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« Interactions stratégiques dans les organisations collectives »
un symposium en mémoire de Louis-André Gérard-Varet

The Bayesian Average Voting Game with a Large Population

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Summary

The average voting procedure reflects the weighted average of expressed opinions in [0,1]. Participants typically behave strategically. We characterize the equilibrium outcome of the bayesian game where voters have incomplete information about other voter's tastes. We show that when the population is sufficiently large, for a given distribution of voter's weights, the equilibrium allocation may be approximated by a simple fixed point relation. Furthermore, if we consider a sequence of games where weights and taste parameters are randomly drawn from some population then the equilibrium allocation of the bayesian game converges almost surely to the limit of the equilibrium allocation in the complete information game.

Résumé

Le vote moyen est défini comme la moyenne pondérée des votes dans $[0,1]$. Les participants se comportent de manière

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stratégique. Nous caractérisons l'équilibre du jeu bayésien quand les votants ont une information incomplète à propos des goûts des autres participants. Nous montrons que quand la population devient grande, pour une distribution donnée des poids des votants, l'allocation d'équilibre peut être approximée par une relation de point fixe dont l'expression est simple. De plus, si nous considérons une suite de jeux où les poids et les paramètres de goûts sont tirés aléatoirement d'une population, alors l'allocation d'équilibre du jeu bayésien converge presque sûrement vers la limite de l'allocation d'équilibre du jeu en information complète.

Keywords: Average voting, Bayesian equilibrium, large population, approximation.

Mots clés : Jeu de vote moyen bayésien avec une grande population.

J.E.L. : D74, H41, I22

1. Introduction

Average voting is a very simple voting scheme that implements a weighted arithmetic mean of votes. Several countries have quite recently adopted procedures for allocating public funds, that may be described by a weighted average vote. In Spain, tax payers may earmark up to 0.5% of their income tax to the catholic church or to non-governmental organizations and similar provisions can be found in Italy or Portugal. In Canadian provinces of Ontario and Saskatchewan, there are publicly financed separate school boards for Catholic schools along with the public school boards; households may choose which school system receives their property taxes. In France, high schools, colleges and universities are partly financed by a "training tax" [1] that firms must pay, although they may decide on its allocation among different teaching institutions or training programs. Typically firms, especially the smaller ones, choose to finance only one institution. These tax mechanisms are formally equivalent to weighted average voting rules. If there are only two possible uses of public funds, the vote of a tax payer is the fraction of her taxes that she chooses to allocate to one of them. Then the outcome of the

1. Payrolls are taxed at a 0.5% rate, which yields a revenue of €1.2 billion in 2002.

vote (the proportion of public funds going to either use) is a weighted average of the votes, where the weight of each voter is her share in total tax contributions. Although the weights represent the individual share in total wealth or in total tax contributions in all actual applications of the average vote that we are aware of, the interpretation of the weights may be broader. For instance, if each voter represents a group (household, constituency, country...), the weight may be the share of this group in the overall population.

Although there are numerous examples of its application, the average voting rule has only attracted limited attention. With sincere voting the average procedure has some obvious properties: it yields an efficient outcome if agent's preferences are Euclidean, in which case the set of Pareto outcomes is identical to the set of weighted average votes. If there are at least five agents, agents have Lipschitz utility functions and the voting space is multidimensional, the average voting rule is shown to be the unique anonymous and unanimous voting rule that satisfies a weakening of strategy-proofness in large voting problems (Ehlers *et al.*, 2004). In Renault and Trannoy (2003) we axiomatizate the true weighted average vote in order to shed some light on its normative properties as a benchmarck. Bilodeau (1994) in his study of tax-earmarking institutions shows that leaving the spending decisions in the hands of individuals yields a unique non-cooperative equilibrium in the core. In Renault and Trannoy (2005) we exhibit circumstances where the average rule may be more suited to protect minorities than majority voting, taking into account the strategic behavior of voters in a complete information setting. In particular, we provide a complete characterization of the Nash outcome of the game in a one dimensional space.

In many circumstances, if not always, incomplete information prevails, that is, players do not know each other's preferences. The purpose of this paper is to study of the average voting game with this information structure and to compare the outcomes of the game under complete and incomplete information. Louis-André Gérard-Varet (d'Aspremont and Gérard-Varet, 1976) has been an early and influential contributor to the analysis of public decision making under incomplete information. In particular, he has shown that the set of possible outcomes may depend on the information structure.

In the average voting game considered here, individuals choose an alternative in the [0,1] interval. We first describe the properties of the Bayesian Nash outcome in Section 2. We consider a sequence of incomplete information voting games in section 3 and we show that, for a large enough population, the outcome of the game may be approximated by a simple fixed point relation depending upon the size of the population. This result is established for a given distribution of weights. In Section 4, we consider a sequence of games where weights and taste parameters are randomly drawn from some distribution. It is proved that the outcome of the game in large populations may be approximated by the same fixed-point formula that approximates the Nash equilibrium in large populations, implying that all the results found in the context of complete information apply to the context of incomplete information as well. This simple fixed point relation involves a function that, for each possible level of the allocation in the [0,1] interval, indicates the expected relative weight of those who favor an outcome above that level. Section 5 concludes. Proofs of results are gathered in the appendix.

2. The Bayesian Average Voting Game

There are n voters with singlepeaked preferences over the choice space which is the unit interval. Each voter i chooses a vote denoted s_i in $[0,1]$ and voting involves no costs. Agents cast their votes simultaneously. The allocation is then defined by

$$y = \sum_{i=1}^{n} w_i s_i, \tag{1}$$

where $w_i \geq 0$ is the relative weight of voter i, for any i, and $\sum_{i=1}^{n} w_i = 1$. The weights are assumed to be exogenously given and non random in this and the next section. It is assumed that two agents with identical bliss points have identical preferences over the set of allocations. Let V be a continuous function defined on $[0,1]^2$ which is such that $V(y, b_i)$ is agent i's utility associated with the allocation y if his bliss point is b_i. The realization of b_i is private information to agent i while the distribution functions as well as other parameters of the game and in particular the weights are common knowledge. Bliss points are assumed to be independent, so that from the point of view of other agents, agent i's bliss point $\mathbf{b}_i$ is a random variable [2] which is assumed to have a finite mean and to be distributed according to a probability distribution with a c.d.f. denoted F_i. A pure strategy profile, s, is a mapping from $[0,1]^n$ into $[0,1]^n$ where $s_i(b_i)$ is agent i's vote when agent i's bliss point is b_i. We look for Bayesian equilibria.

Definition 2.1 –*A pure strategy Bayesian equilibrium is a strategy profile s^* which satisfies for all $i \in N$*

$$s_i^*(b_i) \in \arg\max_{s \in [0,1]} E_{\mathbf{S}^*_{-i}}(V(\mathbf{S}^*_{-i} + w_i s, b_i)) \tag{2}$$

with $\mathbf{b}_{-i} = (\mathbf{b}_1, ..., \mathbf{b}_{i-1}, \mathbf{b}_{i+1}, ..., \mathbf{b}_n)$ *and* $\mathbf{S}^*_{-i} = \sum_{j \neq i} w_j s_j^*(\mathbf{b}_j)$.

The following proposition deals with the issue of existence.

Proposition 2.1 – *There exists a pure strategy Bayesian equilibrium.*

2. Troughout the paper bold characters denote random variables.

We now turn to a characterization of the equilibrium. To this end, we make the following additional assumptions about the utility function.

Assumption 2.1 – *The function V is continuously differentiable. Let V' be the partial derivative of V with respect to y; it is assumed to be strictly decreasing in y and strictly increasing in b on* $[0,1]$.

Thus V is strictly concave in the allocation and the marginal utility of y is all the larger that the bliss allocation is large. We now establish

Proposition 2.2 – *(i) Each component s_i^* of any pure strategy Bayesian equilibrium s^* is continuous and increasing on* $[0,1]$. *(ii) There exist two pivotal bliss points $\underline{b}_i$ and $\overline{b}_i$ with $\underline{b}_i \leq \overline{b}_i$ such that $s_i^*(b_i) = 0$ if and only if $b_i \in [0, \underline{b}_i]$ and $s_i^*(b_i) = 1$ if and only if $b_i \in [\overline{b}_i, 1]$.*

The next section presents a characterization of the limit of the equilibrium outcome when the number of voters goes to infinity.

3. Approximating the Bayesian Outcome

We now consider a sequence of incomplete information voting games and we show that, for a large enough population, the outcome of the game may be approximated by a simple fixed point relation depending upon n. In the following analysis, the weights are assumed to satisfy

Assumption 3.1 – *Let $\overline{w}_n = \max_i w_i$. The sequence $\{\overline{w}_n\}$ converges to* 0.

From Proposition 2.2 there is an interval of types whose vote is strictly between 0 and 1 which is denoted $[\underline{b}_{in}, \overline{b}_{in}]$ in an n-players game. For any n, let us define $\underline{b}_n \equiv \min_i \underline{b}_{in}$ and $\overline{b}_n \equiv \max_i \overline{b}_{in}$. The following lemma shows that the interval between these two bounds shrinks as the population size becomes large.

Lemma 3.1 – Under the above assumptions, $\lim_{n\to\infty}[\overline{b}_n - \underline{b}_n] = 0$.

The intuition behind this result is the following. Recall that from the point of view of each player, an equilibrium outcome for the n-players game denoted $\mathbf{y}_n^*$ is a random variable. The only thing he is certain of, is his own vote, s_i. An increase in s_i causes the distribution of $\mathbf{y}_n^*$ to shift to the right. If his weight in the vote is large, he may wish to fine tune the distribution of the outcome by picking a vote strictly between 0 and 1. As his weight becomes smaller, it becomes more unlikely that such a fine tuning is desirable for him. Either he is better off leaving the distribution as it is by voting 0 or he chooses to throw all his weight into moving the distribution to the right.

Typically, agents with low bliss points vote 0 and agents with high bliss points vote 1. This type of behavior characterized by an overstatement of one's taste is very similar to that exhibited in the complete information setting.

Lemma 3.1 says that, in the limit, almost all agents choose an extreme vote. Then the average vote is approximately equal to the proportion of voters who vote 1 (i.e., those whose bliss point is to the right of $\bar{b}_n$). This result is now used to provide an approximation for the equilibrium outcome for n large enough.

Proposition 3.1 – *Under the above assumptions, letting $\widetilde{y}_n$ be uniquely defined by*

$$\widetilde{y}_n = \widetilde{H}_n(\widetilde{y}_n) \tag{3}$$

where $\widetilde{H}_n(y) = \sum_{i=1}^{n} w_i(1 - F_i(y))$ for all $y \in [0, 1]$, we have

$$P \lim_{n\to\infty} (\mathbf{y}_n^* - \widetilde{y}_n) = 0$$

for any sequence $\{\mathbf{y}_n^*\}$ of equilibrium outcomes.

The value $\widetilde{H}_n(y)$ may be interpreted as the expected weight of those individuals who hold a bliss point above y. The situation where the bliss point distribution is identical for all agents and has c.d.f. F yields a simple expression for the fixed point. Indeed, the function $\widetilde{H}_n$ melts down to $1 - F$ so that $\widetilde{y}_n = 1 - F(\widetilde{y}_n)$ for all n.[3]

It is now shown that the outcome of the average voting game is actually not very sensitive to the information structure when the population is large.

4. Irrelevance of the Information Structure for Large Populations

In the above analysis, we assume a particular realization of weights. Even though these weights are common knowledge for the voters, the observer may only have some aggregate knowledge of the weights vector, i.e., he does not know any more than the weight distribution. We now address the question of how the outcome of the game may be predicted by an observer who is unaware of the weight realizations.

We need to introduce more notations. A vote with n participants is given by n independent draws from a probability distribution $\mathcal{P}$ defined on $[0, 1] \times \mathbb{R}_{++}$

3. Note that convergence in that case would be with probability 1: in the beginning of the proof of Proposition 3.1 we could appeal to a strong law of large numbers argument because $\mathbf{C}_{1n}$ and $\mathbf{C}_{2n}$ defined in the proof are sums of i.i.d. random variables $I(b_i > y)$ which is not true in the general case where variables are only independent.

admitting a continuous density. For each player i, the first component is his bliss point $\mathbf{b}_i$ and the second component is his absolute weight ω_i, which contrary to relative weight $w_i = \omega_i / \sum_{i=1}^{n} \omega_i$ is not restricted to be in [0,1].[4] Let $\mu(b)$ denote the conditional mean and $\overline{\mu}$ denote the unconditional mean of ω_i and let F denote the unconditional c.d.f. of $\mathbf{b_i}$. We now define the decreasing function H on $[0,1]$ as follows

$$H(y) = \overline{\mu}^{-1} \int_y^1 \mu(b_i) dF(b_i). \tag{4}$$

This function is decreasing from $H(0) = 1$ to $H(1) = 0$. It measures the expected relative cumulative weight of individuals with bliss points in excess of y. In the special case where weights are independent from bliss points, we have $H(y) = 1 - F(y)$.

In the general case, the function H may also be related to F thanks to a concentration curve. Whenever we plot shares of a variable X against quantiles in the distribution of a variable Y, the result is called a concentration curve for X with respect to Y. Now define G as the function that, for all $y \in [0,1]$, maps $1 - F(y)$ into $H(y)$ so that

$$H(y) = G(1 - F(y)).$$

Note that $1 - F(y)$ is the expected cumulative proportion of the population with bliss points above y, while $H(y)$ is the expected cumulative relative weight of this subpopulation. The function G may therefore be viewed as the concentration curve for weights with respect to bliss points. Let y^* denote the unique solution to

$$y^* = H(y^*) = G[1 - F(y^*)]. \tag{5}$$

In Renault and Trannoy[5] (2006) we prove that the Nash equilibrium allocation of the average voting game for a sequence of votes where weights and taste parameters are randomly drawn from $\mathcal{P}$ converges almost surely to the fixed point y^*.

Now going back to the Bayesian equilibria of the average voting game, we first establish that, in this setting, Assumption 1 almost always holds if the population is large.

Lemma 4.1 – The sequence $\{\overline{\mathbf{w}}_n\}$ converges to 0 with probability 1.

Therefore, the above results, Lemma 3.1 and Proposition 3.1, hold with probability 1. As under complete information, it is possible to provide a limit characterization of the equilibrium outcome independent of n.

Proposition 4.1 – *The sequence* $\{\mathbf{y}_n^*\}$ *converges to* y^* *with probability* 1.

4. Specifying absolute weights is convenient because, contrary to relative weights, they may be drawn independently, and we may therefore resort to law of large numbers arguments in the proofs.

5. Albeit a similar formula appears in Renault and Trannoy (2005) p.12, the theorem therein is a pointwise convergence theorem and cannot be invoked for a comparison with our limit result for the Bayesian outcome.

In the limit, votes are concentrated at the extremes, 0 or 1, so that the outcome is given by the cumulative weight of those voting 1, $H(y^*)$, and it is also equal to the bliss point of the pivotal individual, y^*. The above proposition shows that in a large economy, the equilibrium outcome may be approximated by the same fixed point relation whether or not players are imperfectly informed about each other's taste.[6]

5. Conclusion

The study of the properties of the equilibrium of the average voting game in an incomplete information setting provides some interesting insights. Although we do not fully describe the set of Bayesian equilibria for a finite population, we provide a simple approximation of the equilibrium outcome which is valid if the population is sufficiently large. The approximation of the equilibrium outcome that is derived here only requires aggregate information, that is the joint distribution of weights and bliss points. Remarkably, under mild assumptions, the approximation formula is independent of the information structure of the voting game. It follows that our analysis of the protection of minorities by the average vote in Renault and Trannoy (2005) is independent of the information structure of the game.

The driving force for the coincidence of the complete and incomplete information outcomes seems to be that the game is anonymous, in the sense that the only relevant information for each player is the distribution of other players' decisions. Therefore the error made by any player when the population is finite vanishes when the sample size becomes large. It raises the question whether the irrelevance of the information structure emerges for a larger class of games sharing this anonymity property such as Cournot's model of oligopoly.

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A. Appendix

A.1. Proof of Proposition 2.1

The existence of a pure strategy equilibrium follows from theorem 3.3.1 in Balder (1995) which proves that a game with a continuum of players has a pure strategy Nash equilibrium under convexity assumptions. In our setting, we may view each player i as a continuum of players indexed according to player $i's$ type. The union of these continua is isomorphic to the $[0,1]$ interval. The expected utility of each player i expressed in (2) can be written as a function of three arguments: *i*) $i's$ type, *ii*) $i's$ strategy *iii*) a function describing how the strategy of a player depends on his type. **Q.E.D.**

A.2. Proof of Proposition 2.2

(i) Continuity follows form strict concavity of V with respect to y and the theorem of the maximum.

$E_{\mathbf{S}^*_{-i}}(V'(\mathbf{S}^*_{-i}+w_i s, b_i))$ is strictly decreasing in s on $[0,1]$ due to assumption 2.1. Let b and b' be two bliss values with $b > b'$. If $s^*(b) = 1$ then we trivially have

$s_i^*(b) \geq s_i^*(b')$. If $s_i^*(b) < 1$ we must have $E_{\mathbf{S}^*_{-i}} V'(\mathbf{S}^*_{-i} + w_i \; s_i^*(b), b) \leq 0$. Since V' is strictly increasing in b, it follows that $E_{\mathbf{S}^*_{-i}} V'(\mathbf{S}^*_{-i} + w_i \; s_i^*(b), b') < 0$. Hence $s_i^*(b) \geq s_i^*(b')$.

Since a sincere vote is a dominant strategy whenever $b_i = 0$ or 1, $s_i^*(0) = 0$ and $s_i^*(1) = 1$ and combined with statement *(i)* we deduce *(ii)* **Q.E.D.**

A.3. Proof of Lemma 3.1

Let us first show that

Step 1. $\lim_{n\to\infty} \max_i[\overline{b}_{in} - \underline{b}_{in}] = 0$.

The proof proceeds by contradiction. Suppose the limit is not zero or does not exist. Then there exists $k > 0$ such that for any N there exists $n > N$ with $\overline{b}_{in} - \underline{b}_{in} > k$ for some i. Let $g(S,b) = V'(S, b+k) - V'(S,b)$. Under assumption 1, g is strictly positive on $[0,1]^2$. Since V' is uniformly continuous in y and $\overline{w}_n$ tends to zero as n tends to infinity, if n is large enough, we have

$$E_{\mathbf{S}^*_{-in}}[V'(\mathbf{S}^*_{-in} + w_{in}, \underline{b}_{in} + k) - V'(\mathbf{S}^*_{-in}, \underline{b}_{in})] > 0. \tag{6}$$

Now, by definition of $\underline{b}_{in}$ we must have $E_{\mathbf{S}^*_{-in}} V'(\mathbf{S}^*_{-in}, \underline{b}_{in}) = 0$. On the other hand, since $\underline{b}_{in} + k \in (\underline{b}_{in}, \overline{b}_{in})$, the optimal vote of an agent of type $\underline{b}_{in} + k$ is strictly below 1 and we must have $E_{\mathbf{S}^*_{-in}} V'(\mathbf{S}^*_{-in} + w_{in}, \underline{b}_{in} + k) < 0$. This yields a contradiction.

To complete the proof we show

Step 2. $\lim_{n\to\infty} \max_{i,j}[\overline{b}_{in} - \overline{b}_{jn}] = 0$, and $\lim_{n\to\infty} \max_{i,j}[\underline{b}_{in} - \underline{b}_{jn}] = 0$.

Once again we proceed by contradiction. Suppose the first limit is not zero or does not exit. Then there exists $k > 0$ such that for any N there exists $n > N$ with $\left|\overline{b}_{in} - \overline{b}_{jn}\right| > k$ for some i,j. *W.l.o.g* we may assume that $\overline{b}_{in} > \overline{b}_{jn}$. As in Step 1, $g(S,b)$ is strictly positive on $[0,1]^2$ and, since V' is uniformly continuous on $[0,1]$ and $\overline{w}_n$ tends to zero as n goes to infinity we have

$$E_{\mathbf{S}^*_{-jn}} V'(\mathbf{S}^*_{-jn} + w_{jn} + w_{in}, \overline{b}_{jn} + k) - V'(\mathbf{S}^*_{-jn} + w_{jn}, \overline{b}_{jn}) > 0, \tag{7}$$

for n large enough. We also have

$$E_{\mathbf{S}^*_{-jn}} V'(\mathbf{S}^*_{-jn} + w_{jn} + w_{in}, \overline{b}_{jn} + k) \leq E_{\mathbf{S}^*_{-in}} V'(\mathbf{S}^*_{-in} + w_{in}, \overline{b}_{jn} + k) < 0, \tag{8}$$

where the first inequality follows from the concavity of V with respect to y and the second from the monotonicity of V' with respect to b and $\overline{b}_{jn} + k < \overline{b}_{in}$. This yields a contradiction. The second limit statement may be proved in a similar fashion. **Q.E.D.**

A.4. Proof of Proposition 3.1

Let $\mathbf{C}_{1n} = \sum_{i=1}^{n} w_i I(b_i \geq \overline{b}_n)$ be the cumulative weight of agents with bliss points above $\overline{b}_n$, where I is the indicator function. It is a random variable and we have

$$E[\mathbf{C}_{1n} - \widetilde{H}_n(\overline{b}_n)] = 0 \quad (9)$$

$$\text{and } var[\mathbf{C}_{1n} - \widetilde{H}_n(\overline{b}_n)] = \sum_{i=1}^{n} w_i^2 F_i(\overline{b}_n)(1 - F_i(\overline{b}_n)). \quad (10)$$

Similarly, let $\mathbf{C}_{2n} = \sum_{i=1}^{n} w_i I(b_i \leq \underline{b}_n)$ denote the cumulative weight of those who have bliss points less than $\underline{b}_n$. We have

$$E[\mathbf{C}_{2n} - [1 - \widetilde{H}_n(\underline{b}_n)]] = 0 \quad (11)$$

$$\text{and } var[\mathbf{C}_{2n} - [1 - \widetilde{H}_n(\underline{b}_n)]] = \sum_{i=1}^{n} w_i^2 F_i(\underline{b}_n)(1 - F_i(\underline{b}_n)). \quad (12)$$

Both variance expressions are bounded above by $\sum_{i=1}^{n} w_i^2$ which in turn is bounded above by $\overline{w}_n^2 + \overline{w}_n$. The latter upper bound is obtained by maximizing the sum expression with respect to the vector $(w_1, ..., w_n)$ subject to the constraint that it sums up to 1 and that each term is less than $\overline{w}_n > \frac{1}{n}$.[7] Hence both variances tend to 0 as n goes to infinity and therefore $\mathbf{C}_{1n} - \widetilde{H}_n(\overline{b}_n)$ and $\mathbf{C}_{2n} - [1 - \widetilde{H}_n(\underline{b}_n)]$ tend to 0 in probability as n goes to infinity. Using the mean value theorem we have

$$\widetilde{H}_n(\underline{b}_n) - \widetilde{H}_n(\overline{b}_n) \leq \beta(\overline{b}_n - \underline{b}_n)$$

where β is the uniform bound on the conditional density $f_i(b)$ (such a uniform bound exists since the unconditional mean of bliss points is assumed to be finite). Then Lemma 3.1 implies that $\lim_{n\to\infty}(\widetilde{H}_n(\overline{b}_n) - \widetilde{H}_n(\underline{b}_n)) = 0$. Thus $\mathbf{C}_{1n} - \widetilde{H}_n(\underline{b}_n)$ must also tend to 0 in probability. Now if $\mathbf{y}_n^*$ is the equilibrium outcome, we have

$$\mathbf{C}_{1n} \leq \mathbf{y}_n^* \leq 1 - \mathbf{C}_{2n}. \quad (13)$$

Rearranging and taking the limit in probability yields

$$\text{p}\lim_{n\to\infty}[\mathbf{y}_n^* - \widetilde{H}_n(\underline{b}_n)] = 0. \quad (14)$$

Hence, to prove the result, it is enough to show that

$$\lim(\underline{b}_n - \widetilde{y}_n) = 0.$$

7. The expression $\overline{w}_n^2 + \overline{w}_n$ is actually an upper bound on the maximal value of the sum. Because the sum is strictly convex, it is maximized by a corner solution in which at most one weight is strictly between zero and $\overline{w}_n$. The upper bound is obtained by noting that the largest number of individuals who may be awarded the largest weight is bounded above by $\frac{1}{\overline{w}_n}$ and the remaining weight is bounded above by $\overline{w}_n$.

We first show

Claim A.1 *There exists a sequence of strictly positive numbers* $\{\eta_n\}$ *such that* $\widetilde{H}_n(\underline{b}_n) \leq \underline{b}_n + \eta_n$ *for all* n*, and* $\lim_{n\to\infty} \eta_n = 0$.

Suppose to the contrary that there exists a $K > 0$ and a subsequence $\underline{b}_{n_k}$ such that $\widetilde{H}_n(\underline{b}_{n_k}) > \underline{b}_{n_k} + K$ for all n_k. Then

$$V(\widetilde{H}_{n_k}(\underline{b}_{n_k}) + s, \underline{b}_{n_k} + \Delta) < V(\widetilde{H}_{n_k}(\underline{b}_{n_k}), \underline{b}_{n_k} + \Delta) \tag{15}$$

for any $s > 0$ and any $\Delta \in (0, K)$. Furthermore using (14), by a standard bounded convergence argument we have

$$\lim_{n_k\to\infty} E_{\mathbf{y}^*_{n_k}}\left[V(\mathbf{y}^*_{n_k} + s, \underline{b}_{n_k} + \Delta)\right] - V(\widetilde{H}_{n_k}(\underline{b}_{n_k}) + s, \underline{b}_{n_k} + \Delta) = 0 \tag{16}$$

for all $s \in [0, 1]$. Thus, using equation (15) and (16), for n_k sufficiently large,

$$E_{\mathbf{y}^*_{n_k}}\left[V(\mathbf{y}^*_{n_k} + s, \underline{b}_{n_k} + \Delta)\right] < E_{\mathbf{y}^*_{n_k}}\left[V(\mathbf{y}^*_{n_k}, \underline{b}_{n_k} + \Delta)\right] \tag{17}$$

It is optimal for any agent of type $\underline{b}_{n_k} + \Delta$ to vote 0. Thus $\underline{b}_{n_k} \geq \underline{b}_{n_k} + \Delta$, which contradicts $\Delta > 0$.

Claim A.2 *There exists a sequence of strictly positive numbers* $\left\{\underline{\epsilon}_n\right\}$ *such that* $\widetilde{y}_n \leq \underline{b}_n + \underline{\epsilon}_n$ *for all* n *and* $\lim_{n\to\infty} \underline{\epsilon}_n = 0$.

Using the definition of $\widetilde{y}_n$ and Claim A.1 we have

$$\widetilde{H}_n(\underline{b}_n) - \widetilde{H}_n(\widetilde{y}_n) + \widetilde{y}_n - \underline{b}_n < \eta_n. \tag{18}$$

If $\underline{b}_n \geq \widetilde{y}_n$, claim A.2 trivially holds. If $\underline{b}_n < \widetilde{y}_n$, the left hand side of (18) is strictly positive since $\widetilde{H}_n$ is decreasing. If this is the case for some subsequences $\{\underline{b}_{n_k}\}$ and $\{\widetilde{y}_{n_k}\}$, then the left-hand side of (18) must tend to 0 when n_k goes to infinity since $\lim_{n_k\to\infty} \eta_{n_k} = 0$. Since both $\widetilde{y}_n - \underline{b}_n$ and $\widetilde{H}_n(\underline{b}_n) - \widetilde{H}_n(\widetilde{y}_n)$ are positive, they must both tend to zero. Thus claim A.2 holds.

Symmetrically it can be shown that there exists a sequence of strictly positive numbers $\{\overline{\varepsilon}_n\}$ such that $\widetilde{y}_n \geq \overline{b}_n + \overline{\varepsilon}_n$ for all n and $\lim_{n\to\infty} \overline{\varepsilon}_n = 0$. Using Lemma (3.1) we have $\lim_{n\to\infty}(\underline{b}_n - \widetilde{y}_n) = 0$. **Q.E.D.**

A.5. Proof of Lemma 4.1

We may rewrite $\overline{\mathbf{w}}_n$ as $\left[n\frac{1}{n}\sum_{i=1}^{n} \frac{\omega_i}{\max_i \omega_i}\right]^{-1}$. The random variable $\frac{\omega_i}{\max_i \omega_i}$ takes on values in [0,1] and has a finite and strictly positive expectation. The results follows from applying the strong law of large numbers.

A.6. Proof of Proposition 4.1

By an argument similar to that of the proof of Proposition 3.2 in Renault and Trannoy (2004) who adapt a proof of Goldie (1977), it can be shown that $\widetilde{\mathbf{H}}_n$ converges to H uniformly with probability 1. Indeed $\widetilde{\mathbf{H}}_n(y) = \frac{[\sum_i \omega_i [1-F(y|\omega_i]/n]}{[\sum_i \omega_i/n]}$. Variables at the top are i.i.d with mean $H_n(y)\overline{\mu}$, while variables at the bottom are i.i.d with mean $\overline{\mu}$. Then the strong law of large numbers may be used in a similar fashion as in the proof of Proposition 3.2 in Renault and Trannoy (2004) to establish uniform convergence. The result follows. **Q.E.D.**