this is so, is not described in the model, but we can imagine that the bargaining power is not equal within the types of individuals. The precise sharing rule adopted in each household is not known outside the family. Under this veil of ignorance, we simply postulate that all households use the same rule of sharing resources among its members. This assumption can be justified by arguing that some common cultural factor shapes the internal relation within households in a given society.

We start by focusing on the simplest possible configuration: given a population of households of the same size, each household is composed of two types of individuals. A given type of individual (not necessarily the same in all households) receives a better treatment but the ethical observer does not know how large the unfairness is. This uncertainty is parallel to the uncertainty concerning the degree of concavity of the utility function in traditional social or stochastic dominance analysis. Taking into account the fact that intra-household decisions are biased, the ethical observer would like to know under which conditions about intra-household behavior the results of welfare and inequality comparisons among household income distributions are preserved at the individual level. It turns out that the 'similarity' of behavior across households is not a sufficient condition. Our main result shows that welfare gains at the household level according to the General Lorenz test translate into welfare gains at the individual stage with the same criterion if and only if the households share their resources among their members according to a *concave sharing rule*. In other terms, a necessary and sufficient condition to neglect intra-household inequality, when we are interested in quasi-orderings on income distributions, is that poorest households are the more egalitarian. In a dynamic perspective, one should say that the pattern of intra-household inequality must be pro-cyclic, if the sharing rule remains unchanged over time.

The next section introduces tools used in the evaluation of welfare and inequality and assumptions on intra-household behavior. The preservation of the General Lorenz dominance criterion is studied in Section 3, while results about the preservation of Absolute and Relative Lorenz rankings are the matter of Section 4. Section 5 shows that the main result of the paper still holds under more general assumptions on household composition or intra-household behavior. Section 6 concludes. All proofs are gathered in the Appendix.

2 The setup

2.1 The normative framework

We consider a population composed of n households indexed by $i = 1, ..., n$ (with $n \geq 2$). Let y_i designate the income of the household i. We assume $y_i \in \mathbb{R}_+$ for all $i = 1, ..., n$. Let $\mathbf{y} = (y_1, y_2, ..., y_n)$ be a generic vector of household incomes with average $\mu_{\mathbf{y}} > 0$ and ordered in an increasing way. The feasible set of households' income distributions is denoted by $\mathbb{D} = \left\{\mathbf{y} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\} \;\mid\; y_1 \leq y_2 ... \leq y_n\right\}$. We denote by $\mathbf{e}_n$ the unit vector in $\mathbb{R}^n_+$ and $\mathbf{y} \geq \mathbf{y}'$ means $y_i \geq y'_i$ for all $i = 1, ., n$. We focus on the Generalized Lorenz (GL) criterion for welfare comparisons (see Shorrocks [11]) and on the Relative (RL) and Absolute Lorenz (AL) criteria for inequality comparisons (see Moyes [9]). For the sake of completeness, we recall the definitions.

Definition 1 *Given* $\mathbf{y}, \mathbf{y}' \in \mathbb{D}$,

i) $\mathbf{y}$ *dominates* $\mathbf{y}'$ *according to the Generalized Lorenz criterion, denoted by* $\mathbf{y} \succcurlyeq_{GL} \mathbf{y}'$, *if*

$$\frac{1}{n}\sum_{i=1}^{k} y_i \geq \frac{1}{n}\sum_{i=1}^{k} y'_i \quad \textit{for } k = 1, .., n.$$

ii) $\mathbf{y}$ *dominates* $\mathbf{y}'$ *according to the Relative Lorenz criterion, denoted by* $\mathbf{y} \succcurlyeq_{RL} \mathbf{y}'$, *if*

$$\frac{1}{n}\sum_{i=1}^{k} \frac{y_i}{\mu_{\mathbf{y}}} \geq \frac{1}{n}\sum_{i=1}^{k} \frac{y'_i}{\mu_{\mathbf{y}'}}, \quad \textit{for } k = 1, ., n.$$

iii) $\mathbf{y}$ *dominates* $\mathbf{y}'$ *according to the Absolute Lorenz criterion, denoted by* $\mathbf{y} \succcurlyeq_{AL} \mathbf{y}'$, *if*

$$\frac{1}{n}\sum_{i=1}^{k} (y_i - \mu_{\mathbf{y}}) \geq \frac{1}{n}\sum_{i=1}^{k} (y'_i - \mu_{\mathbf{y}'}), \quad \textit{for } k = 1, .., n.$$

Households are composed of s individuals. Let $\mathbf{x} = (x_1, x_2, ..., x_{sn})$ be a generic vector of positive individual incomes ordered in an increasing way. Up to Section 5, we assume that individuals are identical from an ethical point of view. In the dominance approach, this assumption is translated by posing that all individuals have the same utility function u. According to an utilitarian social welfare function, the welfare associated to an individual income distribution $\mathbf{x}$ is larger than the welfare associated to the income distribution $\mathbf{x}'$ if $\sum_{j=1}^{sn} u(x_j) \geq \sum_{j=1}^{sn} u(x'_j)$. It is well known (see Shorrocks [11]) that $\mathbf{x} \succcurlyeq_{GL} \mathbf{x}'$ if and only if the above inequality holds for all the class of non-decreasing and concave utility functions u.

2.2 Intra-household allocation

We assume that in every household i, labor and non labor incomes of different individuals are pooled to form the household income y_i, which is then shared among the household members. The income devoted to each individual is supposed to be a good proxy for her or his well-being, an assumption which may be accepted in the absence of family public goods. We assume that individuals are treated in an asymmetric way. More precisely, we suppose that each household is composed of two types of individuals, the *dominant* and the *dominated* ones. In our simplest and favorite example, the couple, there is one person of each type, but the framework is sufficiently general to encompass more complex family structures as tribes or even more to figure out the case of large groups like cities, regions or nations. Each dominated individual receives at most an income share equal to the share received by a dominant individual. Thus dominant individuals are the ‘rich’ within the household and the dominated are the ‘poor’. Moreover, it is assumed that each household is composed of the same number d of dominant individuals and δ of dominated ones. Let p_i be the amount received by each dominated individual in household i. We assume that p_i is determined according to a *sharing function* of the household income y_i

$$p_i = f_p^i(y_i),$$

which represents a reduced form of the intra-household decision making. The amount r_i received by each dominant is consequently defined by $r_i = f_r^i(y_i) = \dfrac{y_i - \delta f_p^i(y_i)}{d}$.

Given a vector $\mathbf{y}$ of household incomes, $\mathbf{p}(\mathbf{y}) = (p_1, .., p_j, .., p_{\delta n})$ designates the income vector for dominated individuals, $\mathbf{r}(\mathbf{y}) = (r_1, ..., r_j, ..., r_{dn})$ the income vector for dominant individuals, and $\mathbf{x}(\mathbf{y}) = (\mathbf{p}(\mathbf{y}), \mathbf{r}(\mathbf{y}))$, rearranged in an increasing way.

We suppose that the sharing functions f_p^i are the same among households, that is, a common bias due to a social norm induces a homogeneous intra-household discrimination in the population considered.

Assumption 1 *The functions* $f_p^i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ *are identical across households and such that* $f_p(y) \leq \dfrac{1}{s} y$.

Let us designate by $\mathcal{F}$ the class of functions satisfying Assumption **1** and by $\mathcal{C} \subset \mathcal{F}$ the class of continuous, non-decreasing and concave sharing functions.

3 The main result

Our main theorem identifies the condition on the sharing function which is necessary and sufficient to preserve the GL dominance relation from the household level to the individual one.

Theorem 1 $f_p \in \mathcal{C} \iff [\mathbf{y} \succcurlyeq_{GL} \mathbf{y}' \implies \mathbf{x}(\mathbf{y}) \succcurlyeq_{GL} \mathbf{x}(\mathbf{y}'),$ *for all* $\mathbf{y}, \mathbf{y}' \in \mathbb{D}]$.

The intuition behind Theorem 1 is the following. It is well known that $\mathbf{y} \succcurlyeq_{GL} \mathbf{y}'$ if and only if $\mathbf{y}$ can be obtained from $\mathbf{y}'$ by a finite sequence of *progressive transfers* (also named

Pigou-Dalton transfers) and *increments* (see Marshall and Olkin [8] C.6, p. 28 and A.9.a, p. 123). When the sharing function is concave, a progressive transfer between households implies a 'double dividend' on social welfare valued at the individual level. Indeed, a transfer from a richer family to a poorer one becomes a transfer from a less egalitarian household to a more egalitarian one as well. An 'intra-household dividend' supplements the traditional 'inter-household dividend'. The Figure 1 illustrates the impact of a transfer of amount Δ from a couple with an initial income y_2 to a couple with an initial income y_1, with $y_1 < y_2$.

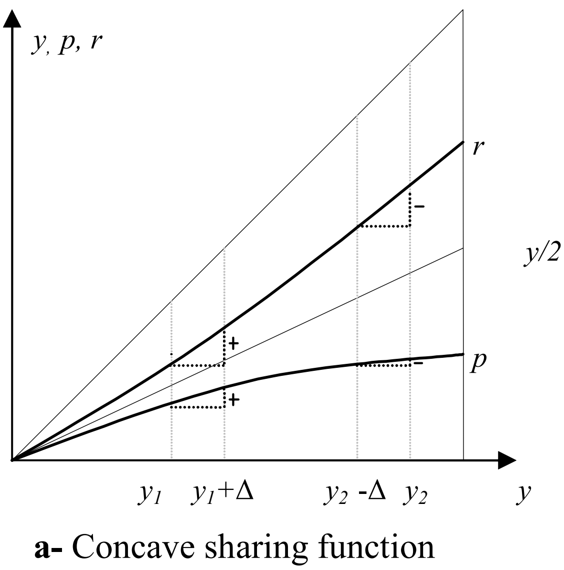

Figure 1: Effect of a progressive transfer between households

The progressive transfer between households induces three progressive transfers among individuals (see Panel **a**). The dominated of the poor family receives a transfer from the two individuals of the rich family. Moreover, the dominant of the rich family loses at the benefit of the poor household's dominant.

An opposite case with a convex sharing function is represented in Panel **b**: the same progressive transfer among households generates an ambiguous effect. The dominant of the poor family receives a *progressive* transfer from his counterpart of the rich household and a *regressive* one from the dominated of the rich household. Then, the social welfare may improve or get worse, depending on the degree of concavity of the individual utility function.

4 Inequality comparisons

In this section, we focus on inequality criteria, which neutralize the differences of the size of the cake.

We start with the relative point of view, where concavity of the sharing function is not sufficient to obtain preservation of the relative Lorenz criterion. Indeed, consider two societies where the more egalitarian according to the relative Lorenz criterion is also the richest on average. Hence, the more egalitarian may be obtained from the less egalitarian by a finite sequence of Pigou-Dalton transfers and increments. On the one hand, the double dividend generated by progressive transfers which we alluded to in Section 3, is still operating. On the other hand, the increments make the households richer. Since a concave sharing function makes rich households more unequal than the poor, increments have a regressive impact on the distribution among individuals. This effect may offset the progressive effect of Pigou-Dalton transfers. Imposing a lower mean for the more egalitarian distribution prevents this outcome to occur.

Corollary 1 $f_p \in \mathcal{C} \iff [\forall\ \mathbf{y}, \mathbf{y}' \in \mathbb{D},\ \textit{with}\ \mu_{\mathbf{y}} \leq \mu_{\mathbf{y}'},\ \mathbf{y} \succcurlyeq_{RL} \mathbf{y}' \implies \mathbf{x}(\mathbf{y}) \succcurlyeq_{RL} \mathbf{x}(\mathbf{y}')]$.

If the 'RL dominant' distribution of household incomes has a higher mean than the 'RL dominated' one, nothing can be immediately concluded about the inequality at the individual level. Nevertheless, in this case, RL dominance implies GL dominance and, via Theorem 1, a higher welfare at the individual level. From a policy point of view, we conclude that, if the concave sharing function remains stable over time, a more and more wealthy society will have to adopt a more and more redistributive policy between households in order to stabilize the level of inequality among individuals.

A further consequence of Corollary 1 concerns the effects of a *progressive taxation* at the *household* level on *individual* inequality. A well-known result, due to Jacobsson [6], states that any after-tax income distribution dominates in the RL sense the before-tax income distribution if and only if the tax system is progressive everywhere. Since the mean of the post-tax household income distribution is lower than the mean of the pre-tax income distribution, Corollary 1 applies. We conclude that when the sharing function is concave, a progressive taxation schedule on household incomes leads to a lower inequality at the individual level (in the sense of the RL dominance).

Now the same kind of questions may be investigated for the absolute Lorenz criterion. We get the analogue of Corollary 1, when the more egalitarian distribution is also the less wealthy one.

Corollary 2 $f_p \in \mathcal{C} \iff [\forall\ \mathbf{y}, \mathbf{y}' \in \mathbb{D},$ *with* $\mu_{\mathbf{y}} \leq \mu_{\mathbf{y}'}, \mathbf{y} \succcurlyeq_{AL} \mathbf{y}' \implies \mathbf{x}(\mathbf{y}) \succcurlyeq_{AL} \mathbf{x}(\mathbf{y}')]$.

5 Extensions

5.1 More than two types

Without loss of generality, we analyze the simplest case of three types in households composed of three individuals. A hierarchy prevails among households, where the dominated always receives less than the median individual, while the share of the dominant always exceeds one third. For notational convenience, f_p (respectively, f_r) is still the dominated (dominant) sharing function and f_m describes the income received by the median individual. f_p satisfies a similar assumption to Assumption 1. $\mathcal{F}$ and $\mathcal{C}$ keep their meaning in this context. We now introduce the *group sharing function* f_g, which gives the income of the group of two first individuals; hence f_g satisfies $f_g = f_p + f_m$. Let $\mathcal{F}^g$ (respectively $\mathcal{C}^g$) designates the set of (resp. concave) group sharing functions.

Proposition 1 *Let* $f_g \in \mathcal{F}^g$, $f_p \in \mathcal{F}$ *and* f_m *non-decreasing.*

$$f_p \in \mathcal{C} \text{ and } f_g \in \mathcal{C}^g \iff [\mathbf{y} \succcurlyeq_{GL} \mathbf{y}' \implies \mathbf{x}(\mathbf{y}) \succcurlyeq_{GL} \mathbf{x}(\mathbf{y}'), \text{ for all } \mathbf{y}, \mathbf{y}' \in \mathbb{D}] .$$

We obtain a 'chain condition', which may easily be extended to more general household structures. If we can rank the s individuals living in a household (or in a tribe) according to their income, and if such a 'hierarchy' is unaffected by the amount of household income, then the concavity of all partial sums of the k poorest individuals for $k = 1, ..., s$ is necessary and sufficient to get the preservation of the GL test.[1]

5.2 Differentiation within a type

Let us consider households with two different dominated individuals and a dominant one. The individual sharing functions f_{p^1} and f_{p^2} describe the income received by the two dominated individuals and they satisfy a similar assumption than Assumption 1, namely, they start from 0 and respect $f_{p^1}(y), f_{p^2}(y) \leq \frac{1}{3}y$.

A complete ranking between the two dominated individuals is not required to obtain a generalization of our main result. For a given household income interval, individual 1 may be the most unfairly treated, while the opposite prevails for some other household income bracket. Let us define the lower contour set of f_{p^j} as $\mathcal{L}_{f_{p^j}} = \left\{(y, x_j) \in \mathbb{R}^2_+ \mid f_{p^j}(y) \geq x_j\right\}$ for $j = 1, 2$. Then, Proposition 1 provides us a preservation result when f_p is replaced by the frontier of the intersection of the two lower contour sets $\mathcal{L}_{f_{p^1}}, \mathcal{L}_{f_{p^2}}$, that gives the part of household income received by the poorest individual in any circumstances.

5.3 Individuals heterogeneous in needs

We now consider the case of a population of couples comprising a *normal* (type 1) and a *needy* individual (type 2), which is the neediest one. An equivalence scale is a scalar $e > 1$ so

[1] The proofs of the results stated in Section 5 are available in the working paper version of the present work: http://www.vcharite.univ-mrs.fr/idep/document/dt/dt0407.pdf.

that the *equivalent income* $\frac{x_2}{e}$ is assumed to be directly comparable with x_1. A fair division of the household income would allocate $\frac{y}{1+e}$ to the normal individual which is assumed to be the dominated individual. Let $\mathcal{F}^1$ (respectively $\mathcal{C}^1$) be the class of (concave) sharing functions such that: $f_p^1(0) = 0$ and $f_p^1(y) \leq \frac{y}{1+e}$. Let $\widehat{\mathbf{x}}$ designate the distribution of equivalent incomes of individuals ordered in an increasing way. In this framework, we use the GL test applied to the distribution of *equivalent incomes* across *equivalent individuals* (individuals weighted by their respective equivalent scale) as a criterion for welfare analysis (see Ebert [4]). Thus, the k^{th} coordinates of this *Equivalent Generalized Lorenz* (EGL) curve are

$$\frac{\sum_{i=1}^{k}\omega_i}{n(1+e)}, \ \frac{\sum_{i=1}^{k}\omega_i\hat{x}_i}{n(1+e)}, \text{ for } k = 1, ...2n \tag{1}$$

with the weights $\omega_i = 1$ (respectively e) if i is of type 1 (resp. 2). We designate by $\widehat{\mathbf{x}}(\mathbf{y}) \succcurlyeq_{EGL} \widehat{\mathbf{x}}(\mathbf{y}')$ the dominance according to the EGL curve.

Corollary 3 *Let* $f_p^1 \in \mathcal{F}^1$.

$$f_p^1 \in \mathcal{C}^1 \iff \left[\mathbf{y} \succcurlyeq_{GL} \mathbf{y}' \implies \widehat{\mathbf{x}}(\mathbf{y}) \succcurlyeq_{EGL} \widehat{\mathbf{x}}(\mathbf{y}'), \ \textit{for all } \mathbf{y}, \mathbf{y}' \in \mathbb{D}\right].$$

6 Concluding remarks

This investigation about the impact of the intra-household inequality on the overall inequality sheds light on the properties of the sharing function. It describes how the part devoted to the more disadvantaged changes as the household income increases along. Concavity of the sharing functions allows to preserve the General Lorenz dominance between any couple of distributions, the Relative and Absolute Lorenz dominance when the more egalitarian distribution is the less wealthy one.

Deeper extensions will relax basic assumptions of the present model. It would be interesting to consider a population composed of families with different size, for instance couples and singles. A substantial improvement would be equally provided by the introduction of family public goods.

In this paper, we resort to a non-structural model of the household: the sharing function is only a reduced form which is compatible with several models of the household behavior. Another direction of research is to explore the micro-economic foundations of concave sharing functions.

At this stage, we do not know the plausibility of the concavity condition in describing the behavior of households in any society. Still, it has the advantage to be served as a testable restriction in an econometric model of the household. In Couprie et al.[3], we use non-parametric methods to estimate the shape of the sharing functions on French data.

7 Appendix

We first state a lemma that will be used in the proof of Theorem 1.

Lemma 1 *Let $u : \mathbb{R} \to \mathbb{R}$ and $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous non-decreasing functions. Let $w : \mathbb{R}_+ \to \mathbb{R}$ be the composite function defined by:*

$$w(y) = \delta u[g(y) - h(y)] + du[g(y) + \frac{\delta}{d}h(y)],$$

with δ and d strictly positive scalars. If u and g are concave and h convex, then w is concave.

Proof. We proceed by *contradiction.* Suppose that w is not concave. Hence, using a result on concavity due to Yaari (see [12], Lemma p.1184) and the concavity of u, the following inequalities hold for some $\alpha, \beta \in \mathbb{R}_+$, with $\alpha < \beta$

$$2\left[\delta u\left(g(\tfrac{\alpha+\beta}{2}) - h(\tfrac{\alpha+\beta}{2})\right) + du\left(g(\tfrac{\alpha+\beta}{2}) + \tfrac{\delta}{d}h(\tfrac{\alpha+\beta}{2})\right)\right] <$$

$$\delta u\left[g(\alpha) - h(\alpha)\right] + du\left[g(\alpha) + \tfrac{\delta}{d}h(\alpha)\right] + \delta u\left[g(\beta) - h(\beta)\right] + du\left[g(\beta) + \tfrac{\delta}{d}h(\beta)\right]$$

$$\leq 2\left[\delta u\left(\tfrac{g(\alpha)+g(\beta)}{2} - \tfrac{h(\alpha)+h(\beta)}{2}\right) + du\left(\tfrac{g(\alpha)+g(\beta)}{2} + \tfrac{\delta}{d}\tfrac{h(\alpha)+h(\beta)}{2}\right)\right].$$

Concavity of g and non-decreasingness of u imply

$$\delta u\left(g(\tfrac{\alpha+\beta}{2})-h(\tfrac{\alpha+\beta}{2})\right)+du\left(g(\tfrac{\alpha+\beta}{2})+\tfrac{\delta}{d}h(\tfrac{\alpha+\beta}{2})\right)<$$
$$\delta u\left(g(\tfrac{\alpha+\beta}{2})-\tfrac{h(\alpha)+h(\beta)}{2}\right)+du\left(g(\tfrac{\alpha+\beta}{2})+\tfrac{\delta}{d}\tfrac{h(\alpha)+h(\beta)}{2}\right). \quad (2)$$

By rearranging the terms of (2), we obtain

$$\delta\left[u\left(b_1\right)-u\left(a_1\right)\right]<d\left[u\left(b_2\right)-u\left(a_2\right)\right], \quad (3)$$

where

$a_1=g(\frac{\alpha+\beta}{2})-\frac{h(\alpha)+h(\beta)}{2}$; $b_1=g(\frac{\alpha+\beta}{2})-h(\frac{\alpha+\beta}{2})$;

$a_2=g(\frac{\alpha+\beta}{2})+\frac{\delta}{d}h(\frac{\alpha+\beta}{2})$; $b_2=g(\frac{\alpha+\beta}{2})+\frac{\delta}{d}\frac{h(\alpha)+h(\beta)}{2}$.

Observe that $b_2-a_2=\frac{\delta}{d}(b_1-a_1)$. Moreover, since h is convex, $a_1\leq b_1<a_2\leq b_2$. If $a_1=b_1$, then $a_2=b_2$ and (3) is impossible. Thus $a_1<b_1<a_2<b_2$. Dividing both terms of (3) by (b_1-a_1), we get $\frac{u(b_1)-u(a_1)}{b_1-a_1}<\frac{u(b_2)-u(a_2)}{b_2-a_2}$, which contradicts the concavity of u (see Marshall and Olkin [8], Proposition B.3.a, p.447). ■

Proof of Theorem 1. $f_p\in\mathcal{C}\implies[\mathbf{y}\succcurlyeq_{GL}\mathbf{y}'\implies\mathbf{x}(\mathbf{y})\succcurlyeq_{GL}\mathbf{x}(\mathbf{y}'),\text{ for all }\mathbf{y},\mathbf{y}'\in\mathbb{D}]$.

Let $f_p\in\mathcal{C}$ and consider $\mathbf{y},\mathbf{y}'\in\mathbb{D}$ such that $\mathbf{y}\succcurlyeq_{GL}\mathbf{y}'$. We prove that $\sum_{j=1}^{sn}u(x_j)\geq\sum_{j=1}^{sn}u(x'_j)$ for all u non-decreasing and concave, which implies $\mathbf{x}(\mathbf{y})\succcurlyeq_{GL}\mathbf{x}(\mathbf{y}')$.

Let $w(y_i)=\delta u(f_p(y_i))+du(f_r(y_i))$ designate the sum of individual utilities within the household i. Then $\sum_{j=1}^{sn}u(x_j)=\sum_{i=1}^{n}w(y_i)$ and $\sum_{j=1}^{sn}u(x'_j)=\sum_{i=1}^{n}w(y'_i)$. We substitute $\frac{y_i}{s}-\psi(y_i)$ for $f_p(y_i)$ and $\frac{y_i}{s}+\frac{\delta}{d}\psi(y_i)$ for $f_r(y_i)$. Since $f_p\in\mathcal{C}$, then ψ is convex. We obtain $w(y_i)=\delta u\left[\frac{y_i}{s}-\psi(\frac{y_i}{s})\right]+du\left[\frac{y_i}{s}+\frac{\delta}{d}\psi(\frac{y_i}{s})\right]$. Applying Lemma 1, by posing $g(y)=\frac{y}{s}$, $h(y)=\psi(\frac{y_i}{s})$ and assuming u non-decreasing and concave, we get w concave. It is easy to see that w is non-decreasing. Since $\mathbf{y}\succcurlyeq_{GL}\mathbf{y}'$, then $\sum_{i=1}^{n}w(y_i)\geq\sum_{i=1}^{n}w(y'_i)$ and therefore $\sum_{j=1}^{sn}u(x_j)\geq\sum_{j=1}^{sn}u(x'_j)$. The reasoning holds for any non-decreasing and concave u.

$[\mathbf{y}\succcurlyeq_{GL}\mathbf{y}'\implies\mathbf{x}(\mathbf{y})\succcurlyeq_{GL}\mathbf{x}(\mathbf{y}'),\text{ for all }\mathbf{y},\mathbf{y}'\in\mathbb{D}]\implies f_p\in\mathcal{C}$.

We prove it in three steps. Notice that each step is useful in proving the following one.

Step 1: f_p must be non-decreasing. Let us suppose by contradiction that there exist some positive scalars a, b such that $a < b$ and $f_p(a) > f_p(b)$. Choosing $\mathbf{y} = (0, .., 0, b)$ and $\mathbf{y}' = (0, .., 0, a)$ trivially entails $\mathbf{y} \succcurlyeq_{GL} \mathbf{y}'$, but $\sum_{j=1}^{sn-d} x_j < \sum_{j=1}^{sn-d} x'_j$. Hence, $\mathbf{x}(\mathbf{y}) \succcurlyeq_{GL} \mathbf{x}(\mathbf{y}')$ is false.

Step 2: f_p must be continuous. Let us consider $\mathbf{y} = (0, .., 0, a, b)$ and $\mathbf{y}' = (0, .., 0, a + b)$. It is easy to see that $\mathbf{y} \succcurlyeq_{GL} \mathbf{y}'$ for any positive a and b. In order to secure $\mathbf{x}(\mathbf{y}) \succcurlyeq_{GL} \mathbf{x}(\mathbf{y}')$, we need $\sum_{j=1}^{sn-d} x_j \geq \sum_{j=1}^{sn-d} x'_j$. Then f_p must satisfy the following property:

$$\delta \left[f_p(a) + f_p(b)\right] + df_r(a) \geq \delta f_p(a + b) \text{ for all positive } a, b. \tag{4}$$

We can rewrite (4) as: $f_p(b + a) - f_p(b) \leq \frac{1}{\delta}a$, for all positive a, b. Given that f_p is non-decreasing, it is bound to be a Lipschitzian function.

Step 3: f_p must be concave. Assume by contradiction that f_p is not concave. From Yaari's Lemma quoted above (which requires the continuity of f_p) it follows that for some $y^* \in \mathbb{R}_{++}$, there exists $\zeta > 0$ such that, for every ε with $0 < \varepsilon < \zeta$

$$2f_p\left(y^*\right) < f_p(y^* - \varepsilon) + f_p(y^* + \varepsilon). \tag{5}$$

Furthermore, (5) combined with $f_p(y^*) \leq \frac{1}{s}y^*$ implies $f_p(y^*) - f_r(y^*) < 0$.[2] Then, by continuity, there exists $\bar{\zeta} > 0$ such that, for every ε satisfying $0 < \varepsilon < \bar{\zeta}$, (5) holds and

$$f_p(y^* + \varepsilon) < f_r(y^* - \varepsilon). \tag{6}$$

We now choose $\mathbf{y} = (0, .., 0, y^*, y^*)$ and $\mathbf{y}' = (0, .., 0, y^* - \varepsilon, y^* + \varepsilon)$. By construction, $\mathbf{y} \succcurlyeq_{GL} \mathbf{y}'$. From (5), we deduce $f_p\left(y_{n-1}\right) + f_p\left(y_n\right) < f_p(\, y'_{n-1}) + f_p(y'_n)$ which gives, combined with (6), $\sum_{j=1}^{sn-2d} x_j < \sum_{j=1}^{sn-2d} x'_j$. Hence, $\mathbf{x}(\mathbf{y}) \succcurlyeq_{GL} \mathbf{x}(\mathbf{y}')$ is contradicted. ■

[2]Indeed, by assumption $f_p(y^*) \leq f_r(y^*)$. Suppose now that $f_p(y^*) - f_r(y^*) = 0$. This is equivalent to $f_p(y^*) = \frac{1}{s}y^*$. From (5) and $f_p(y) \leq \frac{1}{s}y \;\forall y$, we get $\frac{2}{s}y^* < f_p(y^* - \varepsilon) + f_p(y^* + \varepsilon) \leq \frac{2}{s}y^*$, which is impossible.

Proof of Corollary 1. The necessity part may easily obtained as in Theorem 1. Then, it remains to prove: $f_p \in \mathcal{C} \Longrightarrow [\forall\ \mathbf{y}, \mathbf{y}' \in \mathbb{D},\ \text{with}\ \mu_{\mathbf{y}} \leq \mu_{_{\mathbf{y}'}}, \mathbf{y} \succcurlyeq_{RL} \mathbf{y}' \implies \mathbf{x}(\mathbf{y}) \succcurlyeq_{RL} \mathbf{x}(\mathbf{y}')]$.

We divide the proof in two steps, by introducing the distribution $\mathbf{y}'' = \alpha\mathbf{y}'$, where $\alpha = \frac{\mu_{\mathbf{y}}}{\mu_{\mathbf{y}'}}$.

Step 1: $f_p \in \mathcal{C} \Longrightarrow [\forall \mathbf{y}, \mathbf{y}' \in \mathbb{D},\ \text{with}\ \mu_{\mathbf{y}} \leq \mu_{_{\mathbf{y}'}}, \mathbf{y} \succcurlyeq_{RL} \mathbf{y}' \implies \mathbf{x}(\mathbf{y}) \succcurlyeq_{RL} \mathbf{x}(\mathbf{y}'')]$

Since $\mathbf{y} \succcurlyeq_{RL} \mathbf{y}'$ by assumption and $\mathbf{y}'' \sim_{RL} \mathbf{y}'$ by construction, we get $\mathbf{y} \succcurlyeq_{RL} \mathbf{y}''$. We also have $\mu_{\mathbf{y}''} = \mu_{\mathbf{y}}$, then $\mathbf{y} \succcurlyeq_{GL} \mathbf{y}''$. Then, from Theorem 1, we know that f_p concave implies $\mathbf{x}(\mathbf{y}) \succcurlyeq_{GL} \mathbf{x}(\mathbf{y}'')$. Dividing both individual distributions by $\frac{\mu_{\mathbf{y}}}{s}$, we get $\mathbf{x}(\mathbf{y}) \succcurlyeq_{RL} \mathbf{x}(\mathbf{y}'')$.

Step 2: $f_p \in \mathcal{C} \Longrightarrow [\forall \mathbf{y} \in \mathbb{D}$ and $\forall \alpha \in (0,1],\ \ \mathbf{x}(\mathbf{y}'') \succcurlyeq_{RL} \mathbf{x}(\mathbf{y}')]$.

Let $\mathbf{e}_\delta$ and $\mathbf{e}_d$ be the unitary vectors belonging to $\mathbb{R}^\delta$ and $\mathbb{R}^d$, respectively. We claim that for every household i,

$$(f_p(\alpha y_i')\mathbf{e}_\delta,\ f_r(\alpha y_i')\mathbf{e}_d) \succcurlyeq_{GL} (\alpha f_p(y_i')\mathbf{e}_\delta,\ \alpha f_r(y_i')\mathbf{e}_d)\,. \tag{7}$$

Indeed, suppose by contradiction that, for some integer $1 \leq l \leq d-1$,

$$\delta f_p(\alpha y_i') + l f_r(\alpha y_i') < \delta\alpha f_p(y_i') + l\alpha f_r(y_i'). \tag{8}$$

This implies $f_r(\alpha y_i') < \alpha f_r(y_i')$, since the concavity of the sharing function entails $f_p(\alpha y) \geq \alpha f_p(y)$, $\forall \alpha \in [0,1]$ and $\forall y \geq 0$ (see Marshall and Olkin, Proposition B. 9, p.453). By adding $(d-l)\, f_r(\alpha y_i')$ and $(d-l)\,\alpha f_r(y_i')$ respectively to the LHS and the RHS of (8), we get $y_i' < y_i'$, which is impossible. Since (7) holds for any household i, by applying Proposition A.7 (iii), p.121 of Marshall and Olkin [8], we obtain $\mathbf{x}(\alpha\mathbf{y}') \succcurlyeq_{GL} \alpha\mathbf{x}(\mathbf{y}')$. Dividing both vectors by $\alpha\frac{\mu_y'}{s}$, we get: $\dfrac{\mathbf{x}(\alpha\mathbf{y}')}{\alpha\frac{\mu_y'}{s}} \succcurlyeq_{GL} \dfrac{\mathbf{x}(\mathbf{y}')}{\frac{\mu_y'}{s}}$, which is equivalent to $\mathbf{x}(\mathbf{y}'') \succcurlyeq_{RL} \mathbf{x}(\mathbf{y}')$.

By transitivity, we conclude $\mathbf{x}(\mathbf{y}) \succcurlyeq_{RL} \mathbf{x}(\mathbf{y}')$. ■

In the last proof, we make use of the *Lorenz* criterion: $\mathbf{y} \succcurlyeq_L \mathbf{y}'$, if $\mathbf{y} \succcurlyeq_{GL} \mathbf{y}'$ and $\mu_{\mathbf{y}} = \mu_{\mathbf{y}'}$.

Proof of Corollary 2. The necessity part may easily be obtained as in Theorem 1. Then, it remains to prove: $f_p \in \mathcal{C} \Longrightarrow [\forall\ \mathbf{y}, \mathbf{y}' \in \mathbb{D},\ \text{with}\ \mu_{\mathbf{y}} \leq \mu_{_{\mathbf{y}'}}, \mathbf{y} \succcurlyeq_{AL} \mathbf{y}' \implies \mathbf{x}(\mathbf{y}) \succcurlyeq_{AL} \mathbf{x}(\mathbf{y}')]$.

We divide the proof in two steps, by introducing the distribution $\mathbf{y}'' = \mathbf{y} + \alpha \mathbf{e}_n$, where $\alpha = \mu_{\mathbf{y}'} - \mu_{\mathbf{y}}$.

Step 1: $f_p \in \mathcal{C} \Longrightarrow [\forall \mathbf{y}, \mathbf{y}' \in \mathbb{D}$, with $\mu_{\mathbf{y}} \leq \mu_{\mathbf{y}'}$, $\mathbf{y} \succcurlyeq_{AL} \mathbf{y}' \implies \mathbf{x}(\mathbf{y}'') \succcurlyeq_{AL} \mathbf{x}(\mathbf{y}')]$.

Given the assumption $\mathbf{y} \succcurlyeq_{AL} \mathbf{y}'$ and since $\mathbf{y} \sim_{AL} \mathbf{y}''$ by construction, then $\mathbf{y}'' \succcurlyeq_{AL} \mathbf{y}'$. We also have $\mu_{\mathbf{y}''} = \mu_{\mathbf{y}'}$, which implies $\mathbf{y}'' \succcurlyeq_{GL} \mathbf{y}'$. From Theorem 1, f_p concave implies $\mathbf{x}(\mathbf{y}'') \succcurlyeq_{GL} \mathbf{x}(\mathbf{y}')$. By subtracting the vector $\frac{\mu_{\mathbf{y}'}}{s} \mathbf{e}_{sn}$ from both income distributions, we get $\mathbf{x}(\mathbf{y}'') \succcurlyeq_{AL} \mathbf{x}(\mathbf{y}')$.

Step 2: $f_p \in \mathcal{C} \Longrightarrow [\forall \mathbf{y} \in \mathbb{D}$ and $\forall \alpha \geq 0, \ \ \mathbf{x}(\mathbf{y}) \succcurlyeq_{AL} \mathbf{x}(\mathbf{y}'')]$.

Due to the concavity of f_p, it is easy to show that $f_p(y_i + \alpha) - f_p(y_i) \leq \frac{\alpha}{s}$, $\forall \alpha > 0$, which may be rewritten as $f_p(y_i) - \frac{\mu_{\mathbf{y}}}{s} \leq f_p(y_i + \alpha) - \frac{\mu_{\mathbf{y}+\alpha \mathbf{e}_n}}{s}$. By reasoning as in the proof of Corollary 1, we get, for any household i,

$$\left(\left(f_p(y_i) - \tfrac{\mu_{\mathbf{y}}}{s}\right) \mathbf{e}_\delta, \left(f_r(y_i) - \tfrac{\mu_{\mathbf{y}}}{s}\right) \mathbf{e}_d\right) \succcurlyeq_{GL} \left(\left(f_p(y_i+\alpha) - \tfrac{\mu_{\mathbf{y}+\alpha \mathbf{e}_n}}{s}\right) \mathbf{e}_\delta, \ \left(f_r(y_i+\alpha) - \tfrac{\mu_{\mathbf{y}+\alpha \mathbf{e}_n}}{s}\right) \mathbf{e}_d\right). \tag{9}$$

Let $\check{\mathbf{x}}(\mathbf{y})$ designate the centered vector of individual incomes. Then, we can deduce from (9) that $\check{\mathbf{x}}(y_i) \succcurlyeq_L \check{\mathbf{x}}(y_i + \alpha)$ for any i. Proposition A.7 (i), p. 121 of Marshall and Olkin [8] gives $\check{\mathbf{x}}(\mathbf{y}) \succcurlyeq_L \check{\mathbf{x}}(\mathbf{y}+\alpha \mathbf{e}_n)$, that is $\mathbf{x}(\mathbf{y}) \succcurlyeq_{AL} \mathbf{x}(\mathbf{y}'')$.

By transitivity, we conclude $\mathbf{x}(\mathbf{y}) \succcurlyeq_{AL} \mathbf{x}(\mathbf{y}')$. ∎

References

[1] S. Anand, A. Sen, Gender inequality in human development: theory and measurement, Background paper for The Human Development Report 1995, UNDP, Oxford Univ. Press, Oxford 1995.

[2] F. Bourguignon, P.A. Chiappori, The collective approach to household behavior, in: R. Blundell, I. Preston, I. Walker (Eds.), The Measurement of Household Welfare, Cambridge Univ. Press, Cambridge 1994, pp. 70−85.

[3] H. Couprie, E. Peluso and A. Trannoy, From households to individuals: Does the Lorenz criterion still work? Theory and evidence from French data, mimeo (2005), GREQAM-IDEP.

[4] U. Ebert, Using equivalent incomes of equivalent adults to rank income distributions, Soc. Choice Welfare 16 (1999), 233−258.

[5] L. Haddad, R. Kanbur, How serious is the neglect of intra-household inequality?, Econ. J. 100 (1990), 866−881.

[6] U. Jacobsson, On the measurement of the degree of progression, J. Public Econ. 5 (1976), 161-168.

[7] R.Kanbur, L. Haddad, Are better of households more unequal or less unequal?, Oxford Econ. Pap. 46 (1994), 445−458.

[8] A. W. Marshall, I. Olkin, Inequalities: Theory of Majorization and its Applications, Acad. Press, New York, 1979.

[9] P. Moyes, A new concept of Lorenz domination, Econ. Letters 23 (1987), 203−207.

[10] A. Sen, Family and food: sex bias in poverty, in: A. Sen (Ed.), Resources, Values, and Development, Blackwell, Oxford, 1984.

[11] A. F. Shorrocks, Ranking income distributions, Economica 50 (1993), 3−17.

[12] M. E. Yaari, A note on separability and quasi concavity, Econometrica 45 (1977), 1183−1186.